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Optimal Rearrangeable Graphs

By F. R. K. CHUNG

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Many important properties of switching networks can be effectively studied in the more general context of graph theory. In particular, the various rearrangeability properties of a network fall into this category. If G is a graph with vertex set $V = I \cup \Omega$, we say G is rearrangeable if, for all choices of distinct vertices, $i_1, i_2, \cdots, i_t$ in I and $j_1, j_2, \cdots, j_t$ in Ω, there exist vertex disjoint paths between i_k and j_k for all k. In this paper, we determine the minimum number of edges any rearrangeable graph may have for all choices of I and Ω. We also discuss generalizations in which V is strictly greater than $I \cup \Omega$ and/or t is bounded by a predetermined value. The minimal rearrangeable graphs we construct can be used to form efficient rearrangeable (and nearly rearrangeable) switching networks of arbitrary size.

I. INTRODUCTION

Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$.* Let I and Ω be nonempty subsets of $V(G)$ (not necessarily disjoint) and let S denote the set

$$V(G) \setminus (I \cup \Omega) = \{v \in V(G) \mid v \notin I \cup \Omega\}.$$

We use the following terminology:

(*i*) A *request* is an ordered pair (x, y) with $x \in I$, $y \in \Omega$, and $x \neq y$.

(*ii*) A set of requests is called an *assignment* if each vertex in the set occurs once at most.

(*iii*) An assignment A is called *realizable in G* (or we say G *satisfies* A) if we can find a set of vertex-disjoint paths connecting x and y for each pair (x, y) in A.

(*iv*) A graph G is said to be *rearrangeable* if G satisfies *any* assignment.

The problem we consider, first suggested by F. K. Hwang,[2] is to find

* I.e., $E(G)$ consists of a prescribed set of unordered pairs of distinct elements of some finite set $V(G)$. Generally, we follow the terminology of Harary.[1]

rearrangeable graphs with given I and with Ω having the least possible number of edges.

In this paper, we derive lower bounds on the minimum number of edges that a rearrangeable graph can have (see Theorem 1 in Section II). In addition, we also construct rearrangeable graphs which meet these bounds so that these graphs are optimal by this measure. Finally, we consider a generalization of rearrangeability, called k-rearrangeability, and we solve the corresponding problems in this case as well.

This study was motivated by questions of rearrangeability in switching networks (see Ref. 3). The sets I and Ω correspond to the sets of inlets and outlets, respectively; an edge $\{x, y\}$ of G corresponds to a crosspoint between x and y. The optimal rearrangeable graphs we construct can consequently be used to form efficient rearrangeable (and nearly rearrangeable) switching networks of arbitrary size.

II. BASIC PROPERTIES OF REARRANGEABLE GRAPHS

Let G be a rearrangeable graph with distinguished subsets I and Ω, where we assume without loss of generality that $|I| = n \leqq m = |\Omega|$. For the bulk of the paper, we shall restrict ourselves to the special case that S is empty, i.e., $V(G) = I \cup \Omega$.

If $\{x, y\}$ is an edge of G, we say that x and y are *adjacent* and we write $x \sim y$. Similarly, for $T \subseteq V(G)$, the notation $x \sim T$ will denote that $x \sim t$ for some $t \in T$. By the *degree* of $v \in V(G)$, written deg (v), we mean the number of edges of G containing v. More generally, if $X \subseteq V(G)$, then $\deg_X(v)$ denotes

$$|\{\{v, x\} \mid \{v, x\} \in E(G) \text{ and } x \in X\}|.$$

Suppose there is a vertex $v \in I$ with deg $(v) = k < n$, and let $v_1, \cdots, v_k$ denote the vertices that are adjacent to v.

Now consider an assignment A in which all the v_i, $1 \leqq i \leqq k$, occur as well as the pair (v, v'), where $v' \in \Omega$ is not adjacent to v. But this A is not realizable in G, which contradicts the hypothesis that G is rearrangeable. Hence, for all $v \in I$, we must have $\deg(v) \geqq n$. By a similar argument, it can be shown that deg $(v') \geqq n$ for all $v' \in \Omega$. Thus, for any rearrangeable graph G we must have:

Fact 1: For all $v \in V(G)$, deg $(v) \geqq n$.

Let us now state several more elementary facts about rearrangeable graphs G which can be proved in much the same way as Fact 1.

Fact 2: For all $v \in I$, $\deg_\Omega(v) \geqq n$.
Fact 3: For all $v \in V(G)$, $\max[\deg_I(v), \deg_\Omega(v)] \geqq n$.
Fact 4: If $v \sim I$ and $v \sim \Omega$ and $|I \cap \Omega| = 0$, then deg $(v) \geqq n + 1$.

Lemma 1: *In a rearrangeable graph G with vertex set $V(G) = I \cup \Omega$, $|I| = n < m = |\Omega|$, and $|I \cap \Omega| = 0$, there are at least n vertices with degree greater than n.*

Proof: If all $v \in I$ satisfy $\deg_\Omega(v) \geqq n + 1$, then we are done. Suppose there is an element $v \in I$ satisfy $\deg_\Omega(v) = n$, say, v is adjacent to $v_1, v_2, \cdots, v_n$. If any v_i, say, v_1, is not adjacent to Ω, let us consider an assignment in which all the v_i, $2 \leqq i \leqq n$, occur as well as the ordered pair (v, v'), where v' is a vertex in Ω different from any v_i. However, since $|I| = n$, it is impossible for G to satisfy this assignment. Hence, all the v_i are adjacent to both I and Ω. By Fact 4, $\deg v_i \geqq n + 1$ for $i = 1, \cdots, n$, which proves Lemma 1.

