THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 54, No. 7, September 1975
Printed in U.S.A.

Upper Bound on Error Probability for Detection With Unbounded Intersymbol Interference

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(Manuscript received February 27, 1975)

Forney's asymptotic upper bound for per-bit error probability in the detection of pulse-amplitude-modulated digital data in the presence of additive white gaussian noise was obtained for the case where the duration of the intersymbol interference is bounded. In this paper, we show the validity of Forney's bound under much weaker assumptions that allow unbounded intersymbol interference.

I. INTRODUCTION

We consider the situation where a data sequence $a_0, \cdots, a_{N-1}$ of ± 1's is transmitted via pulse amplitude modulation as $\sum_{k=0}^{N-1} h(t - kT)a_k$ and received in the presence of additive white gaussian noise with one-sided spectral density σ^2. In a recent series of papers, Forney,[1] Foschini,[2] and Mazo[3] developed an asymptotic (as $\sigma^2 \to 0$) upper bound on the error probability per data bit P_e:

$$P_e \leqq \exp \left\{ -\frac{d^2(h)}{4\sigma^2} [1 + o(1)] \right\}, \tag{1}$$

where $d(h)$ is the minimum $\mathcal{L}_2$ distance between distinct modulated pulse sequences. This bound holds under the strong assumption that the pulse $h(t)$ is supported on finite interval.

In this paper, we show that (1) is valid for a considerably wider class of $h(t)$. Roughly speaking, our assumptions are little more than that $h(t)$ is in $\mathcal{L}_1(-\infty, \infty)$ and $\mathcal{L}_2(-\infty, \infty)$, and that $H(f)$, the Fourier transform of $h(t)$, does not vanish on an interval. The precise conditions on $h(t)$ under which (1) holds are given below. In particular, (1) is valid when $H(f)$ is a rational function.

In Section II we give a precise statement of our results, and the proof follows in Section III.

II. FORMAL STATEMENT OF PROBLEM AND RESULTS

In this section, we give a precise statement of the problem and the results that were stated informally in Section I.

We begin with some definitions. We denote N vectors by boldface superscripted letters, and components by subscripted letters, e.g.

$\mathbf{u}^N = (u_0, \cdots, u_{N-1})$. When the dimension N is clear from the context, we omit the superscript. Define the sets $\mathcal{A}_N$, $\mathcal{A}^+_{N,k}$, $\mathcal{A}^-_{N,k}$ by

$$\begin{aligned} \mathcal{A}_N &= \{\mathbf{u}^N : u_j = \pm 1, 0 \leqq j \leqq N-1\}, \\ \mathcal{A}^+_{N,k} &= \{\mathbf{u}^N : u_k = +1, u_j = \pm 1, j \neq k\}, \\ \mathcal{A}^-_{N,k} &= \{\mathbf{u}^N : u_k = -1, u_j = \pm 1, j \neq k\}. \end{aligned} \tag{2}$$

Of course, $\mathcal{A}_N = \mathcal{A}^+_{N,k} \cup \mathcal{A}^-_{N,k}$. Again, when N is clear from the context, we write $\mathcal{A}_N = \mathcal{A}$, $\mathcal{A}^+_{N,k} = \mathcal{A}^+_k$, $\mathcal{A}^-_{N,k} = \mathcal{A}^-_k$.

Next, let $f(t)$, $g(t)$, and $-\infty < t < \infty$ be real-valued measurable functions. The inner product of f and g is denoted by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) g(t) dt, \tag{3a}$$

and the norm of f is

$$\|f\| = (\langle f, f \rangle)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} f^2(t) dt \right)^{\frac{1}{2}}. \tag{3b}$$

For a vector $\mathbf{u}^N \in \mathcal{A}_N$, and $f(t)$, $-\infty < t < \infty$, a real-valued function, let the function $h*\mathbf{u} = s$ be defined by

$$s(t) = \sum_{k=0}^{N-1} f(t - kT) u_k,$$

where $T > 0$ is a fixed parameter.

We are concerned with the following modulation scheme. Let $\mathbf{a}^N = (a_0, \cdots, a_{N-1}) \in \mathcal{A}_N$ denote the data to be transmitted. Assume that all the 2^N vectors in $\mathcal{A}_N$ are equally likely. The transmitted signal is the function $h*\mathbf{a}^N$, where the pulse $h(t)$ is a fixed function for which $\|h\| < \infty$. The received signal is

$$y(t) = (h*\mathbf{a}^N)(t) + z(t), \qquad -\infty < t < \infty, \tag{4}$$

where $z(t)$ is a sample from a white gaussian noise process with zero mean and one-sided spectral density σ^2.

