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Filter Response of Nonuniform Almost-Periodic Structures

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The filter response of nonuniform, almost-periodic structures, such as corrugated optical waveguides, is investigated theoretically. The filter process, leading to reflection of a band of frequencies near the Bragg frequency, is treated as a contradirectional coupled-wave interaction and shown to obey a Riccati differential equation. The nonuniformity of the structure is represented by a tapering in the coupling strength (e.g., the depth of the corrugation) and by a chirp in the period of the structure. For small reflectivities, the filter response is a Fourier transform of the taper function. For large reflectivities, the Riccati equation was evaluated numerically and plots are given for the response of filters with linear and quadratic tapers and with linear and quadratic chirps.

I. INTRODUCTION

Recent papers[1,2] have reported the fabrication of wavelength-selective filters in the form of corrugated, thin-film, optical waveguides and have described the evaluation of the filter characteristics by means of tunable dye lasers. These papers include a numerical comparison of experimental and theoretical results on the nature of the filter response caused by a gradual tapering in the corrugation or grating strength (which we call "tapers"), or by a gradual variation of the effective grating period (called "chirping"). The present paper describes in some detail the theory by which these numerical results were obtained; it offers a discussion of the general characteristics of nonuniform grating filters, such as their scaling properties and the symmetry of the filter response; and it provides a more complete collection of plots representing numerical results for linear and quadratic tapers in the coupling strengths as well as linear and quadratic chirps in the effective grating period. Periodic waveguides serve as band-rejection filters with a center frequency determined by the Bragg condition

$$K = 2\beta(\omega), \tag{1}$$

where $K = 2\pi/\Lambda$ is the grating constant or spatial frequency, Λ the grating period, and $\beta(\omega)$ the propagation constant of the waveguide mode of interest. The filter mechanism results from the backward scattering of the light by the periodic structure which we describe as a contradirectional coupled-wave interaction. For its analysis, we use the coupled-wave formalism[3,4] which allows adequately for the depletion of the incident light.

Problems similar to ours have been encountered in other coupled-wave processes. Codirectional coupling in nonuniform structures has been investigated in connection with microwave directional couplers,[5–7] holography,[8] and tapered optical directional couplers.[9] Our present problem of contradirectional coupling in nonuniform periodic structures has been considered by Uchida[10] who analyzed structures with exponential tapers and by Hill and coworkers[11–13] who use a numerical method based on the iteration of a pair of coupled-mode integral equations. Another possible method is to approximate the nonuniform structure by a set of short uniform elements, each represented by a known matrix, and use matrix multiplication to calculate the properties of the compound overall structure. Kermisch[8] has applied such a technique to codirectional coupling in hologram gratings. As described below, we have used yet another method where we reduce the pair of coupled-wave equations to a single Riccati differential equation, which can then be solved by tested numerical techniques such as the Runge-Kutta method.

II. RICCATI EQUATION FOR THE REFLECTION COEFFICIENT

Following coupled-wave theory, we assume a periodic, single-mode waveguide with an electromagnetic field which can be represented by two contradirectional coupled waves in the form

$$E(z) = R(z)e^{-j\beta_0 z} + S(z)e^{j\beta_0 z}, \tag{2}$$

where R and S are the complex amplitudes of the forward- and backward-running mode. These amplitudes are linked by the standard coupled-wave equations[3,4]

$$R' + j\delta R = -j\kappa S e^{-j\phi} \tag{3}$$

and

$$S' - j\delta S = j\kappa R e^{j\phi}, \tag{4}$$

where we have allowed for a phase-shift ϕ of the periodicity relative to the origin ($z = 0$). The prime indicates differentiation with respect to z. The coupling coefficient κ is related to the amplitude of the waveguide perturbation (e.g., the height of the film corrugation), and formulas for specific guides with specific perturbations are given in the literature.[3,4,14] The measure $\delta(\omega)$ indicates the frequency deviation from the Bragg condition and is defined as[4]

$$\delta = \beta - K/2 = \beta - \beta_0 \approx \Delta\omega/v_g, \tag{5}$$

where $\beta_0 = K/2$ is the propagation constant at the center (Bragg) frequency, $\Delta\omega$ is the radian frequency deviation from that frequency, and v_g is the group velocity of the guide.

We consider, now, structures in which both the coupling coefficient $\kappa(z)$ and the grating phase $\phi(z)$ are slowly varying functions of z, indicating the nonuniformity in the grating parameters. We assume that the structure has a length L and extends from $z = -L/2$ to $z = L/2$. The boundary conditions for our scattering problem are then

$$R(-L/2) = 1, \qquad S(L/2) = 0. \tag{6}$$

The key to the reduction of the coupled-wave equations to a single differential equation is the definition of a local reflection coefficient $\rho(z)$,

$$\rho = \frac{S}{R} e^{-j\phi}. \tag{7}$$

The z-derivative of this is

$$\rho' = \left(\frac{S'}{R} - \frac{SR'}{R^2} - j\phi'\frac{S}{R}\right) e^{-j\phi}. \tag{8}$$

Combining the above expressions with (3) and (4), we obtain a Riccati differential equation for ρ which is of the form

$$\rho' = j(2\delta - \phi')\rho + j\kappa(1 + \rho^2). \tag{9}$$

The boundary condition for this equation follows from (6) as

$$\rho(L/2) = 0. \tag{10}$$

Our quantity of interest is the reflection coefficient $\rho(-L/2)$ of the entire structure, or the corresponding reflectivity $\rho\rho^*$. What we call filter response is the dependence of this reflectivity on the frequency (or wavelength) of the light. Results for the response curves of specific taper functions are provided in later sections, but first we study some of their general properties.