Lemma 2: *If G is any rearrangeable graph with vertex set $I \cup \Omega$, which has $|\Omega| > |I| = n$ and $|I \cap \Omega| = 0$, then G has at least $n(p + 1)/2$ edges, where $p = |V(G)|$.*

Proof: The number of edges in G satisfies the following inequality:

$$|E(G)| = e(G) = \tfrac{1}{2} \sum_{v \in G} \deg(v)$$
$$\geqq \tfrac{1}{2}[(p - n)n + (n + 1)n]$$
$$= \tfrac{1}{2}n(p + 1).$$

Lemma 3: *In a rearrangeable graph G with vertex set $V(G) = I \cup \Omega$, which has $|I| = n < m = |\Omega| < 2n$ and $|I \cap \Omega| = 0$, we have*

$$e(G) \geqq nm - \tfrac{1}{2}(m - n - 1)(m - n).$$

Proof: Denote the vertices in I by $i_1, \cdots, i_n$. Suppose the vertex i_j is adjacent to d_j vertices in Ω, where we may assume $d_1 \leqq d_2 \leqq \cdots \leqq d_n$. Let Ω_j be the union of $\{i_j\}$ and the d_j elements in Ω, which are adjacent to i_j. By Fact 3, each element in $\Omega \setminus \Omega_j$ is then adjacent to at least $n - (m - d_j) + 1$ elements in Ω_j. Hence, by counting the total number of edges $e(G)$ and using the fact that $d_1 \leqq d_i$ for all i, it follows that

$$e(G) \geqq nd_1 + (m - d_1)(n - m + d_1 + 1) + \tfrac{1}{2}(m - d_1)(m - d_1 - 1).$$

But the right-hand side is minimized by choosing d_1 as small as possible. Thus, since $m < 2n$, then by Fact 3, we have

$$e(G) \geqq mn - \tfrac{1}{2}(m - n)(m - n - 1),$$

which proves the lemma.

The preceding inequalities are summarized in the following result.

Theorem 1: *In a rearrangeable graph with vertex set $V(G) = I \cup \Omega$, and $|I| = n \leqq m = |\Omega|$, $|\Omega \cap I| = 0$, the number of edges $e(G)$ satisfies*

$$e(G) \geqq \begin{cases} \left\lceil \dfrac{n(m+n+1)}{2} \right\rceil & \textit{if } m \geqq 2n, \\ \lceil mn - \frac{1}{2}(m-n)(m-n-1) \rceil & \textit{if } 2n > m \geqq n, \end{cases}$$

where $\lceil x \rceil$ *denotes the smallest integer which is greater than or equal to* x.

The proof follows at once from Lemma 2 and Lemma 3.

III. OPTIMAL REARRANGEABLE GRAPHS—MANHATTAN GRAPHS

In this section, we give a construction for a class of optimal rearrangeable graphs. The number of edges in these graphs will meet the lower bound in Theorem 1. These graphs will be called *Manhattan* graphs because they resemble a number of bridges connecting a high-density metropolitan area and low-density suburban areas.

A Manhattan graph with vertex set $V(G) = I \cup \Omega$ will be denoted by $M(I, \Omega)$. If $|I| = n$, $|\Omega| = m$, and $|I \cap \Omega| = 0$, $M(I, \Omega)$ is also denoted by $M(n, m)$.

In this section, we give the construction of $M(n, m)$ for any n and m by considering the following cases.

Case 1, $n = m$: The Manhattan graph $M(n, n)$ is the complete bipartite graph $K_{n,n}$, i.e., there is an edge between every pair of vertices (u, v), $v \in I$, $v \in \Omega$.

Case 2, $n < m < 2n$: We shall specify the edges of $M(n, m)$ by giving the subgraph spanned by various subsets of vertices of $M(n, m)$. The *spanning subgraph* of a set $S \subseteq V(G)$ is the subgraph of G with edge set $\{\{x, y\} \mid \{x, y\} \in E(G) \text{ and } x, y \in S\}$.

Let

$$I = \{i_1, i_2, \cdots, i_n\},$$
$$\Omega = \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_t\},$$

where

$$t = m - n.$$

$M(n, m)$ will be constructed as follows:

(*i*) The spanning subgraph of the vertices $I \cup \{x_i | 1 \leqq i \leqq n\}$ in $M(n, m)$ is a complete bipartite graph $K_{n,n}$;

(*ii*) y_j is adjacent to $x_j, x_{j+1}, \cdots, x_{j+(n-t)}$, for $j = 1, 2, \cdots, t$;

(*iii*) The spanning subgraph of the vertices $\{y_1, y_2, \cdots, y_t\}$ in $M(n, m)$ is a complete graph K_t.

The graph $M(n, m)$ is clearly rearrangeable. As an example of this construction, we illustrate $M(3, 5)$ in Fig. 1.