The decoder associates with the received signal y, a vector $D(y) = \hat{\mathbf{a}}^N \in \mathcal{A}_N$. Corresponding to a given decoder function D, let the bit error probability be

$$P_{eN}(D) = \frac{1}{N} \sum_{k=0}^{N-1} \Pr \{\hat{a}_k \neq a_k\}. \tag{5}$$

Also, define the optimum error probability

$$P^*_{eN} = P^*_{eN}(h, \sigma^2) = \inf_D P_{eN}(D). \tag{6}$$

We are concerned here with the asymptotics of $P^*_{eN}(h, \sigma^2)$, as $\sigma^2 \to 0$, i.e., as the signal-to-noise ratio approaches infinity. Accordingly, define

$$E_N(h) = -\liminf_{\sigma^2 \to 0} \sigma^2 \log P^*_{eN}(h, \sigma^2), \tag{7a}$$

so that, as $\sigma^2 \to 0$,

$$P_{eN}^*(h, \sigma^2) \leqq \exp\left\{-\frac{E_N(h)}{\sigma^2}[1 + o(1)]\right\}. \tag{7b}$$

Next, consider a particular decoder that is of special interest here—the maximum-likelihood decoder, denoted D_h. In the present problem, $D_h(y)$ can be taken to be that $\hat{\mathbf{u}} \in \mathcal{A}_N$ such that for all $\mathbf{u} \in \mathcal{A}_N$, $\mathbf{u} \neq \hat{\mathbf{u}}$,

$$\|y_1 - h*\hat{\mathbf{u}}\| < \|y_1 - h*\mathbf{u}\|, \tag{8}$$

where y_1 is the projection of y onto the subspace of $\mathcal{L}_2(-\infty, \infty)$ spanned by the signals $h*\mathbf{u}$, $\mathbf{u} \in \mathcal{A}_N$. With probability 1, (8) will be satisfied for some $\hat{\mathbf{u}} \in \mathcal{A}_N$.

Now, subject to the condition that $h(t)$ has finite support, i.e., there exists a $t_0 > 0$ such that

$$h(t) = 0, \qquad \text{for } |t| > t_0, \tag{9}$$

Forney,[1] Foschini,[2] and Mazo[3] have shown that $E(h) \geqq d^2(h)/4$, where the "minimum distance" $d(h)$ is defined by

$$d(h) = \liminf_{N\to\infty} \min_{\substack{\mathbf{u},\mathbf{v}\in\mathcal{A}_N \\ \mathbf{u}\neq\mathbf{v}}} \|h*\mathbf{u} - h*\mathbf{v}\|. \tag{10}$$

Thus, as $\sigma^2 \to 0$,

$$P_{eN}^*(h, \sigma^2) \leqq \exp\left\{-\frac{d^2(h)}{4\sigma^2}[1 + o(1)]\right\}. \tag{11}$$

Inequality (11) is established by showing that the error probability for the maximum likelihood decoder, $P_{eN}(D_h)$, is overbounded by the right member of (11). This is done by writing (this is not as difficult as it looks)

$$\begin{aligned} P_{eN}(D_h) &= \frac{1}{N}\sum_{k=1}^{N} \sum_{\mathbf{u}\in\mathcal{A}_{N,k}^+} 2^{-(N-1)} \Pr\{D_h(y) \in \mathcal{A}_{N,k}^- \mid y = h*\mathbf{u} + z\} \\ &= \frac{1}{N}\sum_k \sum_{\mathbf{u}\in\mathcal{A}_k^+} 2^{-(N-1)} \Pr\Big\{\bigcup_{\mathbf{v}\in\mathcal{A}_k^-}\{D_h(y) = \mathbf{v}\} \mid y = h*\mathbf{u} + z\Big\} \\ &\leqq \frac{1}{N}\sum_k \sum_{\mathbf{u}\in\mathcal{A}_k^+} 2^{-(N-1)} \\ &\quad \cdot \Pr\Big\{\bigcup_{\mathbf{v}\in\mathcal{A}_k^-}\{\|y_1 - h*\mathbf{u}\| \geqq \|y_1 - h*\mathbf{v}\|\} \mid y = h*\mathbf{u} + z\Big\} \\ &= \frac{1}{N}\sum_k \sum_{\mathbf{u}\in\mathcal{A}_k^+} 2^{-(N-1)} \Pr \bigcup_{\mathbf{v}\in\mathcal{A}_k^-} \{\langle z, h*(\mathbf{v} - \mathbf{u})\rangle \\ &\qquad\qquad \geqq \tfrac{1}{2}\|h*(\mathbf{v} - \mathbf{u})\|\} \\ &\triangleq \psi_N(h, \sigma^2). \end{aligned} \tag{12}$$