III. FOURIER-TRANSFORM RESPONSE AT LOW REFLECTIVITIES

When the reflectivities are low, we expect a simple relationship between the taper function $\kappa(z)e^{j\phi(z)}$ and the response function $\rho(\delta)$.

This relationship follows from the assumption of an undepleted incident wave or the use of a first Born approximation.[15,16] It is easily derived from the Riccati equation (9) by substituting a new variable σ defined by

$$\rho = \sigma e^{j(2\delta z - \phi)}, \tag{11}$$

which obeys the differential equation

$$\sigma' = j\kappa[e^{-j(2\delta z - \phi)} + \sigma^2 e^{j(2\delta z - \phi)}]. \tag{12}$$

When reflectivities are low, the term proportional to σ^2 can be neglected, and we can integrate this equation with the result

$$\sigma(-L/2) = -j \int_{-L/2}^{L/2} dz \cdot \kappa(z) \cdot e^{-j(2\delta z - \phi)}. \tag{13}$$

This shows that, apart from phase factors, the response function $\rho(\delta)$ is the Fourier transform of the taper function $\kappa(z) \cdot e^{j\phi(z)}$.

IV. SYMMETRY OF FILTER RESPONSE

Let us now consider the conditions for which the filter response is symmetric, i.e., for which

$$\rho\rho^*(\delta) = \rho\rho^*(-\delta). \tag{14}$$

To simplify our discussion, we assume that the filter structure can be described by a real and positive function $\kappa(z)$ and an arbitrary phase function $\phi(z)$. Then the Fourier transform relation (13) predicts that the filter response is symmetric (a) if $\phi(z) = 0$, or (b) if both κ and ϕ are symmetric, i.e., if $\kappa(z) = \kappa(-z)$ and $\phi(z) = \phi(-z)$. Relation (13) is valid for small reflectivities only, but we get the same answer from the Riccati equation (9), which is valid for all reflectivities. To demonstrate this, we take the complex conjugate of (9) and write the result in the form

$$(-\rho^*)' = j(-2\delta + \phi')(-\rho^*) + j\kappa[1 + (-\rho^*)^2]. \tag{15}$$

If we replace δ with $-\delta$ and ϕ' with $-\phi'$, this differential equation for $(-\rho^*)$ is the same as the differential equation (9) for ρ, and (as the boundary conditions are also the same) it predicts the same reflectivity $\rho\rho^*$. While replacing δ with $-\delta$ simply means that we are looking at the other side of the center frequency, replacing ϕ' with $-\phi'$ means that we have replaced the original filter structure with another one. For a structure with symmetrical $\kappa(z)$ and $\phi(z)$, we have $\phi' = -\phi'(-z)$, and the above replacement means that we have physically reversed the filter. Since the structure is lossless and reciprocal, the reflectivity is

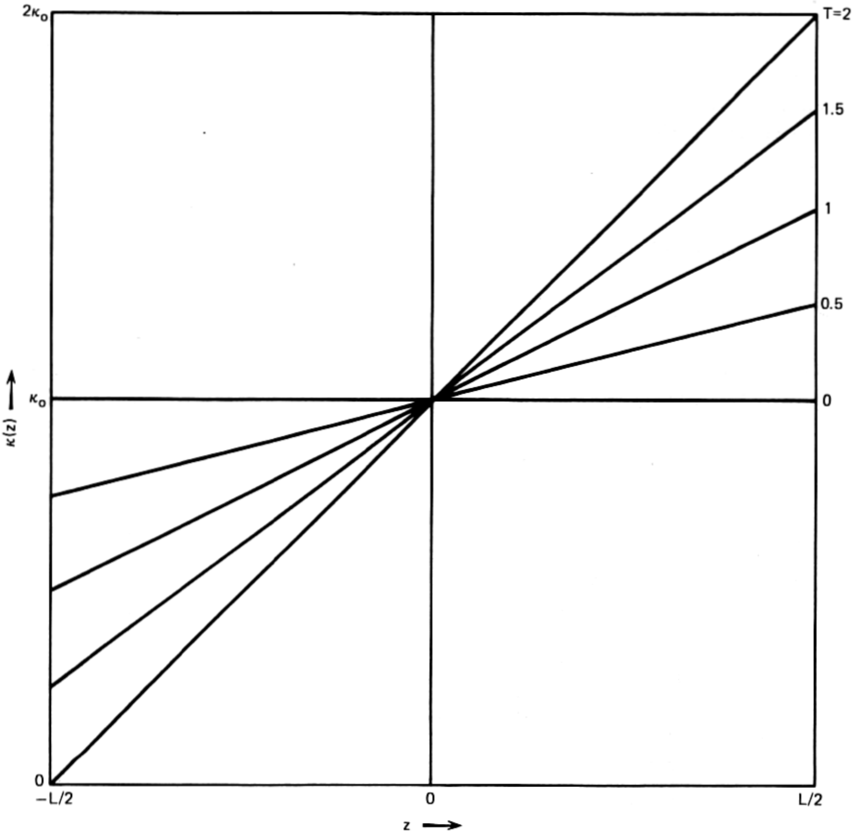

Fig. 1—Linear taper functions $\kappa(z)$ for the taper constants $T = 0, 0.5, 1, 1.5$, and 2.

the same at the input and output port of the filter, and condition (b) is proven. Condition (a) for $\phi' = 0$ is easily proven by inspection of (15).

