THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 54, No. 6, July–August 1975
Printed in U.S.A.

Coupled-Mode Theory for Anisotropic Optical Waveguides

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(Manuscript received December 11, 1974)

The well-known coupled-mode theory of waveguides is extended to include dielectric guides made of anisotropic materials. Exact coupled-wave equations for anisotropic dielectric waveguides are derived, and explicit expressions for the coupling coefficients are given. The coupling coefficients for isotropic waveguides are obtained as a special case. A simple approximation for the coupling coefficients in the case of slight anisotropy and slight departure from an ideal waveguide is presented.

I. INTRODUCTION

The theory of dielectric optical waveguides deals with electromagnetic wave propagation in optical fibers and in the waveguides used for integrated optics. Wave propagation in these structures is described in terms of normal modes.[1–3] However, normal modes preserve their identity only in perfect waveguides without irregularities of either the refractive index distributions or the waveguide geometry. Electromagnetic wave propagation in waveguides with any kind of irregularities must be described by means of coupled-mode theory.[3,4] The electromagnetic waves in imperfect waveguides are expressed as superpositions of all the modes of a perfect waveguide. The mode amplitudes are coupled together by coupling parameters that depend on the nature of the waveguide imperfections. A description of wave propagation by means of coupled-mode theory allows calculation of radiation losses caused by intentional or unintentional fluctuations of the refractive index along the axis of the waveguide or by core-cladding boundary fluctuations.[2,3] Coupling among guided modes is used to design modulators or distributed feedback circuits for lasers or to effect improvements in the multimode dispersion properties of overmoded waveguides. The coupled-mode theory is well developed for waveguides that consist of isotropic dielectric materials.[3,4] Some work has been done to extend this theory to waveguides consisting of anisotropic materials.[5–7] These waveguides are assuming increasing importance in integrated optics as methods are being perfected for fabricating waveguides by diffusing different dopants (or outdiffusion of certain component atoms) into anisotropic crystals.[8–10]

This paper describes the derivation of coupled-wave equations for the modes of waveguides consisting of anisotropic materials. The coupled-wave theory is based on the definition of guided and radiation modes as solutions of Maxwell's equations for idealized structures. An orthogonality relation is derived that is needed to isolate individual terms in the infinite series expansion of the electromagnetic field. The principal result of this theory is the derivation of coupling coefficients that are important for solving coupled-mode problems. Readers not interested in the derivation should look at eqs. (46) and (48). Applications of this theory are not presented here, since they will be the subject of further publications.

II. THE FIELD EQUATIONS FOR ANISOTROPIC MEDIA

The derivation of coupled-wave equations for anisotropic dielectric waveguides follows closely the procedure used for deriving coupled-wave equations for isotropic waveguides.[3] The objective of coupled-wave theory is to construct solutions of Maxwell's equations for waveguiding structures consisting of general refractive-index distributions.

Anisotropic media are characterized by a dielectric tensor,

$$\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}. \tag{1}$$

We assume that the elements of this tensor are real quantities characteristic of lossless materials. It can be shown that conservation of energy requires that the dielectric tensor form a symmetric matrix so that the following relations hold:[11]

$$\epsilon_{xy} = \epsilon_{yx}, \qquad \epsilon_{xz} = \epsilon_{zx}, \qquad \epsilon_{yz} = \epsilon_{zy}. \tag{2}$$

The magnetic properties of the medium are assumed to be the same as that of a vacuum so that we use the (isotropic) magnetic permeability constant μ_0. Maxwell's equations for anisotropic media assume the form

$$\nabla \times \mathbf{H} = i\omega\epsilon\cdot\mathbf{E} \tag{3}$$

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H}. \tag{4}$$

It was assumed that the electric field vector **E** and the magnetic field vector **H** have the time dependence,

$$e^{i\omega t}. \tag{5}$$

The tensor notation $\epsilon \cdot \mathbf{E}$ may be expressed in component form as

$$(\epsilon \cdot \mathbf{E})_i = \epsilon_{ij} E_j. \tag{6}$$

Summation over double indices is understood, and the subscripts i and j assume the values 1, 2, and 3 that represent the x, y, and z components of the vector $\mathbf{E}$ or tensor ϵ.