Case 3, $2n \leqq m < 3n$: The construction scheme for $M(n, m)$ in this case may be described as follows:

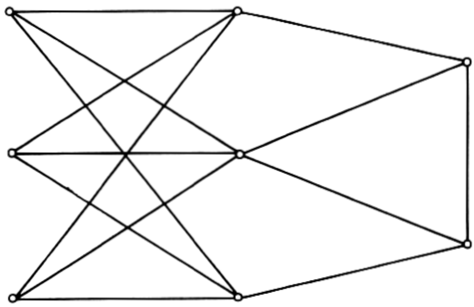


Fig. 1—Graph $M(3, 5)$.

Let

$$I = \{i_1, i_2, \cdots, i_n\},$$
$$\Omega = \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_t\},$$

where

$$t = m - 2n.$$

(*i*) The spanning subgraph of the vertices $I \cup \{x_1, x_2, \cdots, x_n\}$ in $M(n, m)$ is a complete bipartite graph $K_{n,n}$;
(*ii*) x_j is adjacent to y_j for $j = 1, 2, \cdots, n$;
(*iii*) z_j is adjacent to $y_1, y_2, \cdots, y_n$ for $j = 1, 2, \cdots, t$;
(*iv*) The spanning subgraph of vertices $\{y_1, y_2, \cdots, y_n\}$ in $M(n, m)$ is any graph with degree sequence

$$\underbrace{\{n - t - 1, n - t - 1, \cdots, n - t - 1, w\}}_{n - 1 \text{ times},}$$

where

$$w = \begin{cases} n - t - 1 & \text{if } n(n - t + 1) \text{ is even,} \\ n - t & \text{otherwise.} \end{cases}$$

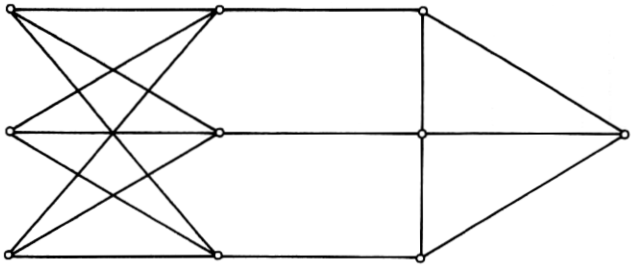
From a well-known theorem of Erdös and Gallai,[4] a graph with this degree sequence can always be constructed. The graphs $M(3, 7)$ and $M(3, 8)$ are shown in Fig. 2 as examples of this construction.

We want to show this graph is rearrangeable. Given an assignment A involving vertices $x_{a_1}, x_{a_2}, \cdots, x_{a_{n_1}}, y_{b_1}, y_{b_2}, \cdots, y_{b_{n_2}}, z_{c_1}, z_{c_2}, \cdots, z_{c_{n_3}}$, it is clear that $n \geqq n_1 + n_2 + n_3$.

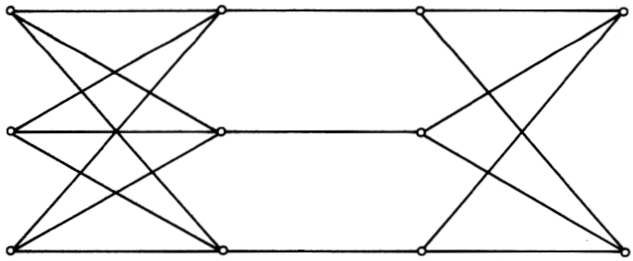
We may assume $n_2 = n_2' + n_2''$, where both x_{a_j} and y_{b_j}, $j = 1, 2, \cdots, n_2'$, appear in A.

If $t - n_3 \geqq n_2'$, it is easy to see this graph is rearrangeable.

Suppose $t - n_3 < n_2'$. Let us consider that the set $S_j = \{y_{b_i} | 1 \leqq i \leqq n_2, y_{b_i}$ is adjacent to y_{b_j}, and both y_{b_i} and x_{a_i} do not occur in $A\}$.



(a)



(b)

Fig. 2—(a) Graph $M(3, 7)$. (b) Graph $M(3, 8)$.

Since $|S_j| \geqq n - t - n_1 - n_2''$ for all $1 \leqq j \leqq n$, we know that at least $n - t - n_3 - n_2''$ of the n_2' requests involving $y_{b_1}, y_{b_2}, \cdots, y_{b_{n_2}}$ can be connected.

If

$$n_2' \geqq n - t - n_1 - n_2'',$$

then

$$\begin{aligned} n_2' - (n - t - n_1 - n_2'') &= t - (n - n_1 - n_2' - n_2'') \\ &\leqq t - n_3. \end{aligned}$$

After the remaining $n_2' - (n - t - n_1 - n_2'')$ requests are connected by a path passing through some of the $t - n_3$ z_i's, which do not occur in A, the requests involving the z_{b_i}'s or the x_{a_i}'s can be easily connected. Thus, we have proved that the graph $M(n, m)$ is rearrangeable.

For the case $m = 3n - 1$, there is another type of Manhattan graph which is a special case of the following class of graphs.