Relation (12) is valid for any $h(t)$. Subject to condition (9), it is then shown that, as $\sigma^2 \to 0$,

$$\psi_N(h, \sigma^2) \leqq \exp\left\{-\frac{d^2(h)}{4\sigma^2}[1 + o_1(1)]\right\}, \tag{13}$$

where $o_1(1)$ does not depend on N. Thus, since $P_{eN}^* \leqq P_{eN}(D_h)$, (11) holds. Further, the $o(1)$ term in (11) does not depend on N. An interesting by-product of these results is that the performance indicated in (11) is achievable via the decoder D_h. This decoder can be instrumented (using the Viterbi algorithm) with a complexity which remains bounded as $N \to \infty$.

We now drop the assumption that $h(t)$ has finite support. Instead, we assume that $h(t)$ satisfies the following conditions:

(*i*) There exists a nonnegative $\mathfrak{L}_1$ function $g_0(t)$, i.e.,

$$\int_{-\infty}^{\infty} g_0(t)dt < \infty,$$

such that

$$|h(t)| \leqq g_0(t), \qquad -\infty < t < \infty, \tag{14}$$

and such that g_0 is monotone in $|t|$.

(*ii*) Let

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt, \qquad -\infty < f < \infty \tag{15}$$

be the Fourier transform of $h(t)$. By (*i*), $\int |h(t)|dt < \infty$, so that $H(f)$ is well defined for all f. We assume that there exists a nonnegative $\mathfrak{L}_1$ function $G_1(f)$ which is monotone in $|f|$, such that

$$|H(f)|^2 \leqq G_1(f), \qquad -\infty < f < \infty. \tag{16}$$

(*iii*) Let the "folded spectrum" of h be

$$S(f) = \sum_{n=-\infty}^{\infty} \left|H\left(f + \frac{n}{T}\right)\right|^2, \qquad 0 \leqq f \leqq \frac{1}{T}. \tag{17}$$

We show in Appendix A that $S(f)$, $0 \leqq f \leqq 1/T$, is finite and continuous. We assume that $S(f) > 0$, $0 \leqq f \leqq 1/T$. Let

$$m = \min_{0 \leqq f \leqq 1/T} S(f) > 0, \tag{18}$$

where the existence of the minimum follows from the continuity of $S(f)$ on the compact interval $[0, 1/T]$.

Remarks:

(1) Condition (*i*) is just slightly stronger than simply requiring h to be in $\mathfrak{L}_1(-\infty, \infty)$. Condition (14) forces $h(t)$ to go to zero as $|t| \to \infty$ in a "well-behaved" manner. Condition (*ii*) imposes a similar condition on $|H(f)|^2$.

(2) For the very important special case where $H(f)$ is a rational function, i.e., $H(f) = P(i2\pi f)/Q(i2\pi f)$, and P, Q are polynomials

with the degree of $P <$ degree of Q, then conditions (*i*) and (*ii*) are satisfied. Since $H(f)$ has only a finite number of zeros, condition (*iii*) is also satisfied.

(3) Suppose that $H(f)$ has no more than a countable number of zeros, but that $S(f) = 0$ for some $f \in [0, 1/T]$. It is easy to see that some arbitrarily small change in T will cause $S(f)$ to be strictly positive for all $f \in [0, 1/T]$. Thus, condition (*iii*) is not especially restrictive.

We now state our main result, the proof of which is in Section III.