V. SCALING OF FILTER RESPONSE

In the general case, the Riccati equation (9) has to be evaluated numerically. To make the numerical results more broadly applicable, it is convenient to introduce the normalized quantities z/L, δL, κL, and $\phi' L$. It follows from (9) that two filters of different length (labeled 1 and 2) have the same reflectivity at the frequencies $\delta_1 L_1 = \delta_2 L_2$ if their taper functions obey the scaling rule

$$\begin{aligned} \kappa_1\left(\frac{z}{L_1}\right) L_1 &= \kappa_2\left(\frac{z}{L_2}\right) L_2 \\ \phi_1'\left(\frac{z}{L_1}\right) L_1 &= \phi_2'\left(\frac{z}{L_2}\right) L_2. \end{aligned} \tag{16}$$

VI. TAPERED FILTERS

Tapered filters are structures where the coupling coefficient $\kappa(z)$ varies along the length of the device. For simplicity, we consider here tapered filters with no chirp ($\phi = 0$). At the center frequency ($\delta = 0$), we can write the Riccati equation (9) of such a filter in the form

$$\frac{d\rho}{1+\rho^2} = j\kappa(z)dz, \tag{17}$$

which is easily integrated to yield for the reflection coefficient

$$\rho(-L/2) = -j \tanh \int_{-L/2}^{L/2} dz.\kappa(z). \tag{18}$$

This is similar to a result found by Kermisch[8] for codirectional interactions in holograms.

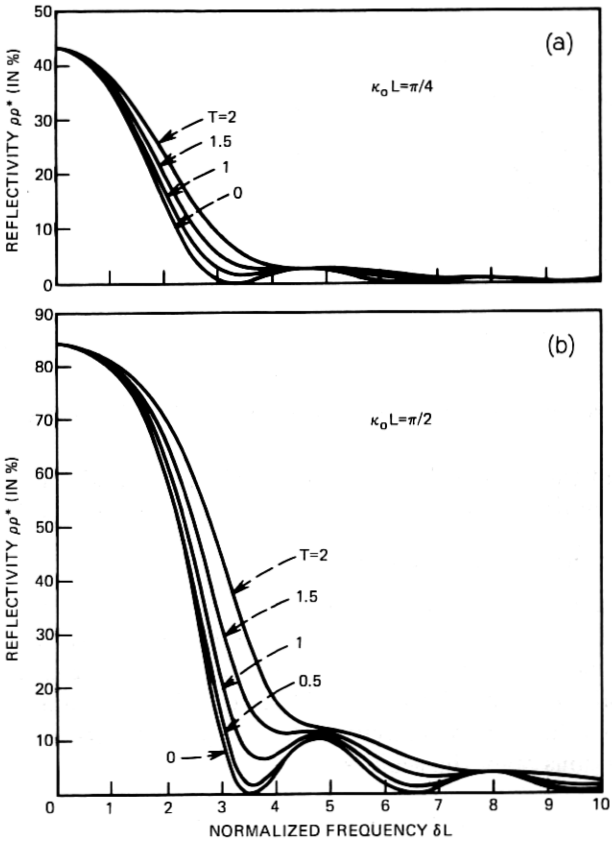

Fig. 2—Frequency response of filters with a linear taper for the taper constants $T = 0, 0.5, 1, 1.5$, and 2 and values of (a) $\kappa_0 L = \pi/4$, (b) $\kappa_0 L = \pi/2$, (c) $\kappa_0 L = 3\pi/4$, and (d) $\kappa_0 L = \pi$.