Case 4, $m = h(n - 1) + 2n$, $h \geqq 1$:
Let

$$I = \{i_1, i_2, \cdots, i_n\},$$
$$\Omega = \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_{(n-1)h}\}.$$

The graph $M(n, m)$ is constructed as follows:

(*i*) The spanning subgraph of vertices $I \cup \{x_i | 1 \leqq i \leqq n\}$ is the complete bipartite graph $K_{n,n}$;

(*ii*) x_j is adjacent to y_j, $j = 1, 2, \cdots, n$;

(*iii*) There is a cycle with vertices $y_1, z_1, z_2, \cdots, z_h, y_2, z_{h+1}, \cdots, z_{2h}, y_3, \cdots, y_n, y_1$;

(*iv*) There is a complete graph K_{n-1} with vertex set $\{z_{i+jh} | j = 0, 1, \cdots, n-2\}$ for $i = 1, 2, \cdots, h$.

(*v*) y_i is adjacent to $y_1, y_2, \cdots, y_{i-2}, y_{i+2}, \cdots, y_n$ for $i = 2, \cdots, n-1$.
y_1 is adjacent to $y_3, y_4, \cdots, y_{n-1}$.
y_n is adjacent to $y_2, y_3, \cdots, y_{n-2}$.

As an example of this construction, we illustrate $M(4, 14)$ in Fig. 3.

To see that this graph is rearrangeable, let us consider an assignment in which $x_{a_1}, x_{a_2}, \cdots, x_{a_{n_1}}, y_{b_1}, y_{b_2}, \cdots, y_{b_{n_2}}$ and $z_{c_1}, z_{c_2}, \cdots, z_{c_{n_3}}$ occur. Because of the structure of this graph, any request involving the x_{a_i} or y_{b_j} can easily be connected after the n_3 requests involving the z_{c_i}'s are connected.

It is clear that $n \geqq n_1 + n_2 + n_3$. First, let us consider the special case $n_1 = n_2 = 0$. Let $P_{i,j}$, $i < j$, denote the path $y_i, z_{(i-1)h+1}, z_{(i-1)h+2}, \cdots, z_{ih}, y_{i+1}, \cdots, y_j$. If each of $P_{1,2}, P_{2,3}, \cdots, P_{n-1,n}$ contains one of the z_{c_i} except for one $P_{i,i+1}$, then the assignment can be satisfied. If more than one of $P_{1,2}, P_{2,3}, \cdots, P_{n-1,n}$ contains more than one z_{c_i}'s, say $P_{1,2}$ contains z_{c_1}, z_{c_2}, and $P_{3,4}$ contains z_{c_3}, z_{c_4}, we know that at least one of the $P_{i,i+1}$'s does not contain any z_{c_i}, say $P_{t,t+1}$. Instead of considering the assignment A, it suffices to consider the assignment involving $\{z_{c_1+(t-1)h}\} \cup \{z_{c_i} | i = 2, 3, \cdots, n_3\}$. Continuing this argument, it is enough to consider an assignment satisfying the property that all the z_{c_i} involved appear in distinct $P_{i,i+1}$'s except for two of them and, therefore, this assignment is realizable.

Now, for arbitrary n_1 and n_2, let $S = \{a_1, a_2, \cdots, a_{n_1}, b_1, b_2, \cdots, b_{n_2}\}$. Relabel S by $S = \{s_1 < s_2 < \cdots < s_{n'}\}$, $n' \leqq n_1 + n_2$, and consider

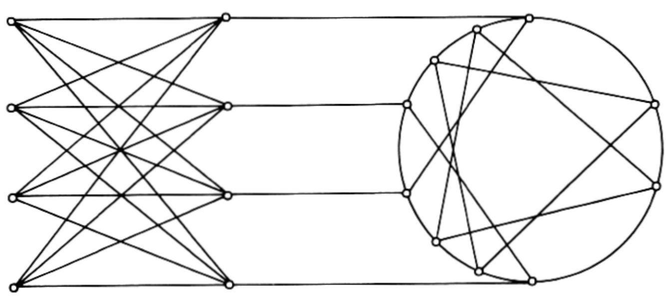


Fig. 3—Graph $M(4, 14)$.

the set $T = \{P_{i,j} | i < j,\ i, j \notin S \text{ and } i+1, i+2, \cdots, j-1 \in S\}$. Then $|T| = n - |S| - 1$.

Let

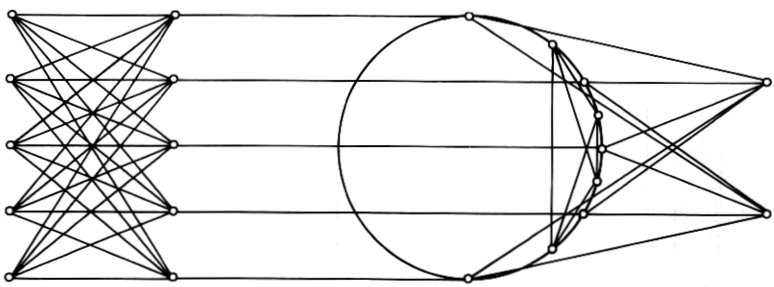
$$R = \{z_i | i = c_1, c_2, \cdots, c_{n_3}\} \cup \{y_j | j \in \{b_1, b_2, \cdots, b_{n_2}\} \setminus \{a_1, \cdots, a_{n_1}\}\}.$$

If all paths in T contain one element of R except one path which contains two elements of R, then the assignment is clearly realizable. Otherwise, we may use an argument similar to the one above to establish the rearrangeability of $M(n, m)$.