Derivation of coupled-wave equations for isotropic media is facilitated by expressing the longitudinal components of $\mathbf{E}$ and $\mathbf{H}$ in terms of the transverse components.[3] This practice is preserved for our derivation of coupled equations for anisotropic media. We single out the z coordinate as the direction of the waveguide axis and express the field vectors and the differential operator ∇ as superpositions of transverse and longitudinal parts. The symbol t indicates the transverse directions x and y. Thus, we have

$$\mathbf{E} = \mathbf{E}_t + \mathbf{E}_z, \tag{7}$$

$$\mathbf{H} = \mathbf{H}_t + \mathbf{H}_z, \tag{8}$$

and

$$\nabla = \nabla_t + \mathbf{e}_z \frac{\partial}{\partial z}. \tag{9}$$

We use the notations $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ to indicate unit vectors in x, y, and z directions.

The transverse part of the vector $\epsilon \cdot \mathbf{E}$ is indicated by the notation $\epsilon_t \cdot \mathbf{E}$ or, in component notation,

$$\epsilon_x \cdot \mathbf{E} = \mathbf{e}_x(\epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z) \tag{10}$$

$$\epsilon_y \cdot \mathbf{E} = \mathbf{e}_y(\epsilon_{yx} E_x + \epsilon_{yy} E_y + \epsilon_{yz} E_z). \tag{11}$$

The longitudinal part is

$$\epsilon_z \cdot \mathbf{E} = \mathbf{e}_z(\epsilon_{zx} E_x + \epsilon_{zy} E_y + \epsilon_{zz} E_z). \tag{12}$$

We may now separate Maxwell's equations into transverse and longitudinal parts. The transverse parts of (3) and (4) are

$$\nabla_t \times \mathbf{H}_z + \mathbf{e}_z \times \frac{\partial \mathbf{H}_t}{\partial z} = i\omega \epsilon_t \cdot \mathbf{E} \tag{13}$$

$$\nabla_t \times \mathbf{E}_z + \mathbf{e}_z \times \frac{\partial \mathbf{E}_t}{\partial z} = -\, i\omega\mu_0 \mathbf{H}_t. \tag{14}$$

Their longitudinal parts may be written as

$$\nabla_t \times \mathbf{H}_t = i\omega(\epsilon_z \cdot \mathbf{E}_t + \epsilon_{zz} \mathbf{E}_z) \tag{15}$$

and

$$\nabla_t \times \mathbf{E}_t = -\, i\omega\mu_0 \mathbf{H}_z. \tag{16}$$

The longitudinal parts of **E** and **H** follow immediately from (15) and (16),

$$\mathbf{E}_z = \frac{1}{i\omega\epsilon_{zz}} \nabla_t \times \mathbf{H}_t - \frac{1}{\epsilon_{zz}} \epsilon_z \cdot \mathbf{E}_t \tag{17}$$

and

$$\mathbf{H}_z = -\frac{1}{i\omega\mu_0} \nabla_t \times \mathbf{E}_t. \tag{18}$$

On the right-hand side of (17) and (18) appear only transverse components of **E** and **H**. It is important to distinguish between the single and double subscript notation of ϵ. A double subscript, like ϵ_{zz}, indicates a single tensor element of ϵ, while a single subscript, like ϵ_z, is defined by (10) through (12). In particular, we have

$$\epsilon_z \cdot \mathbf{E}_t = \mathbf{e}_z(\epsilon_{zx}E_x + \epsilon_{zy}E_y). \tag{19}$$

We now use (17) and (18) to eliminate the z components of **E** and **H** from the transverse parts of Maxwell's equations (13) and (14),

$$-\frac{1}{i\omega\mu_0} \nabla_t \times (\nabla_t \times \mathbf{E}_t) + \mathbf{e}_z \times \frac{\partial \mathbf{H}_t}{\partial z} = i\omega\epsilon_t \cdot \mathbf{E}_t - \frac{i\omega}{\epsilon_{zz}} \epsilon_t \cdot \epsilon_z \cdot \mathbf{E}_t + \frac{1}{\epsilon_{zz}} \epsilon_t \cdot (\nabla_t \times \mathbf{H}_t) \tag{20}$$

and

$$\nabla_t \times \left[\frac{1}{i\omega\epsilon_{zz}} \nabla_t \times \mathbf{H}_t - \frac{1}{\epsilon_{zz}} \epsilon_z \cdot \mathbf{E}_t\right] + \mathbf{e}_z \times \frac{\partial \mathbf{E}_t}{\partial z} = -i\omega\mu_0 \mathbf{H}_t. \tag{21}$$

These two vector equations represent four scalar equations. Once eqs. (20) and (21) are solved, the z components of **E** and **H** can be obtained by simple differentiation from (17) and (18). We have thus achieved a simplification of the original problem by reducing the number of equations from six, in (3) and (4), to only four.