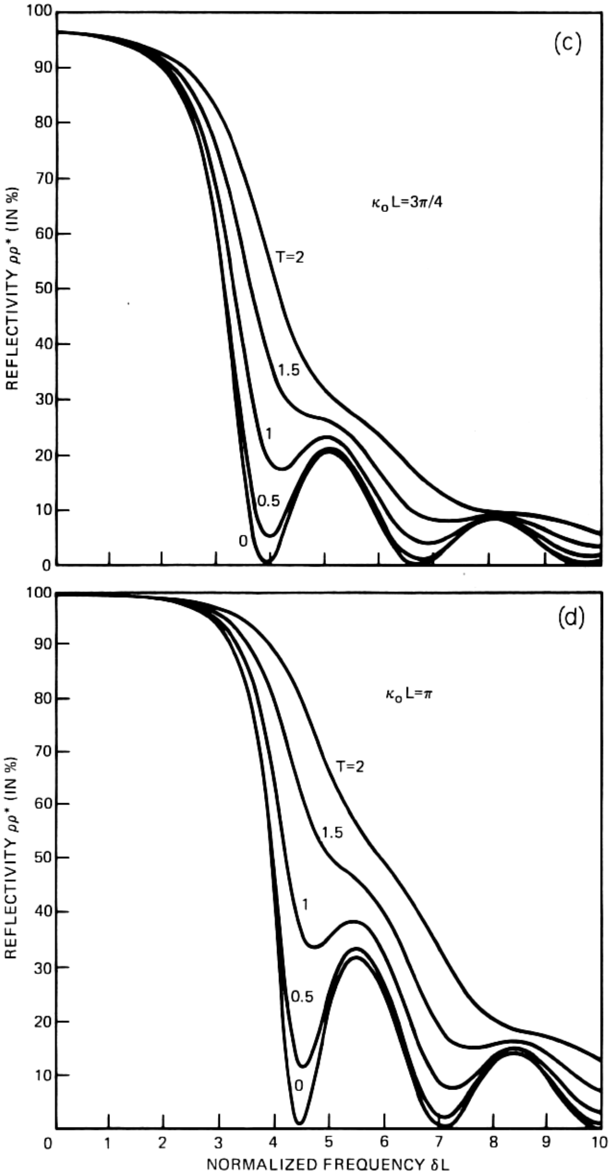

Fig. 2 (continued).

6.1 *Linear tapers*

In a linearly-tapered filter structure, the coupling coefficient $\kappa(z)$ varies as

$$\kappa = \kappa_0(1 + Tz/L), \tag{19}$$

where the constant T indicates the degree of the taper. Figure 1 shows a sketch of this taper function for five values of T including the cases of no tapering ($T = 0$) and full tapering ($T = 2$). From (18) and (19), we calculate the reflectivity of linearly-tapered filters at the center frequency ($\delta = 0$) as

$$\rho\rho^* = \tanh^2(\kappa_0 L). \tag{20}$$

According to rule (a) of the previous section, the filter response of tapered filters is symmetric. Figures 2a through 2b show the filter response of linearly tapered filters for positive frequency deviations and five values of the taper constant T. The curves are the result of a numerical evaluation of (9) using the fourth-order Runge-Kutta method. Response curves are shown for values of $\kappa_0 L = \pi/4$, $\pi/2$, $3\pi/4$, and π, and we notice a washing-out of the zeros in the response curve with increasing amounts of tapering.

6.2 *Quadratic tapers*

A filter with a quadratic taper is characterized by a coupling coefficient $\kappa(z)$ that varies as

$$\kappa = \kappa_0\left(1 - \frac{T}{12} + Tz^2/L^2\right). \tag{21}$$

The particular form of this expression has been chosen to make the reflectivity at center frequency ($\delta = 0$) independent of the constant T, and equal to

$$\rho\rho^* = \tanh^2(\kappa_0 L) \tag{22}$$

as calculated from (18). Figure 3 shows quadratic taper functions for five values of T. For positive T the curves are concave upward, and for negative valued T they are convex upward. Figures 4a through 4d show the numerically evaluated response curves for the same values of T and four typical values of $\kappa_0 L$. We note that the concave upward tapers produce very high side-lobe levels in their response characteristics, while the side-lobe levels can be very low for the convex upward tapers. The side-lobe levels of the tapers with $T = -6$ (emphasized by the thicker lines) are the lowest with about 2 percent in reflectivity.

VII. CHIRPED FILTERS

A variation of the grating period along the length of the filter is called a "chirp." Chirped filter characteristics may also be due to a variation of a waveguide parameter such as the refractive index or