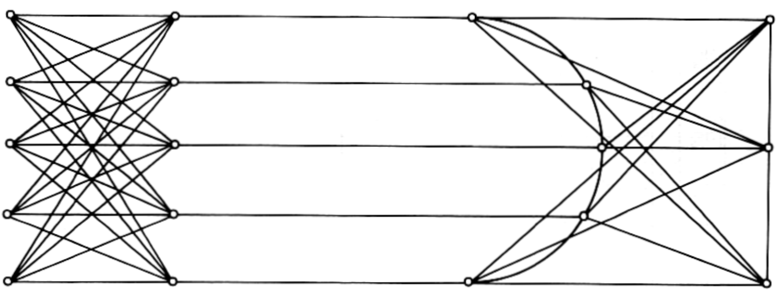
Case 5, $m \geqq 3n$ and $m = 2n + h(n-1) + t$, $0 < t < n-1$: In this case, the graph is a combination of a graph of Case 3 and a graph of Case 4 except for minor modifications. Let

$$I = \{i_1, i_2, \cdots, i_n\},$$
$$\Omega = \{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_{(n-1)h}, w_1, w_2, \cdots, w_t\}.$$

(*i*) If $t \neq n-2$, let us delete all the edges of the form $\{y_i, y_j\}$, $1 \leqq i, j \leqq n$ in $M(n, 2n+t)$ in Case 3. We then construct a cycle of vertices $y_1, z_1, z_2, \cdots, z_{n-1}, y_2, \cdots, y_n, y_1$. The spanning subgraph of vertices $\{z_1, z_2, \cdots, z_{(n-1)h}\}$ are the same as that in $M(n, 2n + (n-1)h)$ in Case 4. The spanning subgraph



(a)



(b)

Fig. 4—(a) Graph $M(5, 16)$. (b) Graph $M(5, 17)$.

of vertices $\{y_1, y_2, \cdots, y_n\}$ with the exception of the edge y_1y_n is any graph with degree sequence

$$\underbrace{(n - t - 3, n - t - 3, \cdots, n - t - 3, w)}_{n - 1 \text{ times,}}$$

where

$$w = \begin{cases} n - t - 3 & \text{if } n(n - t - 3) \text{ is even,} \\ n - t - 2 & \text{otherwise,} \end{cases}$$

and

$$y_i \not\sim y_{i+1}, i = 1, 2, \cdots, n - 1.$$

(*ii*) If $t = n - 2$, the construction scheme is as follows:

(*a*) The spanning subgraph of vertices $I \cup \{x_1, x_2, \cdots, x_n\}$ is $K_{n,n}$;
(*b*) x_j is adjacent to y_j, $j = 1, 2, \cdots, n$;
(*c*) There is a path $y_1, z_1, \cdots, z_h, y_2, z_{k+1}, \cdots, z_{2h}, y_3, \cdots, y_n$;
(*d*) There is a complete graph k_{n-1} with vertex set $\{z_{i+jh} | j = 0, 1, \cdots, n - 2\}$ for $j = 1, 2, \cdots, h$.
(*e*) When $n = 3$, w_1 is connected to any y_i. If $n \neq 3$, we have the following:

w_i, $i = 1, 2, \cdots, n - 2$, is adjacent to all y_j except y_{i+1};
If n is even, then $w_{2i-1} \sim w_{2i}$, $i = 1, 2, \cdots, [n/2] - 1$.
If n is odd, then $w_{2i-1} \sim w_{2i}$, $i = 1, 2, \cdots, [n/2] - 1$, $w_{n-3} \sim w_{n-2}$.

It is an easy exercise to show that the Manhattan graph $M(n, m)$ thus constructed is rearrangeable. As examples of this construction, we illustrate $M(5, 16)$, $M(5, 17)$ in Fig. 4.

By a direct calculation, it is easy to verify that all the Manhattan graphs we constructed in Cases 1 through 5 achieve the lower bounds on all rearrangeable graphs for given I and Ω with $|I \cap \Omega| = 0$. From this, the following result is immediate.

Theorem 2: The Manhattan graphs $M(n, m)$ are optimal rearrangeable graphs.

We note that a complete bipartite graph $K_{n,m}$ has nm edges. Thus, by Theorem 1, a Manhattan graph has precisely $[\frac{1}{2}(m - n - 1) \times \min(n, m - n)]$ fewer edges than $K_{n,m}$. When m is large compared to n, this is approximately $\frac{1}{2}nm$.

IV. MANHATTAN GRAPHS FOR THE CASE OF $|I \cap \Omega| \neq 0$

Let us now consider an optimal rearrangeable graph with vertex set $I \cup \Omega$ and $|I \cap \Omega| \neq 0$.

