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Steady-State Losses of Optical Fibers and Fiber Resonators

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We study the steady-state loss of a fiber with random, nearest-neighbor coupling and compare it with the mode with the lowest loss of a cavity formed from a section of the same type of fiber. We find that the loss of the cavity is not identical with the loss of the steady-state distribution of the fiber with random coupling. In fact, fiber and fiber resonator behave very differently if the fiber mode of highest order is made very lossy. The loss of the steady-state distribution of the fiber with random, nearest-neighbor coupling approaches a weighted average of the losses of its individual modes plus a contribution from the coupling coefficient that couples the highest-order mode to its neighbors. The cavity loss, on the other hand, becomes independent of the coupling coefficients and of the loss of the highest-order mode, provided this loss becomes much higher than the coupling strength. This behavior leads us to conclude that the loss of the cavity is a weighted average of the losses of all those modes whose coupling strength exceeds their (individual, uncoupled) loss coefficients. Two resonator modes with propagation constants β_1 and β_2 remain uncoupled unless they satisfy the condition $\beta_1 - \beta_2 = 2\pi n/L$, where n is an integer and L is twice the length of the resonator.

I. INTRODUCTION

We consider a multimode optical fiber with random imperfections. It is well-known that any type of imperfection built into a fiber causes coupling among its guided modes.[1–3] In a long fiber, the distribution of average power versus mode label approaches a steady state that can be described by a steady-state loss coefficient and a unique distribution function.[3]

Now assume that we take a section of this fiber, place reflectors at either end, and observe the steady-state power distribution of this cavity. Without giving the matter much thought, we might expect the steady-state power distribution of the resonator to be identical to the steady-state power distribution of the long fiber. However, this is not the case. Mode coupling in a resonator has a very different effect on the steady-state power distribution and its loss coefficient than coupling in a long fiber. The reason for this difference in behavior is the fact that the wave traveling back and forth in the resonator experiences a periodic structure whose Fourier transform has a line spectrum. In a resonator of length $L/2$, two modes with propagation constants β_1 and β_2 are effectively coupled only if they satisfy the condition $\beta_1 - \beta_2 = 2\pi n/L$, where n is an integer. The losses and steady-state power distribution of the long fiber and the corresponding fiber resonator are very different. It is the purpose of this paper to clarify these differences.

We dramatize the difference of the fiber and the resonator by considering a fiber supporting only two guided modes. Furthermore, we assume that one of the two modes is relatively very lossy (in the absence of coupling), while the other mode has either no loss at all or very much lower loss. In a long fiber, the loss of the steady-state power distribution turns out to be the sum of the loss coefficient of the (uncoupled) low-loss mode plus the power-coupling coefficient of the two modes. This result is intuitively pleasing. It says that there are two independent loss mechanisms that reinforce each other additively—the loss of the first mode (in the absence of coupling to its high-loss companion) and the coupling of the low-loss mode to the high-loss mode. Since the high-loss mode carries practically no power, coupling of power to this mode appears directly as a loss coefficient.

Naively, it should be expected that the same behavior occurs in the fiber cavity. However, this is not true. In the resonator, the loss of the resonant field distribution is identical to the loss of the low-loss mode alone. Coupling between the two modes has no influence on the loss of the resonator, provided that the loss of the second mode is very high compared to the coupling coefficient. It is hard to understand this situation intuitively. In the periodic structure (the resonator), the field apparently manages to shape itself in such a way that it avoids carrying power in those regions that provide high loss. Since the structure is periodic, the field passes over the same region again and again, adjusting itself to the unfavorable loss situation. In the long fiber with random coupling, no such adjustment is possible. The field does not have the chance to establish a normal mode and is confronted with a new, random-coupling situation in each section of the fiber. In this case, coupling to the high-loss mode simply subtracts power from the low-loss mode that is irretrievably lost.

The results presented in this paper are needed for the discussion of scattering losses in a fiber laser that is the subject of Ref. 4.

II. TWO-MODE CASE

For a fiber supporting only two modes, the problem can be solved easily. We describe each mode by its amplitude coefficient a_1 and a_2. The interaction of the two modes is described by the familiar coupled-wave equations[5] (self-coupling coefficients only modify the real parts of the propagation constants and are therefore omitted),

$$\frac{da_1}{dz} = -i\gamma_1 a_1 + K_{12} a_2 \tag{1}$$

$$\frac{da_2}{dz} = -i\gamma_2 a_2 + K_{21} a_1. \tag{2}$$

The complex propagation constants $\gamma_{1,2}$ contain the loss coefficients $\alpha_{1,2}$ of each mode in the absence of coupling,

$$\gamma_n = \beta_n - i\alpha_n \qquad n = 1, 2. \tag{3}$$

The coupling coefficients obey the symmetry relation[6]

$$K_{21} = -K_{12}^*. \tag{4}$$

It is convenient to express the z-dependence of K_{12} explicitly ($\hat{K}$ is real),

$$K_{12} = i\hat{K} f(z). \tag{5}$$

To first-order perturbation theory, it is only the Fourier component of $f(z)$ at the spatial frequency $\theta = \beta_1 - \beta_2$ that contributes to coupling between the modes.[3] This allows us to write (1) and (2) as

$$\frac{da_1}{dz} = -i\gamma a_1 + i\hat{K} b a_2 e^{-i(\beta_1 - \beta_2)} \tag{6}$$

$$\frac{da_2}{dz} = -i\gamma_2 a_2 + i\hat{K} b a_1 e^{i(\beta_1 - \beta_2)}. \tag{7}$$

We have assumed that

$$f(z) = \sum_{\nu=1}^{\infty} 2b_\nu \cos \theta_\nu z, \qquad \theta_\nu = \frac{2\pi}{L}\nu, \tag{8}$$

with real values of b_ν, and have included in (6) and (7) only the terms in (8) that contribute to mode coupling, dropping the index on b_ν.