Theorem 1: *Let h satisfy conditions (i), (ii), and (iii) above. Then, for all* $\epsilon > 0$, *there exists a* $\tau_0 = \tau_0(\epsilon)$ *sufficiently large so that, for all* $\tau > \tau_0$,

$$P_{eN}(D_{h_\tau}) \leqq \psi_N[h_\tau, \sigma^2(1+\epsilon)^2],$$

where

$$h_\tau(t) = \begin{cases} h(t), & |t| \leqq \tau, \\ 0, & |t| > \tau, \end{cases} \tag{19}$$

is the truncated version of $h(t)$. *The quantity* τ_0 *does not depend on* N.

Since h_τ has finite support, we conclude from Theorem 1 and (13) that, for all $\epsilon > 0$ and τ sufficiently large,

$$\begin{aligned} P^*_{eN}(h, \sigma^2) &\leqq P_{eN}(D_{h_\tau}) \\ &\leqq \exp\left\{-\frac{d^2(h_\tau)}{4\sigma^2(1+\epsilon)^2}[1 + o_2(1)]\right\}, \end{aligned} \tag{20a}$$

[where $o_2(1)$ is independent of N] so that

$$E_N(h) = -\liminf_{\sigma^2\to 0} \sigma^2 \log P^*_e(h, \sigma^2) \geqq \frac{d^2(h_\tau)}{4(1+\epsilon)^2}. \tag{20b}$$

We show in Appendix B that

$$d(h_\tau) \to d(h), \qquad \text{as } \tau \to \infty, \tag{21}$$

so that letting $\epsilon \to 0$ and $\tau \to \infty$ in (20b) yields

$$E_N(h) \geqq \frac{d^2(h)}{4}. \tag{22}$$

We state this as

Corollary 2: *Let h satisfy conditions (i) to (iii) above. Then, as* $\sigma^2 \to 0$,

$$P^*_{eN}(h, \sigma^2) \leqq \exp\left\{-\frac{d^2(h)}{4\sigma^2}[1 + o_3(1)]\right\},$$

where $o_3(1)$ *is independent of* N.

We conclude this section with a remark concerning the relationship of the bound of Forney et al. (11) with the result of Corollary 2. We can rewrite (11) as

$$P^*_{eN}(h, \sigma^2) = K_1(t_0, \sigma^2)e^{-d^2(h)/2\sigma^2}, \tag{23}$$

and the bound of Corollary 2 as

$$P^*_{eN}(h, \sigma^2) \leqq K_2(\|h\|_1, m, \sigma^2)e^{-d^2(h)/4\sigma^2}. \tag{24}$$

Here, we made explicit the dependence of K_1 on the support interval t_0 of $h(t)$ [see (9)], and the dependence of K_2 on $\|h\|_1$, the $\mathcal{L}_1$ norm of h, and on $m = \min S(f)$. Both K_1 and K_2 increase in $1/\sigma^2$ slower than $e^{d^2(h)/4\sigma^2}$. But $K_1(t_0, \sigma^2) \rightarrow \infty$ as $t_0 \rightarrow \infty$, and $K_2(\|h\|_1, m, \sigma^2) \rightarrow \infty$, as $\|h\|_1 \rightarrow \infty$ or as $m \rightarrow 0$. Thus, although it might seem reasonable to assume that all $h(t)$ satisfy (9) for some t_0, the bound of (23) depends on that t_0 and becomes meaningless as $t_0 \rightarrow \infty$. Similarly, although it might be reasonable to assume for any $h(t)$ that $\|h\|_1 < \infty$ and $m = \min S(f) > 0$, the bound of (24) depends on these quantities and also becomes meaningless as $\|h\|_1 \rightarrow \infty$ or $m \rightarrow 0$. Therefore, both bounds have their limitations; the new one, however, is considerably less limited.

III. PROOF OF THEOREM 1

Let h satisfy (*i*) to (*iii*). Let $h_\tau(t)$ be as defined in (19), and let $\tilde{h}_\tau(t) = h(t) - h_\tau(t)$, i.e.,

$$\tilde{h}_\tau(t) = \begin{cases} 0, & |t| \leqq \tau, \\ h(t), & |t| > \tau. \end{cases} \tag{25}$$

Then, if the data sequence is $\mathbf{u} \in \mathcal{A}_N$, the received sequence is $y = h*\mathbf{u} + z = h_\tau*\mathbf{u} + \hat{z}$, where

$$\hat{z} = z + \tilde{h}_\tau*\mathbf{u}. \tag{26}$$

Following the same steps as in (12), we obtain

$$P_{eN}(D_{h_\tau}) \leqq \frac{1}{N}\sum_k \sum_{\mathbf{u}\in\mathcal{A}_k^+} 2^{-(N-1)} \Pr \bigcup_{\mathbf{v}\in\mathcal{A}_k^-} \{\langle\hat{z}, h_\tau*(\mathbf{v}-\mathbf{u})\rangle \geqq \tfrac{1}{2}\|h_\tau*(\mathbf{v}-\mathbf{u})\|\}, \tag{27}$$

where $\hat{z}$ is given in (26).