If $|\Omega \setminus I| \geqq |I|$, then the Manhattan graph $M(I, \Omega)$ will be taken to be the same as $M(I, \Omega \setminus I)$. To prove that the Manhattan graph $M(I, \Omega)$ is an optimal rearrangeable graph for given I, Ω, and $|\Omega \setminus I| \geqq |I|$, we need only show that $M(I, \Omega)$ is rearrangeable. Any request $(x, y) \in A$, $x \in I$, $y \in \Omega \setminus I$, can be connected in $M(I, \Omega \setminus I)$ as well as in $M(I, \Omega)$. If the assignment contains some request (x, y), where both x and y are in I, they can be successfully joined by a path of length 2 via some vertex in $\Omega \setminus I$.

When $|\Omega \setminus I| \leqq |I| - 1$, the following construction suggested by F. K. Hwang[2] suffices for $M(I, \Omega)$.

Let $M(I, \Omega)$ be the union of a complete graph K_l and a three-partite graph $K_{n,m,l}$ as shown in Fig. 5b, where $n = |I \setminus \Omega|$, $m = |\Omega \setminus I|$, $l = |\Omega \cap I|$. To illustrate this construction more clearly, we denote the graph I_n to be the graph of n vertices without any edge. If two graphs G and H are joined by two thick lines, as shown in Fig. 5a, there is an edge connecting any vertex in G to any vertex in H.

We note that *no* edge in the above graph can be deleted without destroying the rearrangeability of the graph for the given I, Ω. Thus, we can state the following result.

Theorem 3: The Manhattan graph $M(I, \Omega)$ is an optimal rearrangeable graph for any given I, Ω.

V. *k*-REARRANGEABLE GRAPHS

A graph is said to be *rearrangeable of capacity* k or k*-rearrangeable* if it satisfies any assignment A of size at most k, i.e.,

$$A = \{(x_1, y_1), \cdots, (x_t, y_t)\}, t \leqq k.$$

A rearrangeable graph is easily seen to be a special case of a k-rearrangeable graph with $k = |I|$.

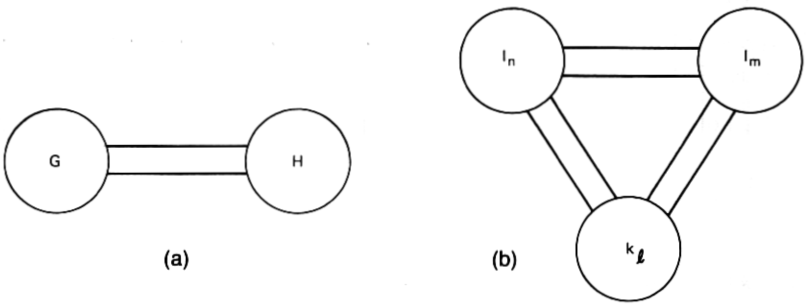


Fig. 5—(a) Complete connection between two graphs. (b) Graph $M(I,\Omega)$ with $n = |I \setminus \Omega|$, $m = |\Omega \setminus I|$, $l = |\Omega \cap I|$.

Assuming $|\Omega| \geqq |I| \geqq k$, a k-rearrangeable graph G with vertex set $V(G) = I \cup \Omega$ has the following properties (which are similar to those of a rearrangeable graph).

Fact 1': For all $v \in V(G)$, $\deg(v) \geqq k$.
Fact 2': If $v \in I$, $\deg_\Omega(v) \geqq k$.
Fact 3': For any $v \in V(G)$, $\max[\deg_I(v), \deg_\Omega(v)] \geqq k$.
Fact 4': If $v \sim I$, $v \sim \Omega$, then $\deg(v) \geqq k + 1$.

Lemma 1': The number of edges in a k-rearrangeable graph with vertex set $V(G) = I \cup \Omega$, $|I| > k$, $|\Omega \setminus I| > k$, $p = |V(G)|$, satisfies

$$e(G) \geqq \left\lceil \frac{k(p+2)}{2} \right\rceil.$$

Proof: If there is an $x \in I$ which is not adjacent to I, then the spanning subgraph G' of the vertex set $V(G) \setminus \{v\}$ in G must be k-rearrangeable. If $|I| = k + 1$, then G' has at least $\frac{1}{2}kp$ edges. If $|I| > k + 1$, then G' has more than $\frac{1}{2}kp$ edges (by induction). In any case, G has at least $\frac{1}{2}k(p + 2)$ edges. Similarly, G must have $\geqq \frac{1}{2}k(p + 2)$ edges if there is an $y \in \Omega$ which is not adjacent to Ω.

If all vertices in I are adjacent to I and all vertices in Ω are adjacent to Ω, consider the sets

$$I' = \{x \in I \mid x \sim \Omega, x \sim I\},$$
$$\Omega' = \{y \in \Omega \mid y \sim I, y \sim \Omega\}.$$

Any element in I' or Ω' has degree $\geqq k + 1$ and also $|I'| \geqq k$, $|\Omega'| \geqq k$. Hence,

$$e(G) \geqq \left\lceil \frac{k(p+2)}{2} \right\rceil.$$

Similar to Theorem 1, we have Theorem 4.