We introduce new variables A_1 and A_2 by the definitions

$$a_1(z) = A_1(z)e^{-\frac{1}{2}i(\beta_1-\beta_2)z} \tag{9}$$

and

$$a_2(z) = A_2(z)e^{\frac{1}{2}i(\beta_1-\beta_2)z}, \tag{10}$$

and obtain, by substitution into (6) and (7),

$$\frac{dA_1}{dz} = -i\delta_1 A_1 + icA_2 \tag{11}$$

and

$$\frac{dA_2}{dz} = -i\delta_2 A_2 + icA_1, \tag{12}$$

with

$$\delta_n = \tfrac{1}{2}(\beta_1 + \beta_2) - i\alpha_n \qquad n = 1, 2 \tag{13}$$

and

$$c = \hat{K}b. \tag{14}$$

Equations (11) and (12) represent two modes coupled by a constant-coupling coefficient. These equations are not exact representations of the starting equations (1) and (2), but they are good approximations. Comparison has been made of the results of this theory with the result of an exact theory of a two-mode model using a straight fiber with discrete offsets alternating periodically in opposite directions. The exact theory agrees with the approximation presented here, provided that the differential loss of the modes is small,

$$|\alpha_1 - \alpha_2| < \frac{4}{L}, \tag{15}$$

and that the following condition holds to an accuracy on the order of $|\alpha_1 - \alpha_2|$:

$$\beta_1 - \beta_2 = \frac{2\pi}{L}n, \tag{16}$$

with n indicating an integer. If (16) cannot be satisfied for any integer n, the two modes remain effectively uncoupled. Our derivation makes it clear that the coupling process is periodic with a period

$$L = n\frac{2\pi}{\beta_1 - \beta_2}. \tag{17}$$

A periodic structure of this type can be used to represent a resonant cavity. It is only necessary to envision the field traveling back and forth in the resonator; when we unfold the resonator of length $L/2$, the periodic structure results.

We now consider the normal modes of the coupled-equation system

(11) and (12) by asking for solutions of the type

$$A_n = B_n e^{-(i/2)(\beta_1+\beta_2)z} e^{-\sigma z} \qquad n = 1, 2, \tag{18}$$

with constant coefficients B_1 and B_2. Substitution of (18) into (11) and (12) results in the equation system

$$(\sigma - \alpha_1)B_1 + icB_2 = 0 \tag{19}$$

and

$$icB_1 + (\sigma - \alpha_2)B_2 = 0. \tag{20}$$

The requirement that the determinant of the equation system (19) and (20) must vanish leads to the determination of the two eigenvalues,

$$\sigma^{(1)} = \tfrac{1}{2}(\alpha_1 + \alpha_2) - \tfrac{1}{2}\sqrt{(\alpha_2 - \alpha_1)^2 - 4c^2} \tag{21}$$

and

$$\sigma^{(2)} = \tfrac{1}{2}(\alpha_1 + \alpha_2) + \tfrac{1}{2}\sqrt{(\alpha_2 - \alpha_1)^2 - 4c^2}. \tag{22}$$

The amplitude coefficients can be expressed as

$$B_2^{(k)} = i\,\frac{\sigma^{(k)} - \alpha_1}{c}\,B_1^{(k)} \qquad k = 1, 2. \tag{23}$$

The actual field amplitudes may now be expressed as a superposition of the two normal modes of the coupled system. Only the normal mode with the lower loss survives for $z \to \infty$ so that the steady-state loss of the resonator is given by the eigenvalue (21).

It is interesting to distinguish two cases. For strong coupling, $c \gg |\alpha_2 - \alpha_1|$, we have, from (21),

$$\operatorname{Re}(\sigma^{(1)}) = \tfrac{1}{2}(\alpha_1 + \alpha_2). \tag{24}$$

For weak coupling, $c \ll |\alpha_2 - \alpha_1|$, we have, instead,

$$\sigma^{(1)} \approx \alpha_1 + \frac{c^2}{\alpha_2 - \alpha_1}. \tag{25}$$

Next we consider the long fiber with two randomly coupled modes. It is possible to derive the solution for this case directly from the coupled wave equations (1) and (2). However, the same result is obtained from the corresponding coupled power equations[3,7,8] ($P_n = \langle |a_n|^2 \rangle$),

$$\frac{dP_1}{dz} = -2\alpha_1 P_1 + h(P_2 - P_1) \tag{26}$$

and

$$\frac{dP_2}{dz} = -2\alpha_2 P_2 + h(P_1 - P_2). \tag{27}$$

The power coupling coefficient is[3]

$$h = \hat{K}^2 \langle |F(\theta)|^2 \rangle, \tag{28}$$

with the Fourier transform of $f(z)$ defined as[3]

$$F(\theta) = \frac{1}{\sqrt{L}} \int_0^L f(z) e^{i\theta z} dz = b\sqrt{L}. \tag{29}$$

The coefficient b is defined by (8) and (14) and L is the length of the periodic structure or twice the length of the resonator. According to (14), (28), and (29), the coupling coefficients c and h are thus related as

$$h = c^2 L. \tag{30}$$

A steady-state solution for the long fiber with random coupling is again obtained with the help of the trial solution,

$$P_n = Q_n e^{-2\rho z}. \tag{31}$$

We find, from (26) and (27),

$$\rho^{(1)} = \tfrac{1}{2}(\alpha_1 + \alpha_2 + h) - \tfrac{1}{2}\sqrt{(\alpha_2 - \alpha_1)^2 + h^2} \tag{32}$$

and

$$\rho^{(2)} = \tfrac{1}{2}(\alpha_1 + \alpha_2 + h) + \tfrac{1}{2}\sqrt{(\alpha_2 - \alpha_1)^2 + h^2}. \tag{33}$$