The components of the ϵ tensor are assumed to be functions of x, y, and z. The ϵ tensor defines the wave-guiding structure. Because of the z dependence of ϵ, eqs. (20) and (21) do not have mode solutions. A normal mode is defined as a solution of Maxwell's equations whose z dependence can be expressed by the simple function

$$e^{-i\beta z}. \tag{22}$$

Such solutions exist only if the dielectric tensor does not depend on the z coordinate. To construct solutions of the general eqs. (20) and (21), we consider solutions of simpler equations that are defined by a tensor $\bar{\epsilon}$ that is similar to ϵ but is independent of z. The choice of $\bar{\epsilon}$ is obviously arbitrary and is determined by convenience. Using (22), we find from (20) and (21) the following equations for the normal modes of the waveguide structure defined by $\bar{\epsilon}$:

$$-\frac{1}{i\omega\mu_0}\nabla_t \times (\nabla_t \times \mathcal{E}_{\nu t}^{(p)}) - i\beta_\nu^{(p)}\mathbf{e}_z \times \mathcal{H}_{\nu t}^{(p)} = i\omega\bar{\epsilon}_t\cdot\mathcal{E}_{\nu t}^{(p)} - \frac{i\omega}{\bar{\epsilon}_{zz}}\bar{\epsilon}_t\cdot\bar{\epsilon}_z\cdot\mathcal{E}_{\nu t}^{(p)} + \frac{1}{\bar{\epsilon}_{zz}}\bar{\epsilon}_t\cdot(\nabla_t \times \mathcal{H}_{\nu t}^{(p)}) \quad (23)$$

and

$$\nabla_t \times \left[\frac{1}{i\omega\bar{\epsilon}_{zz}}\nabla_t \times \mathcal{H}_{\nu t}^{(p)} - \frac{1}{\bar{\epsilon}_{zz}}\bar{\epsilon}_z\cdot\mathcal{E}_{\nu t}^{(p)}\right] - i\beta_\nu^{(p)}\mathbf{e}_z \times \mathcal{E}_{\nu t}^{(p)} = -i\omega\mu_0\mathcal{H}_{\nu t}^{(p)}. \quad (24)$$

The subscript ν indicates a mode label. Equations (23) and (24) admit an infinite number of solutions with different eigenvalues (propagation constants) $\beta_\nu^{(p)}$ and different field vectors $\mathcal{E}_\nu^{(p)}$ and $\mathcal{H}_\nu^{(p)}$. Script letters indicate mode fields, while roman letters **E** and **H** are reserved for general field distributions. The modes are of two different types, guided modes whose fields are confined to the vicinity of the waveguide and radiation modes that extend to infinity in transverse direction to the guide.[2,3] Guided modes have discrete eigenvalues $\beta_\nu^{(p)}$, while the eigenvalues of radiation modes form a continuum. The superscript (p) stands for either $(+)$ or $(-)$, depending on the direction of wave propagation. A wave traveling in the positive z direction has positive (real) values $\beta_\nu^{(+)}$, a wave traveling in the negative z direction has a negative (real) value $\beta_\nu^{(-)}$. In isotropic media, we have the simple relations,

$$\beta_\nu^{(-)} = -\beta_\nu^{(+)}, \quad (25)$$

$$\mathcal{E}_{\nu t}^{(-)} = \mathcal{E}_{\nu t}^{(+)}, \qquad \mathcal{E}_{\nu z}^{(-)} = -\mathcal{E}_{\nu z}^{(+)}, \quad (26)$$

and

$$\mathcal{H}_{\nu t}^{(-)} = -\mathcal{H}_{\nu t}^{(+)}, \qquad \mathcal{H}_{\nu z}^{(-)} = \mathcal{H}_{\nu z}^{(+)}. \quad (27)$$

General anisotropic media are more complicated, so that (26) to (27) do not apply. Modes traveling in one direction may be different from modes traveling in the opposite direction.

