THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 54, No. 6, July–August 1975
Printed in U.S.A.

Power Spectrum of a Digital, Frequency-Modulation Signal

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(Manuscript received December 31, 1974)

We present the power spectrum of a sinusoidal carrier, frequency modulated by a random baseband pulse train in which the signaling-pulse duration is finite and the signal pulses may overlap and have different shapes. Symbols transmitted during different time slots are assumed to be statistically independent and identically distributed. The spectral density appears as a Hermitian form suitable for numerical computation by a digital computer. Simple conditions in terms of the modulation parameters are given under which discrete spectral lines are present in the spectrum. Several examples are given to illustrate the method.

I. INTRODUCTION

In recent years, digital-frequency and phase-modulation techniques have been increasingly important in radio, waveguide, and optical communication systems.

An important parameter in the statistical description of a signal is its spectral density, which defines the average power density of the signal as a function of frequency. In addition to furnishing an estimate of bandwidth requirements, the knowledge of the spectral density is also essential in the evaluation of mutual interference between channels.

In this paper, we extend the techniques developed in Ref. 1 for digital PSK to the case of digital FSK with phase-continuous transitions, such as may be obtained at the output of a voltage-controlled oscillator driven by a digital baseband wave. We assume that the sinusoidal carrier is frequency modulated by a random, baseband pulse train in which the signaling pulse duration is finite and the signal pulses may overlap and have different shapes. It is generally assumed that the symbols transmitted during different time slots are statistically independent and identically distributed.

We express the spectral density of such an ensemble of continuous-phase, constant-envelope, digital, FM waves as a compact Hermitian form that provides an appropriate division between analysis and machine computation. The present work permits simpler numerical computation of digital FSK spectra than earlier studies,[2–5] and contains a simpler statement of the conditions, in terms of the modulation parameters, that determine whether discrete spectral lines are present in the spectrum or not.

Examples give the spectra of binary and quaternary FSK waves with overlapping baseband modulation pulses of several shapes.

II. M-ARY FREQUENCY-MODULATED SIGNALS

We seek the spectrum of the digital frequency-modulated wave:

$$x(t) = \cos\left[2\pi f_c t + \phi(t)\right], \quad f_c > 0. \tag{1}$$

$$\phi(t) = \int^{t} f_d(\mu)d\mu. \tag{2}$$

$$f_d(t) = \sum_{k=-\infty}^{\infty} h_{s_k}(t - kT), \qquad s_k = 1, 2, \cdots, M. \tag{3}$$

The symbol ϕ is the phase and f_d the frequency deviation of the carrier at frequency f_c. The signaling alphabet consists of M waveforms h_1, $h_2, \cdots, h_M$, that may have different shapes; one of these is transmitted for each signaling interval of duration T. The different signaling waveforms in (3) may overlap, but are statistically independent in most of the present work; i.e., s_k is statistically independent of s_l for $k \neq l$.

Define for convenience

$$v(t) \equiv e^{j\phi(t)}; \tag{4}$$

then

$$x(t) = \text{Re}\left\{e^{j2\pi f_c t} v(t)\right\}. \tag{5}$$

The spectral density of $v(t)$ is

$$P_v(f) = \int_{-\infty}^{\infty} \Phi_v(\tau) e^{-j2\pi f\tau} d\tau, \tag{6}$$

where

$$\Phi_v(\tau) = \overline{\Phi_v(t, \tau)} \equiv \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} \Phi_v(t, \tau)dt, \tag{7}^{\S}$$

$$\Phi_v(t, \tau) = \langle v(t + \tau)v^*(t)\rangle = \langle e^{j[\phi(t+\tau)-\phi(t)]}\rangle$$
$$= \left\langle \exp\left(j\int_{t}^{t+\tau} f_d(\mu)d\mu\right)\right\rangle. \tag{8}$$

§ The symbol ―― denotes average on t throughout.