The smaller eigenvalue is the steady-state loss coefficient.[3] In the case of strong coupling, $h \gg |\alpha_2 - \alpha_1|$, we have, from (32),

$$\rho^{(1)} \approx \tfrac{1}{2}(\alpha_1 + \alpha_2) - \frac{(\alpha_2 - \alpha_1)^2}{4h}, \tag{34}$$

while we obtain, in the case of weak coupling, $h \ll |\alpha_2 - \alpha_1|$,

$$\rho^{(1)} \approx \alpha_1 + \tfrac{1}{2}h - \frac{h^2}{4(\alpha_2 - \alpha_1)}. \tag{35}$$

The power coefficients are related in the following way:

$$Q_2 = \frac{h - 2\rho^{(i)} + 2\alpha_1}{h} Q_1. \tag{36}$$

III. DISCUSSION OF THE TWO-MODE CASE

We are now ready to compare the steady-state losses of the long fiber and the fiber resonator. In case of strong coupling, we have approximately

$$\sigma^{(1)} = \tfrac{1}{2}(\alpha_1 + \alpha_2) \tag{37a}$$

for the resonator and

$$\rho^{(1)} = \tfrac{1}{2}(\alpha_1 + \alpha_2) \tag{37b}$$

for the long fiber. Strong coupling ties the two modes together so effectively that the steady-state losses are equal to the average losses of the uncoupled modes in either case. From (21) and (23), we find

$B_2 = B_1$ and from (36) $Q_2 = Q_1$ if the coupling coefficients are very much larger than the loss coefficients. We thus see that both modes carry equal amounts of power in the strong coupling case. There is no difference in loss behavior between the resonator and long fiber if the coupling is strong.

The situation changes for weak coupling. From (25), we have in the limit $c^2/\alpha_2 \rightarrow 0$ for the resonator

$$\sigma^{(1)} = \alpha_1, \tag{38}$$

while (35) yields for the long fiber

$$\rho^{(1)} = \alpha_1 + \tfrac{1}{2}h. \tag{39}$$

In the resonator, coupling to a relatively lossy mode has no effect on the loss of the steady-state field distribution. The solution of the exactly solvable model shows that this is true even if (15) and (16) are not satisfied. The resonator loss becomes equal to the loss of the low-loss mode as though coupling were absent. In the fiber with random coupling, (39) shows that the steady-state loss is equal to the sum of the inherent loss of the low-loss mode plus half the power coupling coefficient. Coupling to the high-loss mode thus expresses itself directly as a loss factor. The power ratios of the two modes are also of interest. From (23), (25), and (30), we find for the weak coupling (or high-loss) resonator case

$$\left|\frac{B_2}{B_1}\right|^2 = \frac{h}{4(\alpha_2 - \alpha_1)}\,\frac{4}{(\alpha_2 - \alpha_1)L}. \tag{40}$$

For the randomly coupled fiber we obtain, from (35) and (36),

$$\frac{Q_2}{Q_1} = \frac{h}{4(\alpha_2 - \alpha_1)}. \tag{41}$$

IV. THE MULTIMODE CASE

We have seen in the section on the two-mode case that we may consider coupled wave equations with constant coupling coefficients if we suitably redefine the mode amplitudes. In addition, we shall assume that only modes that are nearest neighbors are coupled in the resonator or long fiber. This assumption is justified by the observation that the Fourier components of the coupling function $f(z)$ tend to drop off very rapidly with increasing spatial frequencies so that coupling of modes that are not nearest neighbors (such coupling is caused by Fourier components with higher spatial frequencies) is much weaker than nearest-neighbor coupling. In addition only modes satisfying (16) are coupled to each other. Consider the coupled equation system,

$$\frac{dA_\nu}{dz} = -i\gamma_\nu A_\nu + K_{\nu,\nu-1}A_{\nu-1} + K_{\nu,\nu+1}A_{\nu+1} \qquad \nu = 1, 2, \cdots, N. \tag{42}$$

We have shown for the two-mode case that we may assume the real parts of all γ_νs are identical,

$$\gamma_\nu = \beta - i\alpha_\nu. \tag{43}$$

A normal mode solution of (42) is obtained with the help of the trial solution,

$$A_\nu = B_\nu e^{-i\beta z}e^{-\sigma z}. \tag{44}$$

Substitution of (44) into (42) results in a homogeneous algebraic equation system whose determinant must vanish. For $N = 6$, the determinantal equation assumes the form

$$\begin{vmatrix} \alpha_1 - \alpha & -K_1 & 0 & 0 & 0 & 0 \\ K_1 & \alpha_2 - \alpha & -K_2 & 0 & 0 & 0 \\ 0 & K_2 & \alpha_3 - \alpha & -K_3 & 0 & 0 \\ 0 & 0 & K_3 & \alpha_4 - \alpha & -K_4 & 0 \\ 0 & 0 & 0 & K_4 & \alpha_5 - \alpha & -K_5 \\ 0 & 0 & 0 & 0 & K_5 & \alpha_6 - \alpha \end{vmatrix} = 0. \tag{45}$$

For strong coupling with $K_\nu \gg \alpha_\mu$; $\nu, \mu = 1, 2, \cdots, 5$ but $a_6 \gg \alpha_\nu$ and $\alpha_6 \gg K_\nu$, the smallest real root of this equation may be approximated by

$$\alpha = \frac{\alpha_1 K_2^2 K_4^2 + \alpha_3 K_1^2 K_4^2 + \alpha_5 K_1^2 K_3^2 + K_1^2 K_3^2 K_5^2/(\alpha_6 - \alpha_1)}{K_2^2 K_4^2 + K_1^2 K_4^2 + K_1^2 K_3^2}. \tag{46}$$