III. ORTHOGONALITY RELATIONS

The modes of anisotropic dielectric waveguides are mutually orthogonal.[2] For the purpose of deriving orthogonality relations, it is simpler to use Maxwell's equations in the form (3) and (4) instead of the form (23) and (24). Separating the z derivatives from the ∇ operator, we write (3) for a mode labeled ν and (4) for a mode labeled μ,

$$\nabla_t \times \mathcal{H}_\nu^{(p)} - i\beta_\nu^{(p)}\mathbf{e}_z \times \mathcal{H}_\nu^{(p)} = i\omega\bar{\epsilon}\cdot\mathcal{E}_\nu^{(p)} \quad (28)$$

and

$$\nabla_t \times \mathcal{E}_\mu^{(q)} - i\beta_\mu^{(q)}\mathbf{e}_z \times \mathcal{E}_\mu^{(q)} = -i\omega\mu_0\mathcal{H}_\mu^{(q)}. \quad (29)$$

Next, we take the complex conjugate of (28), multiply the resulting equation by $\mathcal{E}_\mu^{(q)}$, multiply (29) by $-\mathcal{H}_\nu^{(p)*}$, add the two equations, and integrate over the infinite cross section ($\bar{\epsilon}$ is assumed real):

$$\iint \{\mathcal{E}_\mu^{(q)}\cdot\nabla_t\times\mathcal{H}_\nu^{(p)*} - \mathcal{H}_\nu^{(p)*}\cdot\nabla_t\times\mathcal{E}_\mu^{(q)} + i\beta_\nu^{(p)*}\mathcal{E}_\mu^{(q)}\cdot\mathbf{e}_z\times\mathcal{H}_\nu^{(p)*} + i\beta_\mu^{(q)}\mathcal{H}_\nu^{(p)*}\cdot\mathbf{e}_z\times\mathcal{E}_\mu^{(q)}\}dxdy = -i\omega\iint[\mathcal{E}_\mu^{(q)}\cdot\bar{\epsilon}\cdot\mathcal{E}_\nu^{(p)*} - \mu_0\mathcal{H}_\nu^{(p)*}\cdot\mathcal{H}_\mu^{(q)}]dxdy. \quad (30)$$

The first two terms on the left-hand side of (30) can be expressed as

$$-\iint\nabla_t\cdot(\mathcal{E}_\mu^{(q)}\times\mathcal{H}_\nu^{(p)*})dxdy = -\int(\mathcal{E}_\mu^{(q)}\times\mathcal{H}_\nu^{(p)*})\cdot\mathbf{n}ds. \quad (31)$$

The two-dimensional divergence theorem was used to convert the integral over the infinite cross section in the x-y plane to an integral over the infinite circle with outward normal direction $\mathbf{n}$ and line element ds. The integral on the right-hand side vanishes if at least one of the two modes is a guided mode. If both modes ν and μ are radiation modes, the integral vanishes in the sense of a delta function of nonzero argument.[2] Using this fact and a well-known vector identity, we can express (30) as

$$(\beta_\mu^{(q)} - \beta_\nu^{(p)*})\iint\mathbf{e}_z\cdot(\mathcal{E}_\mu^{(q)}\times\mathcal{H}_\nu^{(p)*})dxdy = -\omega\iint[\mathcal{E}_\mu^{(q)}\cdot\bar{\epsilon}\cdot\mathcal{E}_\nu^{(p)*} - \mu_0\mathcal{H}_\mu^{(q)}\cdot\mathcal{H}_\nu^{(p)*}]dxdy. \quad (32)$$

Because of the symmetry of the $\bar{\epsilon}$ tensor, the following relation holds:

$$\mathcal{E}_\nu^{(p)*}\cdot\bar{\epsilon}\cdot\mathcal{E}_\mu^{(q)} = \mathcal{E}_\mu^{(q)}\cdot\bar{\epsilon}\cdot\mathcal{E}_\nu^{(p)*}. \quad (33)$$

We take the complex conjugate of (32), interchange the superscripts p and q as well as the subscripts ν and μ and, using (33), subtract the new expression from (32) with the result:

$$(\beta_\mu^{(q)} - \beta_\nu^{(p)*})\iint\mathbf{e}_z\cdot[\mathcal{E}_\mu^{(q)}\times\mathcal{H}_\nu^{(p)*} + \mathcal{E}_\nu^{(p)*}\times\mathcal{H}_\mu^{(q)}]dxdy = 0. \quad (34)$$

Equation (34) is the desired orthogonality relation. It is obvious that this expression holds also for isotropic media. However, in the isotropic case it is possible to use (25) through (27) to prove that each term in (34) must vanish separately.[2] For the general anisotropic case, (34) cannot be simplified further. We infer from (34) that the integral vanishes if $\beta_\mu^{(q)} - \beta_\nu^{(p)*} \neq 0$. This means that the integral vanishes even in the case $\nu = \mu$ if p and q indicate opposite signs, and a wave is orthogonal to its backward traveling counterpart (if $\beta_\nu^{(q)}$ is real) if orthogonality means vanishing of the integral in (34).