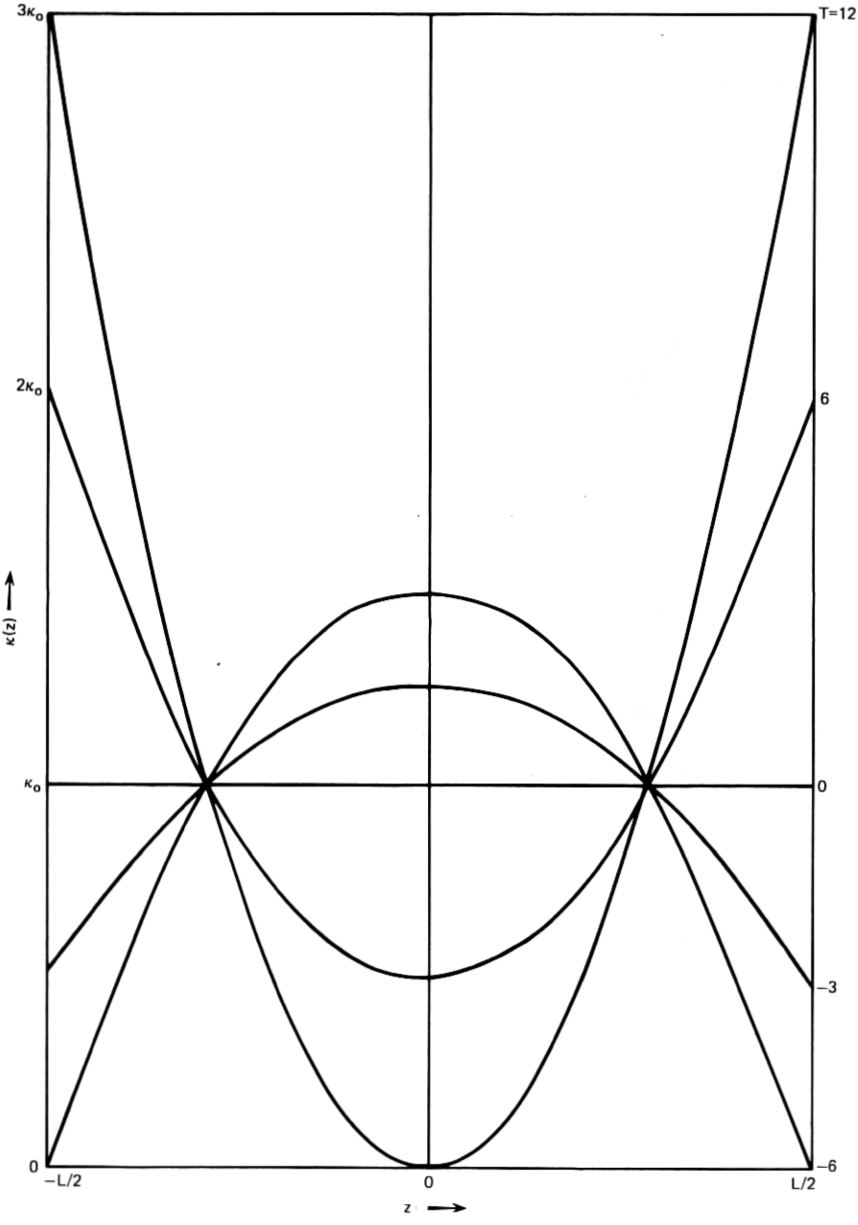

Fig. 3—Quadratic taper functions $\kappa(z)$ for the taper constants $T = -6, -3, 0, 6,$ and 12. The curves intersect at the points $z = \pm L/2 \cdot \sqrt{3}$ and $\kappa = \kappa_0$.

the guide height or width. We have chosen the phase function $\phi(z)$ to represent all variations of this kind. We are, therefore, dealing with a perturbation proportional to cos $(Kz + \phi)$ with constant spatial frequency K and a variable phase shift $\phi(z)$, which can also be viewed as a perturbation with a variable, or chirped, spatial frequency $K + \Delta K(z)$, where

$$\Delta K(z) = \phi'(z), \tag{23}$$

or as a perturbation of variable period $\Lambda + \Delta\Lambda(z)$, where

$$\frac{\Delta\Lambda(z)}{\Lambda} = -\frac{\Delta K(z)}{K}. \tag{24}$$

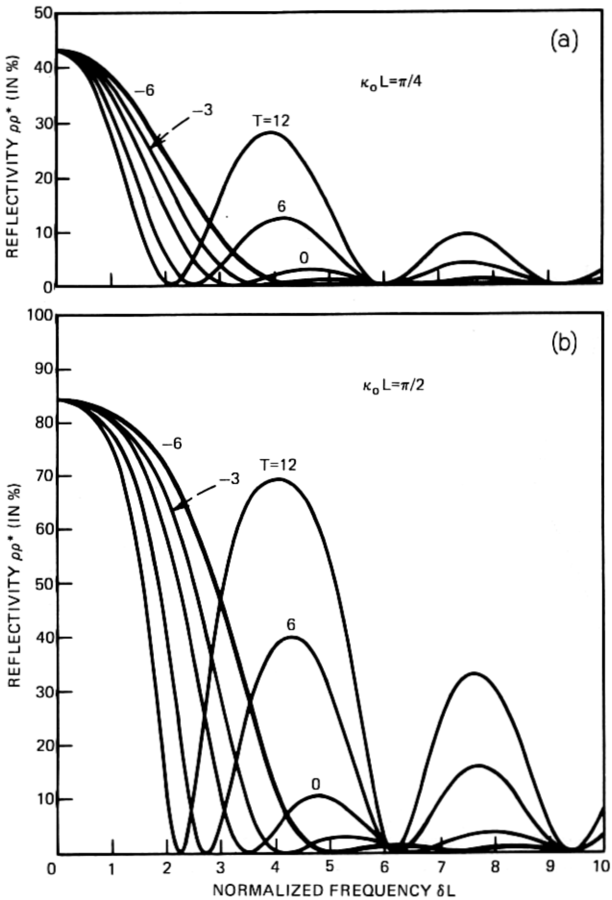

Fig. 4—Frequency response of filters with a quadratic taper for the taper constants $T = -6, -3, 0, 6$, and 12, and values of (a) $\kappa_0 L = \pi/4$, (b) $\kappa_0 L = \pi/2$, (c) $\kappa_0 L = 3\pi/4$, and (d) $\kappa_0 L = \pi$.