Theorem 4: The number of edges in a k-rearrangeable graph with vertex set $V(G) = I \cup \Omega$, $|I| = n$, $|\Omega| = m$, $k < n \leqq m$, $|I \cap \Omega| = 0$, satisfies

$$e(G) \geqq \begin{cases} \lceil \frac{1}{2}k(m+n+2) \rceil & \text{if } 2k \leqq n \leqq m, \\ \lceil \frac{1}{2}\{k(m+n+1) + t(k-t+1)\} \rceil & \\ \qquad \text{if } k < n < 2k \leqq m, \quad n = k + t, & \\ \lceil \frac{1}{2}\{k(m+n) + t(k-t+1) + t'(k-t'+1)\} \rceil & \\ \qquad \text{if } k < n \leqq m < 2k, \quad n = k+t, \quad m = k + t'. & \end{cases}$$

If I and Ω are disjoint, an optimal k-rearrangeable graph can be constructed by combining two optimal rearrangeable graphs $M(k, n)$, $M(k, m)$ by overlapping $K_{k,k}$ as shown in Fig. 6. These are called

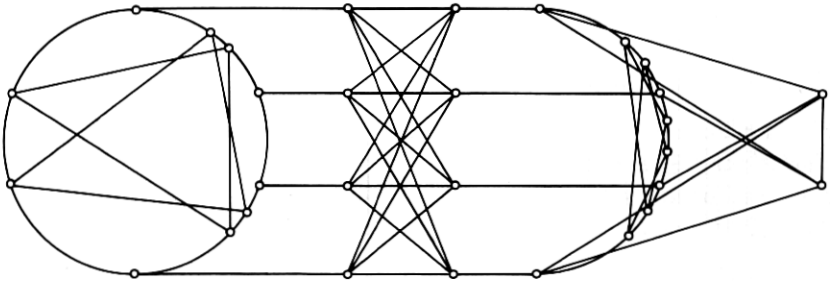


Fig. 6—Graph $M_4(14,16)$.

Manhattan k-graphs and are denoted by $M_k(n, m)$, where $n = |I|$, $m = |\Omega|$.

Because $M(k, n)$, $M(k, m)$ are rearrangeable, the k-rearrangeability of $M_k(n, m)$ follows immediately.

If $|\Omega \cap I| = 1$, then $M_k(I, \Omega)$ is the same as $M_k(|I|, |\Omega| - 1)$. If $|\Omega \cap I| = 2$ and $|\Omega \setminus I| \geqq k$, then $M_k(I, \Omega)$ is $M_k(|I| - 1, |\Omega| - 1)$.

We notice that $M_n(n, m) = M(n, m)$.

Theorem 5: Manhattan k-graphs are optimal rearrangeable graphs of capacity k for given I, Ω where $|\Omega| \geqq |I| > k$, $|\Omega \cap I| \leqq 2$, $|\Omega \setminus I| \geqq k$.

As we noted earlier, Manhattan graphs have considerably fewer edges than the corresponding complete bipartite graphs with the same vertex sets. This is also the case for Manhattan k-graphs as well. In particular, the number of edges saved is

$$[mn - \tfrac{1}{2}\{k(m + n) + \max[k, (n - k)(2k - n + 1)] + \max[k, (m - k)(2k - m + 1)]\}].$$

When $|\Omega \cap I|$ is large, alternate constructions of k-rearrangeable graphs for given I and Ω can be given by adding k additional vertices,

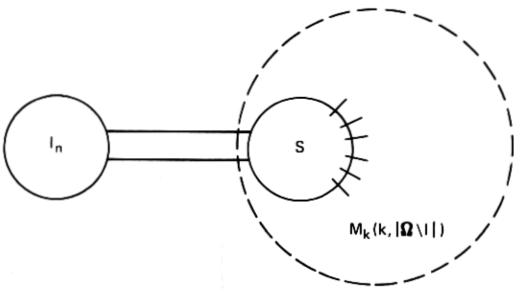


Fig. 7—k-rearrangeable graph with Steiner vertices.

called *Steiner vertices*. For example, we may consider the following graph with vertex set $I \cup \Omega \cup S$ and $|S| = k$ as shown in Fig. 7.

(*i*) The spanning subgraph of I and S is a complete bipartite graph $K_{n,k}$.
(*ii*) The spanning subgraph of S and $\Omega \setminus I$ is precisely the Manhattan graph $M_k(k, |\Omega \setminus I|)$.

This graph is clearly k-rearrangeable.

VI. GRAPH REPRESENTATIONS OF A SWITCHING NETWORK

Consider a graph G with vertex set $V(G)$. Let I and Ω be nonempty subsets of $V(G)$ and let $S = V(G) \setminus (I \cup \Omega)$, which we shall call the *Steiner set* of G.

The graph G corresponds to a switching network in the following way:

(*i*) $I \leftrightarrow$ inlet lines.
(*ii*) $\Omega \leftrightarrow$ outlet lines.
(*iii*) Edge $\{x, y\} \leftrightarrow$ a crosspoint between x and y.
(*iv*) $S \leftrightarrow$ additional lines.

For example, the rectangle network in Fig. 8 corresponds to the complete bipartite graph $K_{3,4}$.

The Manhattan graph $M(3, 5)$ of Fig. 1 corresponds to the rearrangeable network shown in Fig. 9.

An example of a network derived from the graph in Fig. 10a with a nontrivial Steiner set is shown in Fig. 10b.

In this way, a switching network can be represented by a graph. A rearrangeable graph then corresponds to a rearrangeable network. A k-rearrangeable graph corresponds to a rearrangeable network of capacity k.