The integral in (34) expresses the total power flow if $\nu = \mu$ and $p = q$. We may therefore use the orthonormality relation,

$$\int\int \mathbf{e}_z \cdot [\mathcal{E}_{\mu t}^{(q)} \times \mathcal{H}_{\nu t}^{(p)*} + \mathcal{E}_{\nu t}^{(p)*} \times \mathcal{H}_{\mu t}^{(q)}]dxdy = 2s_\nu^{(q)} \frac{\beta_\nu^{(q)} + \beta_\nu^{(p)*}}{|\beta_\nu^{(q)}|} P\delta_{\nu\mu}, \quad (35)$$

to express mode orthogonality and normalization. The subscripts t indicating the transverse parts of the modes were added since the z components of the fields do not contribute to (35). P is a normalizing factor common to all modes that is used to adjust the arbitrary amplitudes of the normal modes. For real values of $\beta_\nu^{(q)}$, we have

$$s_\mu^{(q)} = 1. \quad (36)$$

In this case, the sign of the integral is expressed correctly by the fact that $\beta_\nu^{(q)}$ reverses its sign if q goes from $(+)$ to $(-)$. For opposite signs of p and q, the right-hand side of (35) vanishes as required by (34) if $\beta_\nu^{(q)}$ is real. For imaginary $\beta_\nu^{(q)}$, (35) vanishes for $q = p$. The orthogonality relation also holds for imaginary values of $\beta_\nu^{(q)}$. Imaginary values of the propagation constants occurs only for evanescent "radiation" modes.[2,3] In the case of imaginary $\beta_\nu^{(q)}$, the sign of the right-hand side of (35) is not certain. For this reason, we have introduced the factor $s_\mu^{(q)}$ that must be adjusted so that P is a positive real quantity. This means that $s_\mu^{(q)}$ may have to be negative, $s_\mu^{(q)} = -1$. However, this case can arise only in connection with evanescent "radiation" modes. The $\delta_{\nu\mu}$ symbol in (35) indicates Kronecker's delta if both modes are guided. When one mode is guided while the other is a radiation mode, we have $\delta_{\nu\mu} = 0$. If both modes are radiation modes, $\delta_{\nu\mu}$ must be interpreted as the Dirac delta function.

IV. DERIVATION OF COUPLED-WAVE EQUATIONS

Any arbitrary field distribution compatible with Maxwell's equations can be expressed as the superposition of all the modes of the idealized structure defined by the dielectric tensor $\bar{\epsilon}$. Because the complete set of modes consists of a finite number of guided modes plus a continuum of radiation modes, we express the transverse parts of a general field by the expansion

$$\mathbf{E}_t = \sum_{\nu,p} a_\nu^{(p)} \mathcal{E}_{\nu t}^{(p)} + \sum_p \int_0^\infty a^{(p)}(\rho)\mathcal{E}_t^{(p)}(\rho)d\rho \quad (37)$$

and

$$\mathbf{H}_t = \sum_{\nu,p} a_\nu^{(p)} \mathcal{H}_{\nu t}^{(p)} + \sum_p \int_0^\infty a^{(p)}(\rho)\mathcal{H}_t^{(p)}(\rho)d\rho. \quad (38)$$

The longitudinal parts follow from (17) and (18). The superscripts assume the values (+) and (−) indicating waves traveling in positive and negative z direction. The first terms in (37) and (38) represent the contribution of the finite number of guided modes labeled ν. The second terms indicated combinations of sums and integrals. The integration ranges over the entire region of continuous-mode labels ρ and includes radiation modes with real as well as imaginary values of $\beta^{(p)}(\rho)$. The summation symbol in front of the integral sign indicates that, in addition to modes traveling in positive and negative z direction, various types of radiation modes exist and must be added to obtain the complete set of modes. For the purpose of deriving coupled-wave equations, the notation of (37) and (38) is too cumbersome. We use an abbreviated notation by omitting the integration sign, leaving it understood that the summation symbol includes summation over guided modes and summation as well as integration over radiation modes. We thus write

$$\mathbf{E}_t = \sum_{\nu,p} a_\nu^{(p)} \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} \tag{39}$$

and

$$\mathbf{H}_t = \sum_{\nu,p} a_\nu^{(p)} \boldsymbol{\mathcal{H}}_{\nu t}^{(p)}. \tag{40}$$