We have assumed that the guided mode of highest order, $\nu = 6$ in this case, is coupled very strongly to the radiation modes so that its loss coefficient is much larger than that of all the other guided modes and also larger than the coupling coefficients among the guided modes. This assumption is usually made in the analysis of fibers with many coupled guided modes.[3] For the special case $K_1 = K_2 = \cdots K_5 = \text{const.}$, (46) simplifies to

$$\alpha = \tfrac{1}{3}(\alpha_1 + \alpha_3 + \alpha_5) + \frac{K^2}{3(\alpha_6 - \alpha_1)}. \tag{47}$$

For weak coupling, $K_\nu \ll \alpha_\mu$, we have the approximation

$$\alpha = \alpha_1 + \frac{K_1^2}{\alpha_2 - \alpha_1}. \tag{48}$$

Here we assumed that $\alpha_1 < \alpha_2 < \cdots \alpha_6$, in this case, (48) represents the smallest solution of (45).

The corresponding coupled power equations may be expressed in the form

$$\frac{dP_\nu}{dz} = -(2\alpha_\nu + h_{\nu-1} + h_\nu)P_\nu + h_{\nu-1}P_{\nu-1} + h_\nu P_{\nu+1}. \tag{49}$$

The trial solution

$$P_\nu = e^{-2\alpha z}Q_\nu \tag{50}$$

leads to the algebraic equation system

$$(2\alpha_\nu + h_{\nu-1} + h_\nu - 2\alpha)Q_\nu - h_{\nu-1}Q_{\nu-1} - h_\nu Q_{\nu+1} = 0. \tag{51}$$

The eigenvalue α is obtained as the solution of a determinantal equation which, for six modes, assumes the form

$$\begin{vmatrix} (2\alpha_1+h_1-2\alpha) & -h_1 & 0 & 0 & 0 & 0 \\ -h_1 & (2\alpha_2+h_2+h_1-2\alpha) & -h_2 & 0 & 0 & 0 \\ 0 & -h_2 & (2\alpha_3+h_3+h_2-2\alpha) & -h_3 & 0 & 0 \\ 0 & 0 & -h_3 & (2\alpha_4+h_4+h_3-2\alpha) & -h_4 & 0 \\ 0 & 0 & 0 & -h_4 & (2\alpha_5+h_5+h_4-2\alpha) & -h_5 \\ 0 & 0 & 0 & 0 & -h_5 & (2\alpha_6+h_5-2\alpha) \end{vmatrix} = 0. \tag{52}$$

For strong coupling (in the sense used in the cavity case), we obtain the following approximation from (52) in the special case $h_\nu = h = \text{const.}$,

$$\alpha = \frac{1}{15}(5\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5) + \frac{h}{30} - \frac{h^2}{60\alpha_6}. \tag{53}$$

For weak coupling we find

$$\alpha = \alpha_1 + \tfrac{1}{2}h_1. \tag{54}$$

V. DISCUSSION OF THE MULTIMODE CASE

If we consider that the losses α_ν are caused by random coupling between guided modes and radiation modes, we may assume the following dependence[9,10] on the mode label ν:

$$\alpha_\nu = \alpha_1\nu^2. \tag{55}$$

If the Fourier amplitudes b_ν in (8) were independent of the spatial frequency, we would have[3]

$$K_\nu = K_1\nu^2 \qquad \text{and} \qquad h_\nu = h_1\nu^4. \tag{56}$$

However, the assumption of nearest-neighbor coupling becomes questionable in this case. It is natural to consider the case

$$K_\nu = K_1 = \text{const.} \qquad \text{and} \qquad h_\nu = h_1 = \text{const.}, \tag{57}$$

and in our discussion of numerical examples we also include the case of decreasing coupling strength between neighboring modes of higher order,

$$K_\nu = \frac{K_1}{\nu} \qquad \text{and} \qquad h_\nu = \frac{h_1}{\nu^2}. \tag{58}$$

Numerical solutions of the eigenvalue equations (45) and (52) were obtained by computer. The relation between the coupling coefficients K_ν and h_ν is given by (30):

$$h_\nu = LK_\nu^2, \tag{59}$$

but for the purpose of comparing the long fiber with the fiber resonator it seems more realistic to choose instead

$$K_1 = \tfrac{1}{2}h_1, \tag{60}$$

because this choice yields the same ratios of coupling coefficients to attenuation coefficient for mode 1 and mode 2 according to (45) and (52).

Table I lists numerical values of the lowest eigenvalues of (45) and (52). It was found that, for strong coupling with $K_\nu =$ const., the lowest eigenvalue of (45) is of the form

$$\alpha = \alpha_r \pm i\alpha_i. \tag{61}$$

The imaginary part of this expression is simply a correction to the propagation constant of the normal mode solution, while the real part has the meaning of the loss of the normal mode of the cavity. However, since the solutions of the cavity loss coefficients are not real, our approximate solution (46) does not apply because the approximation (46) yields the smallest real eigenvalue of (45).