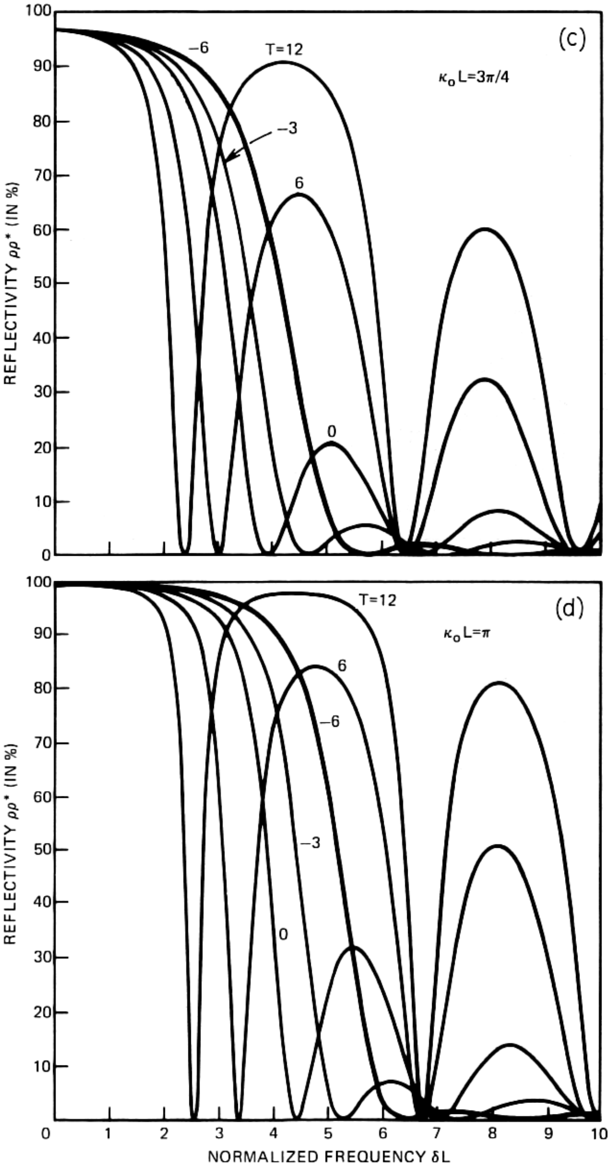

Fig. 4 (continued).

A chirped grating period implies that the (local) Bragg frequency changes along z, which results in a general broadening of the filter response.

7.1 Linear chirp

A periodic structure with a linear chirp is described by a linearly varying spatial frequency which we write in the form

$$\Delta K(z) = \phi'(z) = 2Fz/L^2, \tag{25}$$

where the constant F is a measure for the degree of the chirp. The

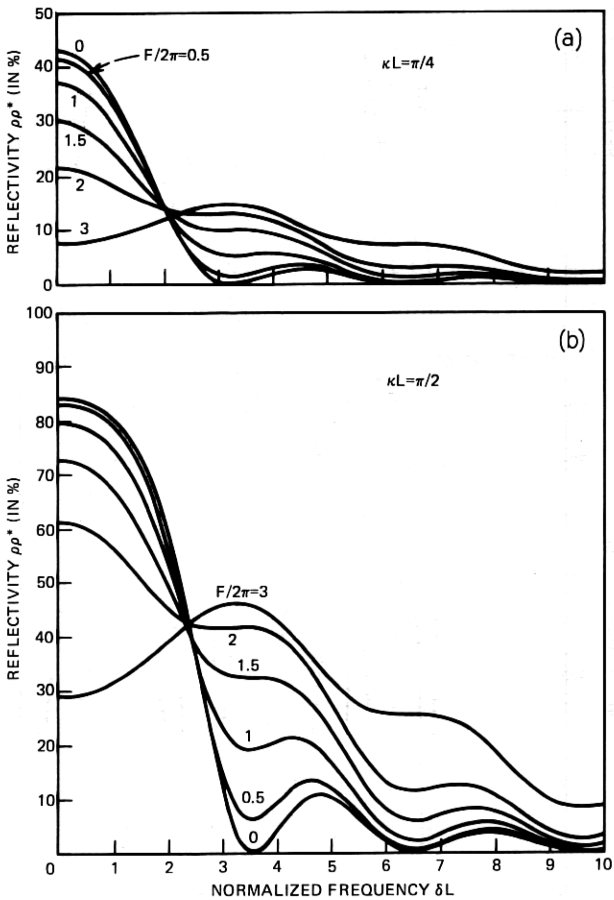

Fig. 5—Frequency response of filters with a linear chirp for the chirp constants $F/2\pi = 0, 0.5, 1, 1.5, 2$, and 3, and for κL-values of (a) $\pi/4$, (b) $\pi/2$, (c) $3\pi/4$, and (d) π.