Many problems in switching networks can in this way be viewed as graph-theoretic problems. Instead of minimizing the number of

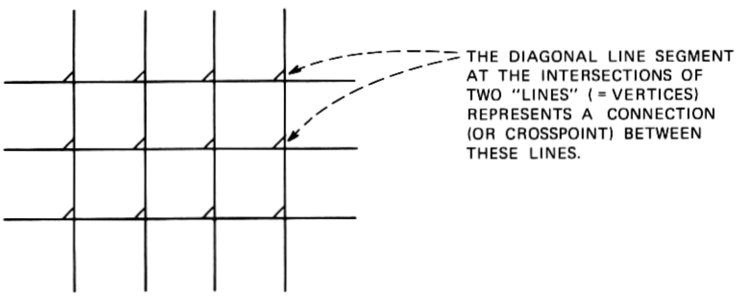


Fig. 8—Rectangle network of size 3 × 4.

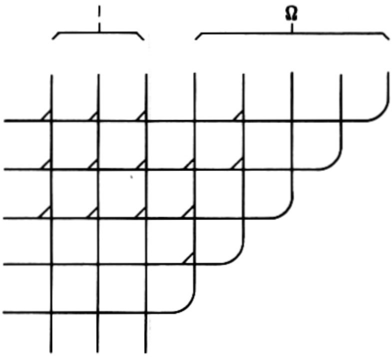


Fig. 9—Rearrangeable network corresponding to graph $M(3, 5)$.

crosspoints to reduce the cost of building a network, we consider the problem of finding a graph with the least possible number of edges. The size of S in the graph representation of a switching network determines how many lines we have to use in addition to the inlet and outlet lines. The Manhattan graph we have constructed then provides a model of a rearrangeable network with a minimum number of crosspoints for the case that the size of S is 0.

VII. CONCLUDING REMARKS

Almost all previous results on rearrangeable networks dealt with rearrangeable graphs having $|I| = |\Omega|$ and $|I \cap \Omega| = 0$. Beneš[3]

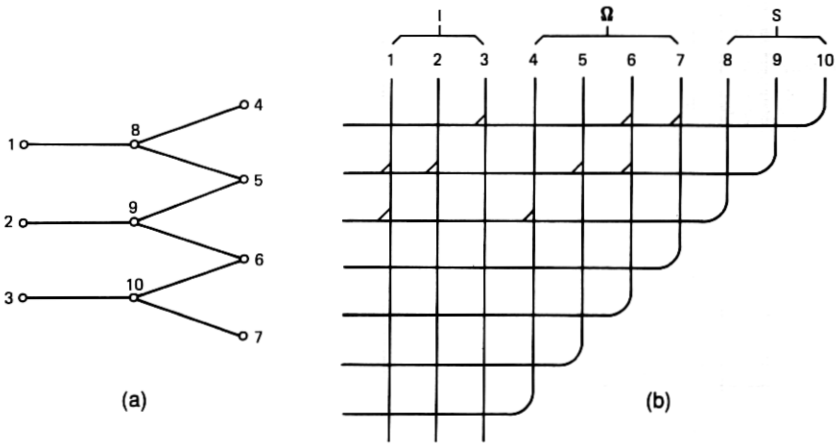


Fig. 10—(a) Graph with $I = \{1, 2, 3\}$, and $\Omega = \{4, 5, 6, 7\}$, and $S = \{8, 9, 10\}$. (b) Network corresponding to (a).

has shown that a rearrangeable network for $|I| = |\Omega| = n$ can be constructed with slightly more than $0(n \log n)$ crosspoints, which is just the information-theoretic lower bound. However, $|S|$ is required to be arbitrarily large to approach the $0(n \log n)$ bound. This result was later refined by Waksman[5] and Joel.[6]

When just one middle stage is allowed, Preparata[7] gave a lower bound on the number of crosspoints in a k-rearrangeable network and showed several optimal designs for arbitrary sizes of I and Ω.

With regard to nonblocking networks, we can define *nonblocking graphs* as those which satisfy the following property: The vertex set of a nonblocking graph G is $V(G) = I \bigcup \Omega \bigcup S$, where S is disjoint from I and Ω. For any assignment $A = \{(x_i, y_i) | i = 1, 2, \cdots, t\}$, we can find a path connecting x_i and y_i without disturbing the existing paths already connecting x_j and y_j, $1 \leqq j < i$. In other words, there is always a path connecting x_i and y_i whose vertices and edges are disjoint from those of the previous paths.

If the vertex set of a nonblocking network is the union of I and Ω, one class of nonblocking graphs we can construct is formed from the union of a three-partite graph $K_{n,m,l}$ and a complete graph K_l, where $|I \bigcap \Omega| = l$, $|I| = n$, and $|\Omega| = m$, as shown in Fig. 5.

Bassalygo and Pinsker[8] have shown by a nonconstructive argument that there exist nonblocking networks with $0(n \log n)$ crosspoints, where $|I| = |\Omega| = n$ and the size of S approaches infinity. The best known construction, due to Cantor,[9] requires $0[n(\log n)^2]$ crosspoints.

VIII. ACKNOWLEDGMENT

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