$\boldsymbol{\mathcal{E}}_{\nu t}^{(p)}$ and $\boldsymbol{\mathcal{H}}_{\nu t}^{(p)}$ are independent of z, but $a_\nu^{(p)}$ is a function of z. Substitution of (39) and (40) into (20) and (21) and use of the mode eqs. (23) and (24) leads to

$$\begin{aligned}\sum_{\nu,p} \left(\frac{da_\nu^{(p)}}{dz} + i\beta_\nu^{(p)} a_\nu^{(p)} \right) (\mathbf{e}_z \times \boldsymbol{\mathcal{H}}_{\nu t}^{(p)}) \\ = \sum_{\nu,p} a_\nu^{(p)} \Bigg\{ i\omega(\boldsymbol{\epsilon}_t - \bar{\boldsymbol{\epsilon}}_t) \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} - i\omega \left(\frac{1}{\epsilon_{zz}} \boldsymbol{\epsilon}_t \cdot \boldsymbol{\epsilon}_z - \frac{1}{\bar{\epsilon}_{zz}} \bar{\boldsymbol{\epsilon}}_t \cdot \bar{\boldsymbol{\epsilon}}_z \right) \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} \\ + \left(\frac{1}{\epsilon_{zz}} \boldsymbol{\epsilon}_t - \frac{1}{\bar{\epsilon}_{zz}} \bar{\boldsymbol{\epsilon}}_t \right) \cdot (\nabla_t \times \boldsymbol{\mathcal{H}}_{\nu t}^{(p)}) \Bigg\}\end{aligned} \tag{41}$$

and

$$\begin{aligned}\sum_{\nu,p} \left(\frac{da_\nu^{(p)}}{dz} + i\beta_\nu^{(p)} a_\nu^{(p)} \right) (\mathbf{e}_z \times \boldsymbol{\mathcal{E}}_{\nu t}^{(p)}) \\ = -\sum_{\nu,p} a_\nu^{(p)} \Bigg\{ \nabla_t \times \Bigg[\frac{1}{i\omega} \left(\frac{1}{\epsilon_{zz}} - \frac{1}{\bar{\epsilon}_{zz}} \right) \nabla_t \times \boldsymbol{\mathcal{H}}_{\nu t}^{(p)} \\ - \left(\frac{1}{\epsilon_{zz}} \boldsymbol{\epsilon}_z - \frac{1}{\bar{\epsilon}_{zz}} \bar{\boldsymbol{\epsilon}}_z \right) \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} \Bigg] \Bigg\}.\end{aligned} \tag{42}$$

We take the scalar product of (41) with $-\boldsymbol{\mathcal{E}}_{\mu t}^{(q)*}$ and of (42) with $\boldsymbol{\mathcal{H}}_{\mu t}^{(q)*}$, then we add the two equations, integrate over the infinite cross section, and use the orthogonality relation (35). The result of this procedure is

$$2s_\mu^{(r)} \frac{\beta_\mu^{(q)} + \beta_\mu^{(r)*}}{|\beta_\mu^{(r)}|} P \left(\frac{da_\mu^{(r)}}{dz} + i\beta_\mu^{(r)} a_\mu^{(r)} \right)$$

$$= \sum_{\nu,p} a_\nu^{(p)} \iint \Bigg\{ -i\omega\, \boldsymbol{\mathcal{E}}_{\mu t}^{(q)*} \cdot (\epsilon_t - \bar{\epsilon}_t) \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)}$$

$$+ i\omega\, \boldsymbol{\mathcal{E}}_{\mu t}^{(q)*} \cdot \left(\frac{\epsilon_t \cdot \epsilon_z}{\epsilon_{zz}} - \frac{\bar{\epsilon}_t \cdot \bar{\epsilon}_z}{\bar{\epsilon}_{zz}} \right) \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} - \boldsymbol{\mathcal{E}}_{\mu t}^{(q)*} \cdot \left(\frac{\epsilon_t}{\epsilon_{zz}} - \frac{\bar{\epsilon}_t}{\bar{\epsilon}_{zz}} \right) \cdot (\nabla_t \times \boldsymbol{\mathcal{H}}_{\nu t}^{(p)})$$