We will show that, for arbitrary $\epsilon > 0$, there exists a $\tau_0 = \tau_0(\epsilon, h)$ (τ_0 independent of N), such that for $\tau \geqq \tau_0$, the event

$$\{\langle\hat{z}, h_\tau*(\mathbf{v}-\mathbf{u})\rangle \geqq \tfrac{1}{2}\|h_\tau*(\mathbf{v}-\mathbf{u})\|^2\} \subseteq \{\langle z, h_\tau*(\mathbf{v}-\mathbf{u})\rangle \geqq \frac{(1+\epsilon)}{2}\|h_\tau*(\mathbf{v}-\mathbf{u})\|^2\}, \tag{28}$$

for all $\mathbf{u} \in \mathcal{A}_k^+$, $\mathbf{v} \in \mathcal{A}_k^-$. Substituting (28) into (27) yields, on comparison with (12),

$$P_{eN}(D_{h_\tau}) \leqq \psi_N[h_\tau, \sigma^2(1+\epsilon)^2],$$

which is Theorem 1. It remains to establish (28).

Relation (28) will follow immediately when we show the existence of a $\tau_0(\epsilon, h)$ such that, for $\tau \geqq \tau_0$ and all $\mathbf{u}$, $\mathbf{v}$,

$$|\langle \tilde{h}_\tau * \mathbf{u}, h_\tau * (\mathbf{v} - \mathbf{u})\rangle| \leqq \frac{\epsilon}{2} \| h_\tau * (\mathbf{v} - \mathbf{u}) \|^2. \tag{29}$$

If (29) holds, the event in the left member of (28)

$$\begin{aligned} \{\langle \hat{z}, h_\tau * (\mathbf{v} - \mathbf{u})\rangle &\geqq \tfrac{1}{2} \| h_\tau * (\mathbf{v} - \mathbf{u})\|^2\} \\ &= \{\langle z, h_\tau * (\mathbf{v} - \mathbf{u})\rangle \geqq \tfrac{1}{2} \| h_\tau * (\mathbf{v} - \mathbf{u})\|^2 - \langle \tilde{h}_\tau * \mathbf{u}, h_\tau * (\mathbf{v} - \mathbf{u})\rangle\} \\ &\subseteq \left\{ \langle z, h_\tau * (\mathbf{v} - \mathbf{u})\rangle \geqq \frac{1 + \epsilon}{2} \| h_\tau * (\mathbf{v} - \mathbf{u})\|^2 \right\}, \end{aligned} \tag{30}$$

which is the right member of (28). Thus, it remains to establish (29).

Let $\mathbf{w} = (w_0, \cdots, w_{N-1}) = \mathbf{v} - \mathbf{u}$. The entries of $\mathbf{w}$ are $0, \pm 2$. Also set $q = \tilde{h}_\tau * \mathbf{u}$, and $r = h_\tau * \mathbf{w}$. Then

$$\begin{aligned} |\langle \tilde{h}_\tau * \mathbf{u}, h_\tau * (\mathbf{v} - \mathbf{u})\rangle| = |\langle q, r\rangle| &\leqq \int_{-\infty}^{\infty} |q(t)|\,|r(t)|\,dt \\ &\leqq [\sup_{-\infty < t < \infty} |q(t)|] \int_{-\infty}^{\infty} |r(t)|\,dt. \end{aligned} \tag{31}$$

Consider

$$\begin{aligned} \int_{-\infty}^{\infty} |r(t)|\,dt &= \int_{-\infty}^{\infty} \left| \sum_{k=0}^{N-1} h_\tau(t - kT) w_k \right| dt \\ &\leqq \sum_k |w_k| \int_{-\infty}^{\infty} |h_\tau(t - kT)|\,dt \leqq \|h\|_1 \sum_{k=0}^{N-1} |w_k|, \end{aligned} \tag{32}$$

where $\|h\|_1 = \int |h(t)|\,dt < \infty$, by condition (i).