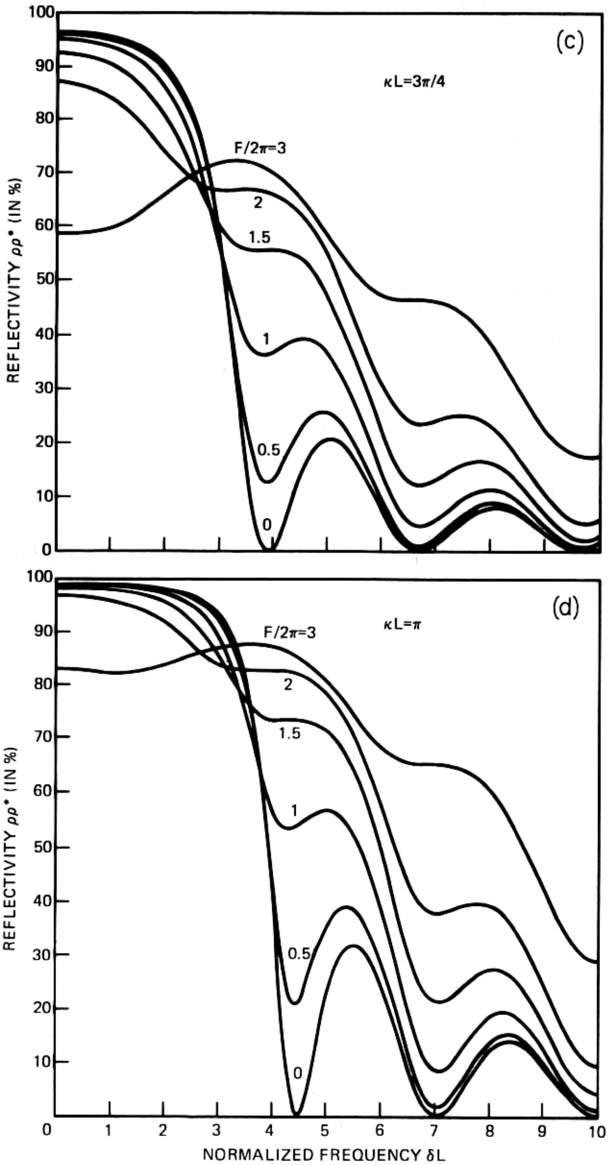

Fig. 5 (continued).

corresponding phase function is

$$\phi(z) = Fz^2/L^2. \tag{26}$$

For the grating period we have

$$\frac{\Delta\Lambda(z)}{\Lambda} = -\frac{\Delta K(z)}{K} = -\frac{F}{2\pi}\cdot\frac{\Lambda}{L}\cdot\frac{2z}{L}. \tag{27}$$

For the difference $\Delta\Lambda$ of the grating periods at the center ($z = 0$) and end ($z = L/2$) of the device, we obtain

$$\frac{\Delta\Lambda}{\Lambda} = \frac{\Lambda(0) - \Lambda(L/2)}{\Lambda} = \frac{F}{2\pi}\cdot\frac{\Lambda}{L}, \tag{28}$$

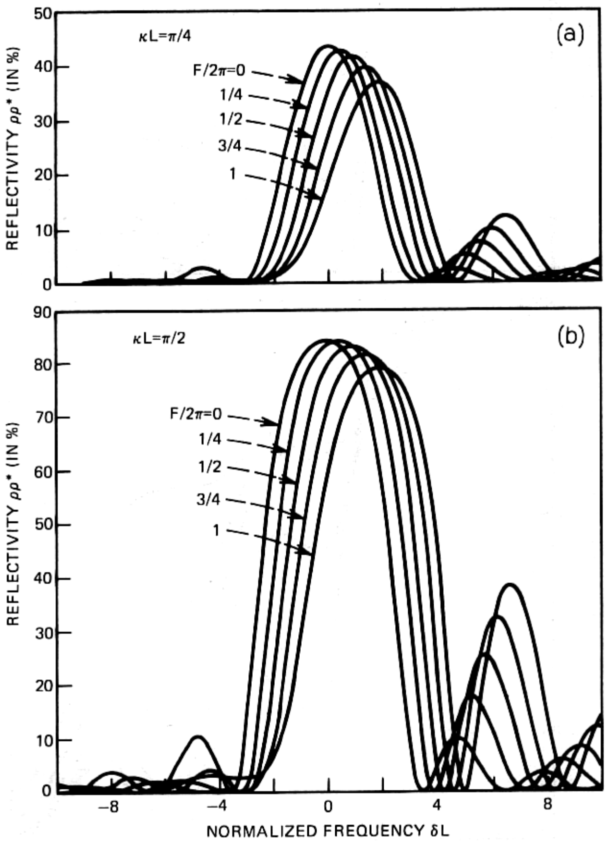

Fig. 6—Frequency response of filters with a quadratic chirp for the chirp constants $F/2\pi = 0$, 0.25, 0.5, 0.75, and 1, and for κL-values of (a) $\pi/4$, (b) $\pi/2$, (c) $3\pi/4$, and (d) π.