$$- \boldsymbol{\mathcal{H}}_{\mu t}^{(q)*} \cdot \nabla_t \times \Bigg[\frac{1}{i\omega} \left(\frac{1}{\epsilon_{zz}} - \frac{1}{\bar{\epsilon}_{zz}} \right) \nabla_t \times \boldsymbol{\mathcal{H}}_{\nu t}^{(p)}$$

$$- \left(\frac{\epsilon_z}{\epsilon_{zz}} - \frac{\bar{\epsilon}_z}{\bar{\epsilon}_{zz}} \right) \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} \Bigg] \Bigg\} dx dy. \quad (43)$$

On the left-hand side of (43), we have used a superscript (r). This notation is necessary to distinguish between the case of real and imaginary propagation constants $\beta_\mu^{(q)}$. If $\beta_\mu^{(q)}$ is real, we have $r = q$. If $\beta_\mu^{(q)}$ is imaginary, as it is for evanescent radiation modes, we must choose for r the sign opposite to q. We now write (43) in the abbreviated form

$$\frac{da_\mu^{(r)}}{dz} = - i\beta_\mu^{(r)} a_\mu^{(r)} + \sum_{\nu,p} K_{\mu\nu}^{(r,p)} a_\nu^{(p)}. \quad (44)$$

The coupling coefficient is defined by (43). We may eliminate the transverse magnetic-mode field vector from the coupling coefficient by using (17) (applied to the mode field) and the identity (which is obtained by partial integration),

$$\iint \boldsymbol{\mathcal{H}}_{\mu t}^{(q)*} \cdot (\nabla_t \times \mathbf{F}) dx dy = \iint (\nabla_t \times \boldsymbol{\mathcal{H}}_{\mu t}^{(q)*}) \cdot \mathbf{F} dx dy. \quad (45)$$

The coupling coefficient can be expressed as

$$K_{\mu\nu}^{(r,p)} = \frac{i\omega |\beta_\mu^{(r)}|}{4s_\mu^{(r)} \beta_\mu^{(r)*} P} \iint \Bigg\{ \boldsymbol{\mathcal{E}}_{\mu t}^{(q)*} \cdot \left[\left(\frac{\epsilon_t \cdot \epsilon_z}{\epsilon_{zz}} - \frac{\bar{\epsilon}_t \cdot \bar{\epsilon}_z}{\bar{\epsilon}_{zz}} \right) - (\epsilon_t - \bar{\epsilon}_t) \right] \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)}$$

$$- \boldsymbol{\mathcal{E}}_{\mu t}^{(q)*} \cdot \left(\frac{\bar{\epsilon}_{zz}}{\epsilon_{zz}} \epsilon_t - \bar{\epsilon}_t \right) \cdot \left(\frac{\bar{\epsilon}_z}{\bar{\epsilon}_{zz}} \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} + \boldsymbol{\mathcal{E}}_{\nu z}^{(p)} \right)$$

$$+ (\bar{\epsilon}_z \cdot \boldsymbol{\mathcal{E}}_{\mu t}^{(q)*} + \bar{\epsilon}_{zz} \boldsymbol{\mathcal{E}}_{\mu z}^{(q)*}) \cdot \Bigg[\left(\frac{\bar{\epsilon}_{zz}}{\epsilon_{zz}} - 1 \right) \left(\frac{\bar{\epsilon}_z}{\bar{\epsilon}_{zz}} \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} + \boldsymbol{\mathcal{E}}_{\nu z}^{(p)} \right)$$

$$- \left(\frac{\epsilon_z}{\epsilon_{zz}} - \frac{\bar{\epsilon}_z}{\bar{\epsilon}_{zz}} \right) \cdot \boldsymbol{\mathcal{E}}_{\nu t}^{(p)} \Bigg] \Bigg\} dx dy. \quad (46)$$

V. IMPORTANT SPECIAL CASES

In its complete form, (46), the coupling coefficient is very complicated. For isotropic media, where the ϵ tensor degenerates to a multiple of the unit tensor, (46) simplifies to the exact form,

$$K_{\mu\nu}^{(r,p)} = \frac{\omega|\beta_\mu^{(r)}|}{4is_\mu^{(r)}\beta_\mu^{(r)*}P}\int\int(\epsilon - \bar{\epsilon})\left[\boldsymbol{\mathcal{E}}_{\mu t}^{(q)*}\cdot\boldsymbol{\mathcal{E}}_{\nu t}^{(p)} + \frac{\bar{\epsilon}_{zz}}{\epsilon_{zz}}\boldsymbol{\mathcal{E}}_{\mu z}^{(q)*}\cdot\boldsymbol{\mathcal{E}}_{\nu z}^{(p)}\right]dxdy, \quad (47)$$

in complete agreement with the well-known result.[12]