We obtain an upper bound on $\sum |w_k|$ as follows. Since $w_k = 0, \pm 2$, we have $\sum |w_k| = \frac{1}{2} \sum w_k^2$. Now, let $H_\tau(f) = \int_{-\infty}^{\infty} h_\tau(t) e^{i2\pi f t} dt$ be the Fourier transform of $h_\tau(t)$, and let

$$S_\tau(f) = \sum_{n=-\infty}^{\infty} \left| H_\tau\left(f + \frac{n}{T}\right) \right|^2, \qquad 0 \leqq f \leqq \frac{1}{T} \tag{33}$$

be the corresponding folded spectrum. Then, from Parseval's theorem,

$$\begin{aligned} \|r\|^2 = \|h_\tau * \mathbf{w}\|^2 &= \int_{-\infty}^{\infty} |H_\tau(f)|^2 |\sum_k w_k e^{i2\pi kTf}|^2 df \\ &= \int_0^{1/T} S_\tau(f) |\sum_k w_k e^{i2\pi kTf}|^2 df \\ &\geqq [\inf_{0 \leqq f \leqq 1/T} S_\tau(f)] \int_0^{1/T} |\sum w_k e^{i2\pi kTf}|^2 df \\ &= [\inf_f S_\tau(f)] \sum_k w_k^2 = [\inf_f S_\tau(f)] \sum |w_k| \cdot 2. \end{aligned} \tag{34}$$

Therefore,

$$\sum_k |w_k| \leqq \tfrac{1}{2} \|r\|^2 [\inf_{0 \leqq f \leqq 1/T} S_\tau(f)]^{-1}. \tag{35}$$

Combining (31), (32), and (35), we have

$$|\langle q, r\rangle| \leqq [\sup_{-\infty<t<\infty} |q(t)|][\inf_{0\leqq f\leqq 1/T} S_\tau(f)]^{-1} \tfrac{1}{2}\|h\|_1\|r\|^2. \tag{36}$$

Now, we show in Appendix A that

$$\liminf_{\tau\to\infty} [\inf_{0\leqq f\leqq 1/T} S_\tau(f)] \geqq m > 0, \tag{37}$$

where $m = \min S(f) > 0$ [see condition (*iii*)]. Further, using condition (*i*) [particularly the monotonicity of $g_0(t)$], we have

$$\begin{aligned} |q(t)| &= |\sum_k \tilde{h}_\tau(t-kT)u_k| \leqq \sum_{k=0}^{N-1} |\tilde{h}_\tau(t-kT)| \\ &\leqq \sum_{k=-\infty}^{\infty} |\tilde{h}_\tau(t-kT)| = \sum_{k:|t-kT|\geqq\tau} |h(t-kT)| \\ &\leqq \sum_{k:|t-kT|\geqq\tau} g_0(t-kT) \leqq \sum_{j=0}^{\infty} [g_0(\tau+jT) + g_0(-\tau-jT)] \\ &\leqq \frac{1}{T}\int_{\tau-T}^{\infty} g_0(t)dt + \frac{1}{T}\int_{-\tau+T}^{\infty} g_0(t)dt \to 0, \quad \text{as } \tau\to\infty. \end{aligned} \tag{38}$$

Combining (36), (37), and (38), we obtain

$$\frac{|\langle q, r\rangle|}{\|r\|^2} = \frac{|\langle \tilde{h}_\tau * \mathbf{u}, h_\tau * (\mathbf{v}-\mathbf{u})\rangle|}{\|h_\tau * (\mathbf{v}-\mathbf{u}\|^2} \to 0,$$

as $\tau\to\infty$. This is equivalent to (29), so that the proof of Theorem 1 is complete.

IV. ACKNOWLEDGMENTS

I would like to acknowledge with thanks the help of J. E. Mazo and H. J. Landau.