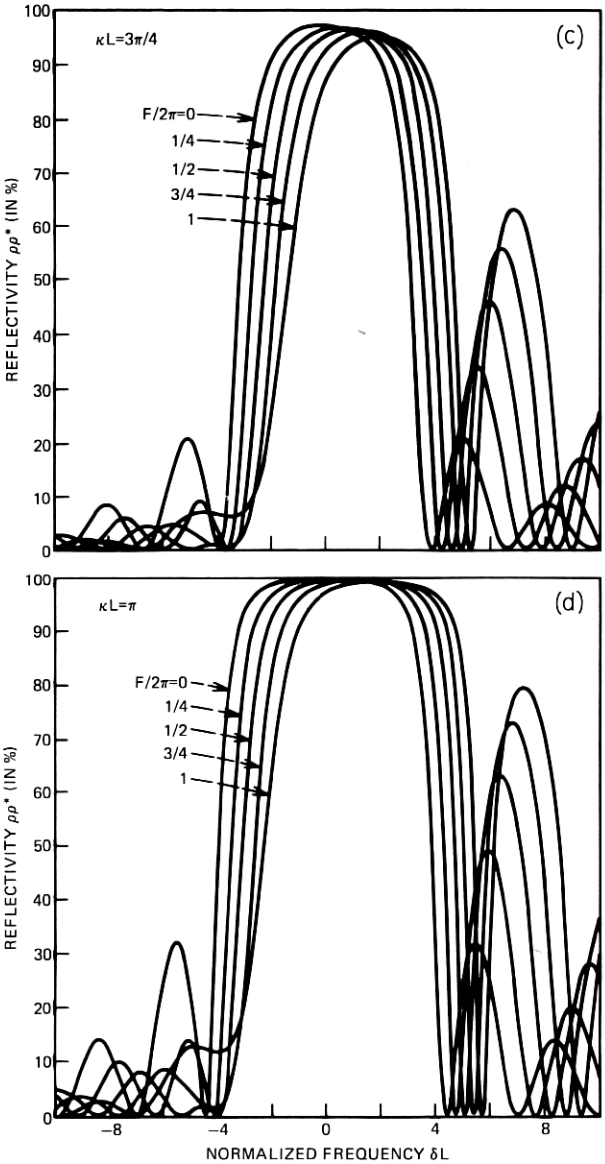

Fig. 6 (continued).

which allows us to express the chirp parameter F in terms of the total number of periods $N = L/\Lambda$ in the form

$$F = 2\pi \cdot N \cdot \Delta\Lambda/\Lambda. \tag{29}$$

According to symmetry rule (b), the response of a filter with a linear chirp is symmetric. The discussion in Section IV implies that a reversal of the sign of F will not change the filter response since this operation simply means that we have turned the two-port structure around to measure the reflectivity at the other port. The filter response as obtained from a numerical evaluation of (9) using the fourth-order Runge-Kutta method is shown in Figs. 5a through 5d. Again, we have chosen for κL the values $\pi/4$, $\pi/2$, $3\pi/4$, and π. We have included the curves for the unchirped filters ($F = 0$) and shown the responses of five chirped filters up to a chirp constant $F/2\pi = 3$. We note that the chirp washes out the zeros and broadens the response.

7.2 Quadratic chirp

For a periodic structure with a quadratic chirp, we write the spatial frequency variation in the form

$$\Delta K(z) = \phi' = 12Fz^2/L^3 \tag{30}$$

using F as a chirp parameter. With this we get for the phase function

$$\phi = 4F(z/L)^3. \tag{31}$$

The variation of the grating period is then given by

$$\frac{\Delta\Lambda(z)}{\Lambda} = -3\frac{F}{2\pi}\cdot\frac{\Lambda}{L}\cdot\left(\frac{2z}{L}\right)^2, \tag{32}$$

and the difference between the periods in the middle of the device ($z = 0$) and the device ends ($z = \pm L/2$) becomes

$$\frac{\Delta\Lambda}{\Lambda} = \frac{\Delta\Lambda(0) - \Delta\Lambda(L/2)}{\Lambda} = 3\cdot\frac{F}{2\pi}\cdot\frac{\Lambda}{L}. \tag{33}$$

Again, we can express the chirp parameter F in terms of the total number of periods by

$$3F = 2\pi \cdot N \cdot \Delta\Lambda/\Lambda. \tag{34}$$

The filter response of a periodic structure with a quadratic chirp is asymmetric. If we reverse the sign of F in this case, we are dealing with a different structure which has a different filter response. According to the discussion of Section IV, this new response curve is the mirror image of the response of the original filter relative to the center frequency. The response, as obtained by a numerical evaluation of (9), is shown in Figs. 6a through 6d for four values of κL. Curves are given for chirp parameters up to $F/2\pi = 1$. We note that with increasing chirps the asymmetry of the response curve increases with the side-lobe levels increasing on one side and decreasing on the other side of the center frequency. Most of the zeros in the response are preserved and shifted somewhat, and there is a small shift in the frequency of peak reflectivity. There appears to be no significant broadening of the main lobe of the response for the parameters selected.

VIII. CONCLUSIONS

We have described the filter characteristics of nonuniform periodic waveguides on the basis of coupled-wave theory. We have also shown that the filter response is described by a Riccati differential equation and have presented solutions for linear and quadratic tapers of the coupling coefficient and for linear and quadratic chirps in the grating period. The results can be used as an aid in filter design and to predict the effect of imperfections introduced during device fabrication. Quadratic tapers appear to be a promising choice when filter characteristics with low side-lobe levels are desired.

IX. ACKNOWLEDGMENT

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