In many practical applications, the anisotropy of the dielectric medium is only slight, and the difference between the actual dielectric tensor ϵ and the ideal tensor $\bar{\epsilon}$ is small. In that case, (46) can be substantially simplified. A reasonable approximation of (46) for slight anisotropy and small values of $\epsilon - \bar{\epsilon}$ is

$$K_{\mu\nu}^{(r,p)} = \frac{\omega|\beta_\mu^{(r)}|}{4is_\mu^{(r)}\beta_\mu^{(r)*}P}\int\int\boldsymbol{\mathcal{E}}_\mu^{(q)*}\cdot(\epsilon - \bar{\epsilon})\cdot\boldsymbol{\mathcal{E}}_\nu^{(p)}dxdy. \quad (48)$$

Note that the whole vectors of the electric mode fields enter (48) and not just the transverse or longitudinal parts. The approximation (48) is obtained by considering off-diagonal elements and differences between diagonal elements of ϵ and $\bar{\epsilon}$ as quantities that are small of first order. Products of two first-order quantities have been neglected. For readers who did not follow the detailed derivation, we repeat here briefly the definitions of symbols appearing in (46) through (48). The symbol ω is the angular frequency of the electromagnetic field. The script symbols $\boldsymbol{\mathcal{E}}_\nu^{(p)}$ indicate the electric-field vectors of normal modes of an idealized waveguide that is defined by the dielectric tensor $\bar{\epsilon} = \bar{\epsilon}(x, y)$, the subscript ν is a mode label, and the superscript (p) stands for either $(+)$ or $(-)$, indicating the direction of wave propagation. The propagation constants $\beta_\mu^{(r)}$ of the modes are labeled in the same way as the field vectors. The superscript r is usually identical with the superscript q. Only in the case of imaginary $\beta_\mu^{(r)}$ (this case happens for coupling to a nonpropagating radiation mode and is of little practical interest) does r indicate the sign opposite to q. Likewise, $s_\mu^{(r)} = 1$ for most cases of interest. Only for imaginary values of $\beta_\mu^{(r)}$ may it become necessary to choose $s_\mu^{(r)} = -1$ to keep the power normalization coefficient P positive in (35). The asterisk indicates complex conjugation. Subscripts t and z occurring in (46) and (47) refer to the transverse and longitudinal parts of the vectors to which they are attached. Similar subscripts attached to ϵ are defined by (10) through (12) and (19). The dielectric tensor $\epsilon = \epsilon(x, y, z)$ defines the actual waveguide (in contrast to the ideal guide that is only a mathematical fiction). The integrals are extended over the infinite transverse cross section of the guide. Equation (48) assumes the same limit as (47) if the dielectric tensor degenerates into a multiple of the unit tensor since, in the spirit of the approximation (48), we must use $\bar{\epsilon}_{zz}/\epsilon_{zz} = 1$. Equation (48), even though it is only an approximation, is likely to be of most importance in practical applications because of its simple form. For many practical problems, the approximation is justified and leads to sufficiently accurate results.

VI. CONCLUSIONS

We have derived coupled-wave equations representing exact solutions of the electromagnetic field problem for dielectric waveguides that consist of anisotropic materials whose dielectric tensor is a function of the z coordinate. The field of the general waveguide is expressed in terms of ideal modes of a hypothetical dielectric waveguide defined by a dielectric tensor whose elements are independent of the z coordinate. The main result of this paper is the expression (46) for the coupling coefficients. For many practical applications, the exact coupling coefficient can be approximated in the simple form (48).

The coupled-mode theory for anisotropic dielectric waveguides is essential for the solution of problems of mode propagation in integrated-optics guides with random or systematic irregularities. A particularly important area of applications are guides that are made anisotropic by an externally applied dc voltage or whose anisotropy is changed by such a voltage. Instead of an applied voltage, an acoustical wave may cause an anisotropic change of the refractive index of a dielectric waveguide. These cases cannot be handled by the simpler isotropic coupled-mode theory, but require the extension to anisotropic media presented here.

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