APPENDIX A

The Folded Spectrum S(f)

We first show that $S(f)$ as given in (17) is always finite, i.e., the series in (17) converges for all $f \in [0, 1/T]$. From condition (*ii*), using the monotonicity of G_1,

$$\begin{aligned} \xi(n_0, f) &\triangleq \sum_{|n|\geqq n_0} \left|H\left(f+\frac{n}{T}\right)\right|^2 \leqq \sum_{|n|\geqq n_0} G_1\left(f+\frac{n}{T}\right) \\ &= \sum_{n\leqq -n_0} G_1\left(f+\frac{n}{T}\right) + \sum_{n\geqq n_0} G_1\left(f+\frac{n}{T}\right), \\ &\leqq \sum_{n\leqq -n_0} G_1\left(\frac{n+1}{T}\right) + \sum_{n\geqq n_0} G_1\left(\frac{n}{T}\right) \leqq T\int_{-(n_0-2)/T}^{\infty} G_1(x)dx \\ &\qquad + T\int_{n_0-1/T}^{\infty} G_1(x)dx \to 0, \quad \text{as } n_0\to\infty, \end{aligned} \tag{39}$$

so that the series in (17) converges.

To establish the continuity of $S(f)$, write

$$S(f) = \sum_{|n| \leqq n_0} \left| H\left(f + \frac{n}{T}\right) \right|^2 + \xi(n_0, f), \quad 0 \leqq f \leqq 1/T. \tag{40}$$

For arbitrary δ, $0 \leqq f \leqq 1/T$,

$$|S(f) - S(f + \delta)| \leqq \left| \sum_{n \leqq n_0} \left[\left| H\left(f + \frac{n}{T}\right) \right|^2 - \left| H\left(f + \delta + \frac{n}{T}\right) \right|^2 \right] \right| + |\xi(n_0, f)| + |\xi(n_0, f + \delta)|. \tag{41}$$

Now since $h(t) \in \mathfrak{L}_1(-\infty, \infty)$, $H(f)$ is continuous. To make the right member of ineq. (41) $\leqq \epsilon$, first let n_0 be sufficiently large so that the last two terms of the right member of ineq. (41) $\leqq \epsilon/2$; then choose $|\delta|$ sufficiently small so that the first term of the right member of inequality (41) $\leqq \epsilon/2$. This establishes the continuity of $S(f)$.

We next verify (37), which concerns $S_\tau(f)$. Since h_τ is in $\mathfrak{L}_1(-\infty, \infty)$, $H_\tau(f)$ exists for all $f \in (-\infty, \infty)$. Thus, $S_\tau(f)$ as defined in (33) is meaningful, though perhaps infinite on a set of measure zero. With $\xi(n_0 f)$ as in (39), write

$$S_\tau(f) - S(f) = \sum_{n=-\infty}^{\infty} \left| H_\tau\left(f + \frac{n}{T}\right) \right|^2 - \sum_{n=-\infty}^{\infty} \left| H\left(f + \frac{n}{T}\right) \right|^2$$

$$\geqq \sum_{|n| \leqq n_0} \left[\left| H_\tau\left(f + \frac{n}{T}\right) \right|^2 - \left| H\left(f + \frac{n}{T}\right) \right|^2 \right] - \xi(n_0, f). \tag{42}$$

Now let $\epsilon > 0$ be arbitrary. From (39) we can choose n_0 sufficiently large such that $\xi(n_0, f) \leqq \epsilon/2$, for $f \in [0, 1/T]$. With n_0 so chosen,

$$S_\tau(f) - S(f) \geqq \sum_{|n| \leqq n_0} \left[\left| H_\tau\left(f + \frac{n}{T}\right) \right|^2 - \left| H\left(f + \frac{n}{T}\right) \right|^2 \right] - \frac{\epsilon}{2}. \tag{43}$$

Now let $\tilde{H}_\tau(f)$ be the Fourier transform of $\tilde{h}_\tau$. Then

$$|H(f)| = |H_\tau(f) + \tilde{H}_\tau(f)| \leqq |H_\tau(f)| + |\tilde{H}_\tau(f)|.$$

Therefore

$$|H(f)|^2 \leqq |H_\tau(f)|^2 + 2|H_\tau(f)|\,|\tilde{H}(f)| + |\tilde{H}_\tau(f)|^2$$
$$\leqq |H_\tau(f)|^2 + 2\|h_\tau\|_1\|\tilde{h}_\tau\|_1 + \|\tilde{h}_\tau\|_1^2,$$

where $\| \ \|_1$ denotes $\mathfrak{L}_1$ norm. Since $\|\tilde{h}_\tau\|_1 \to 0$, as $\tau \to \infty$, if τ is sufficiently large, then

$$|H_\tau(f)| - |H(f)| \geqq -\frac{\epsilon}{2} \cdot \frac{1}{(2n_0 + 1)}, \quad f \in (-\infty, \infty). \tag{44}$$