Table I shows that (with two exceptions) the cavity losses are generally lower than the losses of the corresponding fiber with random coupling. This fact is in agreement with the two-mode case. Furthermore, the numbers in the table show that an increase of the loss of mode 6 increases the steady-state loss of the fiber with random mode coupling while it decreases the loss of the fiber cavity. This behavior is in qualitative agreement with approximate formulas (47) and (53). In addition to the exact solutions of eigenvalue equations (45) and (52), Table I also contains entries for the approximate solutions obtained from one of the appropriate formulas (46), (47), (48), (53), or (54). In comparing the approximate and exact solutions for the cavity, we must remember that approximations (46) and (47) do not necessarily yield the eigenvalue with the lowest numerical value. In fact, only for the case $K_\nu = K_1\nu^2$ and for small values of $K_\nu = K_1 \ll \alpha_\nu$ do the approximations apply to the solution with the lowest loss. However, comparison of the exact and approximate values in Table I makes it apparent that the approximation provides a

Table I — Loss values of the cavity mode with lowest loss and steady-state loss of the corresponding fiber with random coupling. Exact solutions of eigenvalue equations (45) and (52) are compared with approximate solutions

h_ν	K_ν	λK_1	α_6	$\alpha\lambda$			
				Exact		Approximate	
				Cavity	Fiber	Cavity	Fiber
$h_1\nu^4$	$K_1\nu^2$	10^{-5}	$\alpha_1 6^2$ $\alpha_5 10^3$	4.484×10^{-6} 1.923×10^{-6}	8.626×10^{-6} 1.022×10^{-5}	1.95×10^{-6}	
h_1/ν	K_1/ν	10^{-5}	$\alpha_5 10^3$	3.223×10^{-6}	3.129×10^{-6}	1.77×10^{-5}	
h_1	K_1	10^{-7}	$\alpha_5 10^3$	1.003×10^{-6}	1.097×10^{-6}	1.00×10^{-6}	1.10×10^{-6}
h_1	K_1	10^{-6}	$\alpha_5 10^3$	1.361×10^{-6}	1.758×10^{-6}	1.00×10^{-6}	2.00×10^{-6}
h_1	K_1	10^{-5}	$\alpha_1 6^2$ $\alpha_5 10^6$	6.383×10^{-6} 6.415×10^{-6}	4.321×10^{-6} 4.322×10^{-6}	1.17×10^{-5}	7.67×10^{-6}
h_1	K_1	10^{-4}	$\alpha_5 10^6$	1.015×10^{-5}	1.347×10^{-5}	1.17×10^{-5}	1.37×10^{-5}
h_1	K_1	10^{-3}	$\alpha_5 10^6$	1.017×10^{-5}	8.681×10^{-5}	1.17×10^{-5}	7.37×10^{-5}
h_1	K_1	10^{-2}	$\alpha_5 10^6$ $\alpha_5 10^3$	1.050×10^{-5} 2.931×10^{-4}	8.159×10^{-4} 7.112×10^{-4}	1.30×10^{-5}	6.73×10^{-4}

Note: $\alpha_\nu = \alpha_1\nu^2$, $\alpha_1 = 10^{-6}/\lambda$, $K_1 = \frac{1}{2}h_1$, and $\nu \neq 6$.

reasonable order-of-magnitude estimate of the loss values of the cavity modes and gives at least an upper bound to the exact values.

The approximate solutions for the fiber case with randomly coupled modes do apply to the solution with the lowest loss. Comparison of the exact and approximate values in Table I show that the approximations (53) and (54) are not very precise but again may be regarded as order-of-magnitude estimates.

Table II shows the complete solution of the eigenvalue equation (45) for the fiber cavity for a typical case: $K_\nu = K_1 = 10^{-4}/\lambda$, $\alpha_1 = 10^{-6}/\lambda$, $\alpha_\nu = \alpha_1\nu^2$, $\alpha_6 = \alpha_5 10^3$. As in all cases with $K_\nu = K_1 \gg \alpha_\mu$,

Table II — Complete solution of eigenvalue equation of the fiber cavity (45) for a particular case and comparison of the exact solution to approximation (47)

i	α_r	α_i	α_{appr}
1	1.018×10^{-5}	1.726×10^{-4}	
2	1.018×10^{-5}	-1.726×10^{-4}	
3	1.162×10^{-5}	1.001×10^{-4}	
4	1.162×10^{-5}	-1.001×10^{-4}	
5	1.180×10^{-5}	0.0	1.180×10^{-5}
6	2.500×10^{-2}	0.0	

Note: $K_\nu = K_1 = 10^{-4}/\lambda$, $\alpha_\nu = \alpha_1\nu^2$, $\alpha_1 = 10^{-6}/\lambda$, and $\alpha_6 = \alpha_5 10^3 = 2.5 \times 10^{-2}$.