Inequalities (43) and (44) imply that, for all $\epsilon > 0$, there exists a $\tau_0(\epsilon)$ such that for all $\tau \geqq \tau_0(\epsilon)$,

$$S_\tau(f) \geqq S(f) - \epsilon, \quad 0 \leqq f \leqq \frac{1}{T}. \tag{45}$$

Thus, for $\tau \geqq \tau_0(\epsilon)$,

$$\inf_{0 \leqq f \leqq 1/T} S_\tau(f) \geqq \inf_{0 \leqq f \leqq 1/T} S(f) - \epsilon = m - \epsilon. \tag{46}$$

Letting $\tau \to \infty$ and $\epsilon \to 0$ in (46) yields (37).

APPENDIX B

Convergence of the Minimum Distance

In this appendix, we shall verify (21), i.e.,

$$d(h_\tau) \to d(h), \quad \text{as } \tau \to \infty. \tag{47}$$

From the definition of $d(h_\tau)$ (10), for arbitrary $\epsilon > 0$, we are assured of the existence of a $\mathbf{w} = \mathbf{u} - \mathbf{v}$ such that $\mathbf{u}, \mathbf{v} \in \mathcal{A}_N$, and

$$\|h_\tau * \mathbf{w}\| \leqq d(h_\tau) + \epsilon. \tag{48}$$

Repeating the steps in (34), we obtain

$$\|h_\tau * w\|^2 = \int_0^{1/T} S_\tau(f) \left| \sum_k w_k e^{i2\pi kTf} \right|^2 df \geqq 2[\inf_f S_\tau(f)] \sum_{k=0}^{N-1} |w_k|. \tag{49}$$

From (37) we can choose τ sufficiently large so that

$$\inf_f S_\tau(f) \geqq m. \tag{50}$$

Hence, for such a choice of τ,

$$\sum_{k=0}^{N-1} |w_k| \leqq \frac{[d(h_\tau) + \epsilon]^2}{2m}. \tag{51}$$

Now

$$d(h) \leqq \|h * \mathbf{w}\| = \|h_\tau * \mathbf{w} - \tilde{h}_\tau * \mathbf{w}\| \leqq \|h_\tau * \mathbf{w}\| + \|\tilde{h}_\tau * \mathbf{w}\|$$
$$\leqq d(h_\tau) + \epsilon + \|\tilde{h}_\tau * \mathbf{w}\|. \tag{52}$$

Since

$$(\tilde{h}_\tau * \mathbf{w})(t) = \sum_k h_\tau(t - kT) \cdot w_k,$$

we have, with τ large enough to satisfy (51),

$$\|\tilde{h}_\tau * \mathbf{w}\| \leqq \sum_k |w_k| \, \|\tilde{h}_\tau\| = \|\tilde{h}_\tau\| \sum_k |w_k| \leqq \frac{\|\tilde{h}_\tau\|}{2m} [d(h_\tau) + \epsilon]^2. \tag{53}$$

Combining (52) and (53) yields for τ sufficiently large (and $\epsilon > 0$ arbitrary)

$$d(h) \leqq d(h_\tau) + \epsilon + \frac{\|\tilde{h}_\tau\|}{2m} [d(h_\tau) + \epsilon]^2. \tag{54}$$

Letting $\tau \to \infty$ and $\epsilon \to 0$ yields

$$d(h) \leqq \liminf_{\tau \to \infty} d(h_\tau). \qquad (55)$$

The identical argument with h and h_τ reversed yields for all $\tau > 0$, $\epsilon > 0$,

$$d(h_\tau) \leqq d(h) + \epsilon + \frac{\|\tilde{h}_\tau\|}{2m} [d(h) + \epsilon]^2,$$

so that (letting $\tau \to \infty$, $\epsilon \to 0$)

$$\limsup_{\tau \to \infty} d(h_\tau) \leqq d(h). \qquad (56)$$

Inequalities (55) and (56) yield (47) or (21), completing the proof.

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