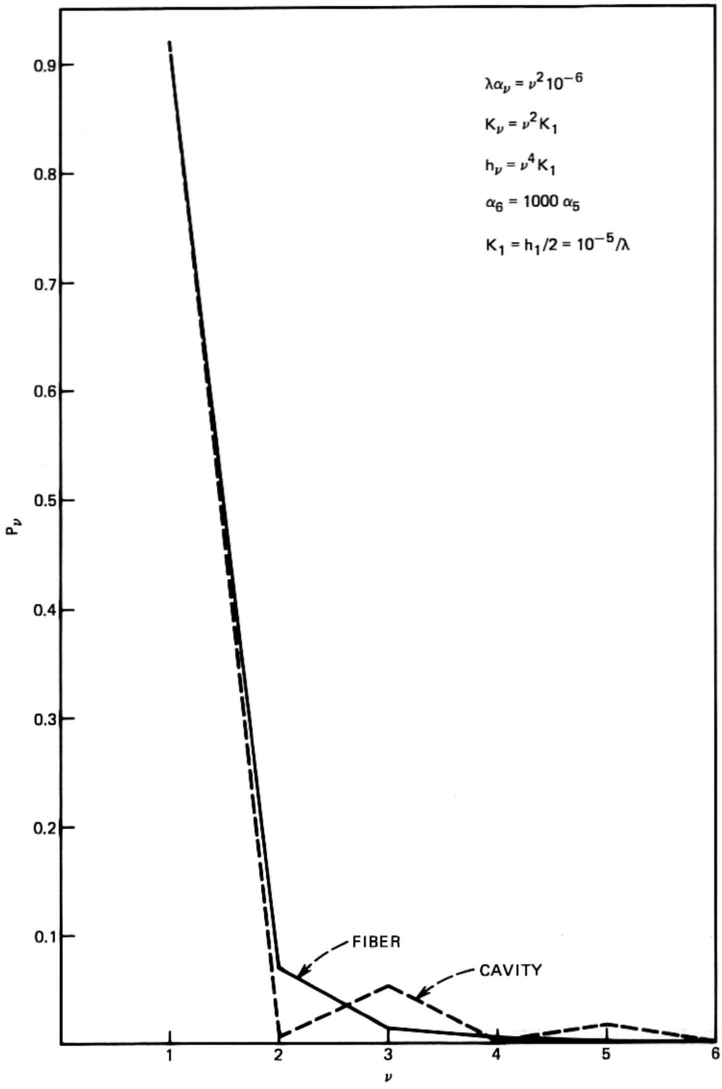

Fig. 1—Normalized power versus mode number distribution for the case of a fiber with random coupling (solid line) and the lowest loss mode of the fiber cavity (dotted line). $K_\nu = K_1\nu^2$, $h_\nu = h_1\nu^4$, $K_1 = 0.5h_1 = 10^{-5}/\lambda$, $\alpha_\nu = \alpha_1\nu^2$ with $\alpha_1 = 10^{-6}/\lambda$, $\alpha_6 = 1000\alpha_5$.

$\mu < 6$, there are two sets of complex, conjugate solutions and two single, real solutions. Approximation (47) yields the smallest of the real solutions to a remarkable accuracy.

It is interesting to compare the distribution of power versus mode number for the cavity and fiber cases. Figure 1 (and all subsequent figures) shows the normalized power P_ν as a function of the mode label ν. For Fig. 1 we used $K_\nu = K_1\nu^2$ for the fiber cavity and $h_\nu = h_1\nu^4$ for the fiber with random-mode coupling with $K_1 = h_1/2 = 10^{-5}/\lambda$. This and all the other figures were computed with $\alpha_\nu = 10^{-6}\nu^2/\lambda$, for $\nu = 1, 2, \cdots, 5$. In Fig. 1 we assumed $\alpha_6 = \alpha_5 10^3$. Even though the coupling strength is increasing for nearest neighbors with increasing mode number, mode 1 carries by far the most power. Of course, we have only plotted the power distribution for the mode with the lowest loss. In the cavity case, there are solutions with the maximum power in any one of the six modes. The coupled power problem of the fiber with random coupling also has six different solutions. However, only the solutions with the lowest loss value have physical significance[3] as the steady-state power distribution. This solution is shown in Fig. 1 and the subsequent figures.

Figure 2 was drawn for almost the same condition as Fig. 1, except that we used the law $\alpha_\nu = \alpha_1\nu^2$ for all six values of ν. This has the consequence that the loss of the mode of highest order, $\nu = 6$, is now much lower than in Fig. 1 so that more power is carried by the higher-order modes.

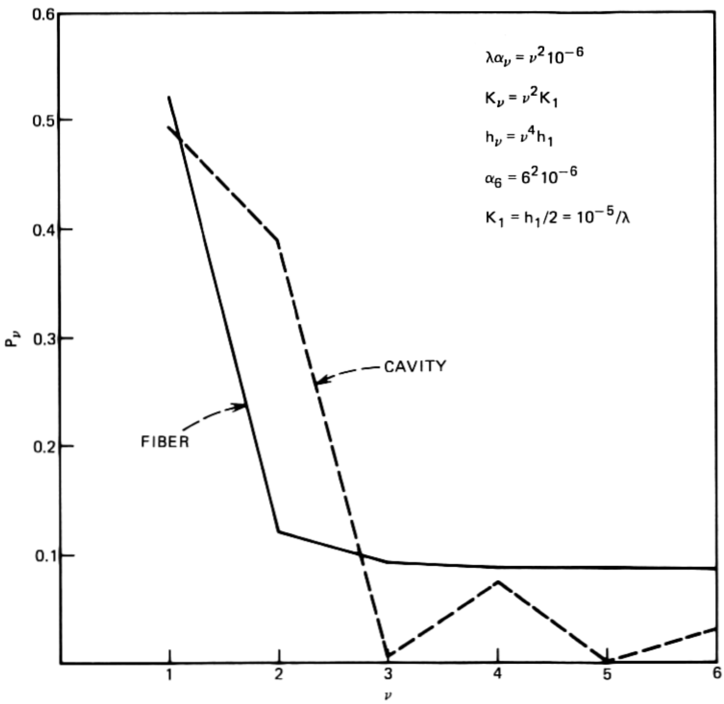

Fig. 2—Same as Fig. 1 but with $\alpha_6 = \alpha_1 6^2$.

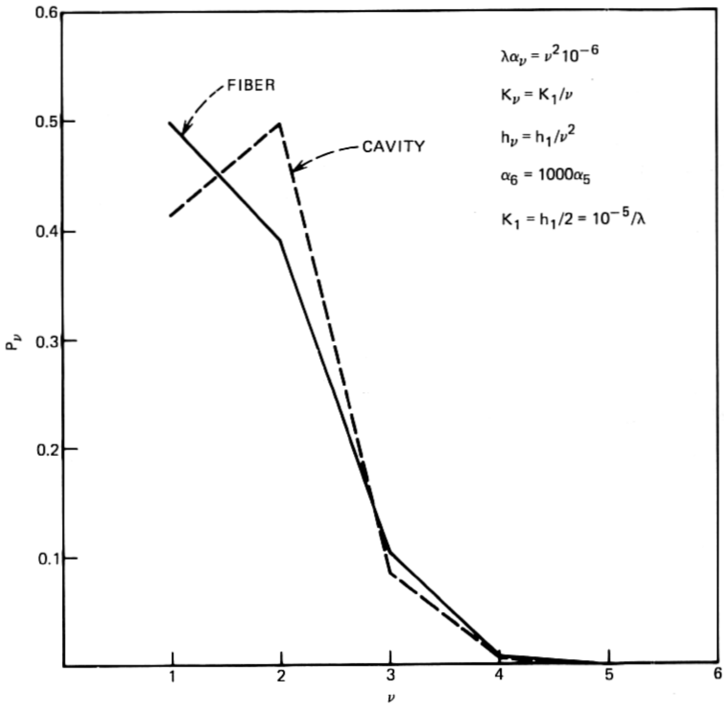

Fig. 3—Same as Fig. 1 but with $K_\nu = K_1/\nu$.

Figure 3 applies to the case $K_\nu = K_1/\nu$, $h_\nu = h_1/\nu^2$, with $K_1 = h_1/2 = 10^{-5}/\lambda$. Contrary to the cases in Figs. 1 and 2, the coupling strength is now decreasing with increasing mode number. It is interesting to observe that the cavity as well as the fiber with random coupling now carries more power in modes 2 and 3. The cavity solution with the least loss now has higher loss than in the case in Fig. 1 (see Table I). The loss of the steady-state power distribution of the fiber is, however, reduced compared to the case in Fig. 1 (again see Table I).

The remaining Figs. 4 through 7 describe the case of constant coupling, $K_\nu = K_1$ with different values of K_1 and α_6. We see that for very weak coupling most power resides in the modes with the lowest loss, $\nu = 1$. As the coupling strength is increased, more power is carried in higher-order modes. If we did not insist on making the last mode ($\nu = 6$) very lossy, there would be equal power in all the modes of the fiber with random coupling. It is interesting to observe that there is a saturation effect; comparison of Figs. 6 and 7 shows that the power distribution remains unchanged for a further increase of the coupling strength. Another interesting phenomenon is the different shape of the power distribution for the cavity mode with the lowest loss and the steady-state power distribution in the fiber with random coupling. Naively, one may have expected that the steady-state power distribution of the fiber would also apply to the cavity case. Figures 5 through 7 show that this is not the case. In spite of the fact that the cavity carries more power in the higher-order modes, Table I shows that the cavity losses are generally lower than the fiber losses. The cavity loss becomes high only when the highest-order mode has relatively low loss.

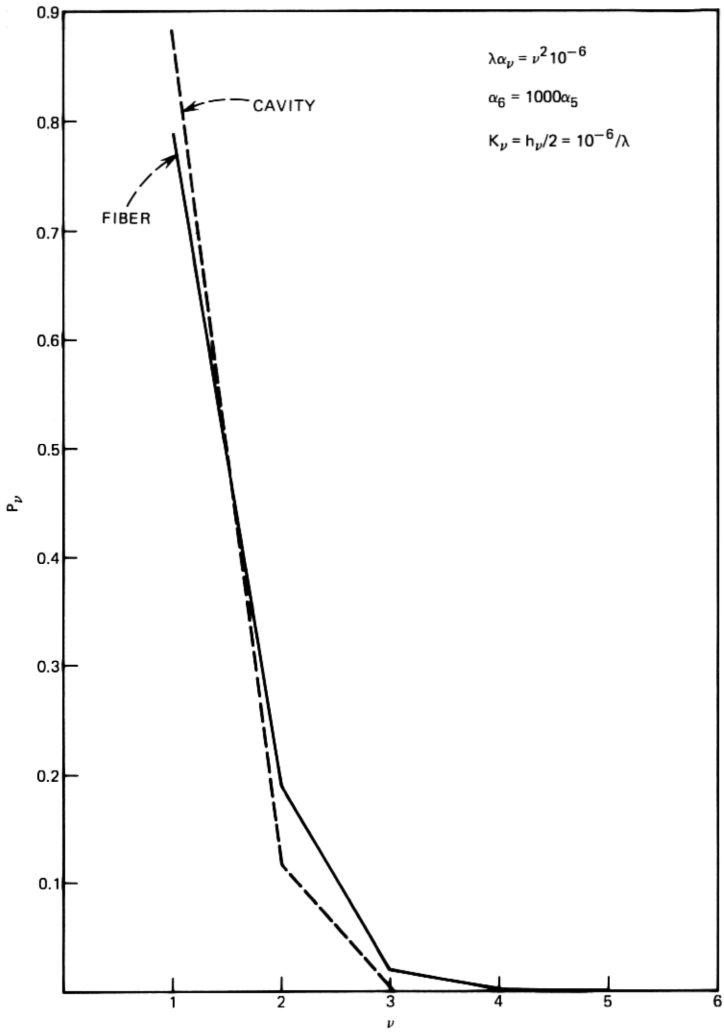

Fig. 4—Same as Fig. 1 but with $K_\nu = K_1$, $h_\nu = h_1$, $K_1 = 0.5h_1 = 10^{-6}/\lambda$.

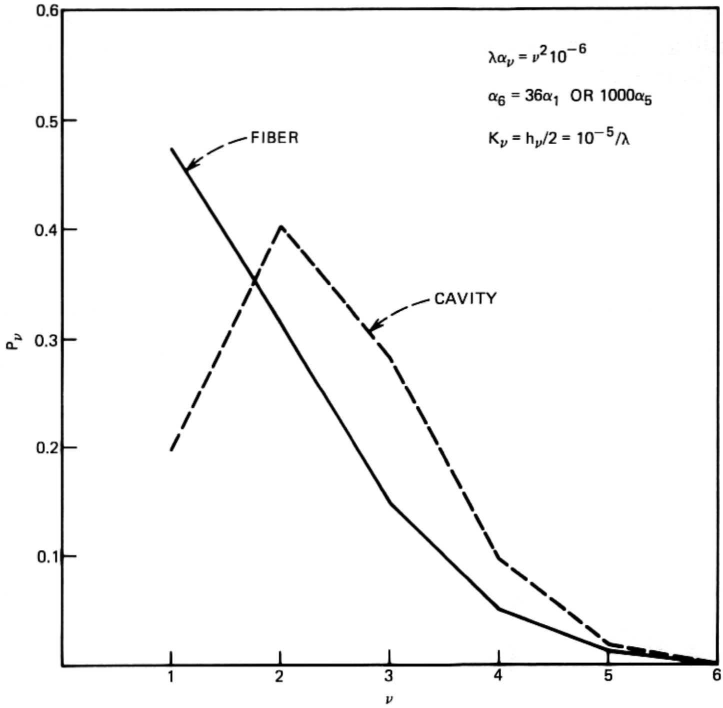

Fig. 5—Same as Fig. 4 but with $K_1 = 0.5h_1 = 10^{-5}/\lambda$. These curves are practically independent of the loss value of α_6; the curves for $\alpha_6 = 36\alpha_1$ and $\alpha_6 = 1000\alpha_5$ are indistinguishable on the scale of this figure.

VI. CONCLUSIONS

We have compared the losses and power distribution of a fiber with random coupling and of a cavity made of a section of the same fiber. We have shown that these two systems behave quite differently. While the losses of the fiber increase with an increase of the loss of the highest-order mode, the cavity losses decrease as the loss of the highest-order mode approaches infinity. This behavior has been studied with the help of exact numerical solutions of the eigenvalue equations of these systems for six modes and is also apparent from approximate solutions.

We may generalize our results for the fiber cavity as follows. We have seen that the losses of the solution with the lowest eigenvalue are higher than the loss of the lowest-order (uncoupled) mode. The approximate formula (46) or (47) shows that the cavity loss is an average of the losses of the individual, uncoupled modes. However, the last mode, $\nu = 6$ in our examples, did not participate in this average since its loss far exceeded the coupling strength. This behavior

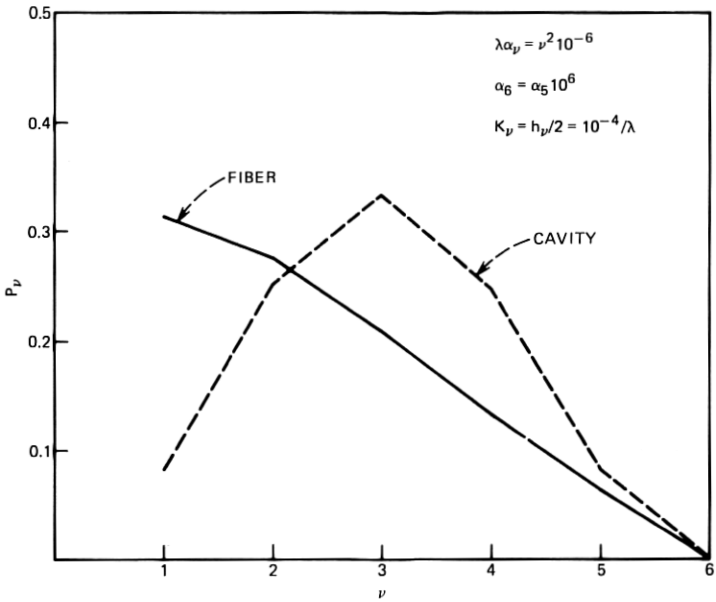

Fig. 6—Same as Fig. 4 but with $K_1 = 0.5h_1 = 10^{-4}/\lambda$ and $\alpha_6 = \alpha_5 10^6$.

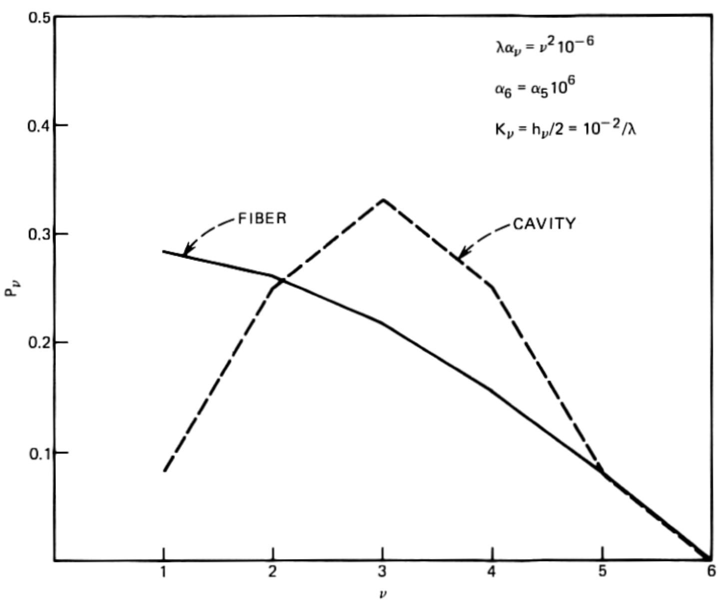

Fig. 7—Same as Fig. 6 but with $K_1 = 0.5h_1 = 10^{-2}/\lambda$.

leads us to conclude that the cavity losses are the average of the mode losses of all those modes whose (uncoupled) loss values are less than the coupling strength of neighboring modes. Modes whose losses exceed the coupling strength do not contribute appreciably to the cavity loss.

We have also indicated that two modes are coupled only if their propagation constants satisfy relation (16). The likelihood that this happens increases with increasing resonator length. In very short resonators, most modes remain effectively uncoupled just because they fail to satisfy condition (16). In long resonators, more modes have a chance to satisfy the additional coupling condition (16), but even here effective coupling ceases for modes whose losses exceed the coupling strength.

VII. ACKNOWLEDGMENT

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