An arbitrary constant of integration is implicit in the notation of (2), which defines the phase. This is of little consequence in the remainder of the present paper, since most of our study is directed toward the spectrum of the complex wave $v(t)$; as seen by the final line of (8), the absolute phase is irrelevant in determining $P_v(f)$. The absolute phase must be rendered explicit only for the following three purposes in the present work:

(*i*) Relating $P_x(f)$ to $P_v(f)$ (the spectra of the real FSK wave $x(t)$ of (1) and of the complex FSK wave of (4), respectively).
(*ii*) Separating possible line-frequency components from the FSK wave.
(*iii*) Specializing the present FSK treatment to the prior PSK results.[1]

To make the absolute phase explicit, we write

$$\phi(t) = \int_0^t f_d(\mu)d\mu + \phi(0); \quad v(0) = e^{j\phi(0)}. \tag{9}$$

The term $\phi(0)$ is the phase of the FSK wave at $t = 0$. We consider three representative assumptions for $\phi(0)$:

(*i*) $\phi(0)$ deterministic, e.g.:

$$\phi(0) = 0. \tag{10}$$

(*ii*) $\phi(0)$ random, uniform, and independent of the modulation s_k:

$$\Pr[\phi < \phi(0) \leqq \phi + d\phi] = \frac{d\phi}{2\pi}, \quad 0 \leqq \phi < 2\pi. \tag{11}$$

(*iii*) $\phi(0)$ dependent only on the modulation parameters (or signaling pulses) that contribute to $f_d(0+)$§ in (3):

$$\phi(0) = \sum_{k:h_{s_k}(-kT+)\neq 0} \int_{-\infty}^{-kT} h_{s_k}(\mu)d\mu. \tag{12}§$$

The signal-pulse duration is assumed finite; consequently, in (12), the $\sum$ has a finite number of terms, and the lower limit on the $\int$ becomes finite. Specific examples of (12) appear in Section IV below. In other words, the net phase contributed by all past signaling pulses that are over by $t = 0$, i.e., for which $h_{s_k}(-kT+) = 0$,§ is normalized to 0.

§ The +'s indicate that f_d and h_{s_k} are to be evaluated an infinitesimal time later than 0 and $-kT$, respectively, if any of the signal pulses have discontinuities at their upper limits at a time-slot boundary (see Section IV). If the signal pulses and, hence, the instantaneous frequency deviation $f_d(t)$ are continuous, these two +'s may be dropped.

In the appendix we show that, except for special modulations with low carrier frequencies f_c that take on special values, the following simple relation gives the spectrum of the real wave $x(t)$ of (1):

$$P_x(f) = \tfrac{1}{4} P_v(f - f_c) + \tfrac{1}{4} P_v(-f - f_c). \tag{13}$$ §

The first term of (13) is the spectrum of the complex baseband wave $v(t)$ shifted to the carrier frequency $+f_c$; the second term is the spectrum of $v^*(t)$ shifted to $-f_c$. More specifically, (13) is valid if any of the following are true:

(*i*) Eq. (11) holds, independently of any other considerations.
(*ii*) Eq. (10) or (12) holds, and $P_v(f)$ [and hence $P_x(f)$] has no line components.
(*iii*) Eq. (10) or (12) holds, $P_v(f)$ [and hence $P_x(f)$] has (equally spaced) line components, and f_c is low enough so that the two terms of (13) overlap, but f_c is such that the line components of the two terms of (13) do not coincide. The discrete values forbidden to f_c under these conditions are given by (146) in the appendix.
(*iv*) The carrier frequency is so large that the two terms of (13) do not significantly overlap, independently of any other considerations.

Consequently, we study only $P_v(f)$ throughout the remainder of this paper. This suffices for all cases except that of a low carrier frequency f_c that takes on special discrete values related to the baud rate $1/T$ and the modulation, for very special modulations that result in line-spectral components (PSK is one such case, but there are others).

The condition expressed by (12) is assumed in much of what follows. This results in no significant loss in generality in treating the spectrum of the real wave $x(t)$, as discussed above. It permits economy of notation in the general case, a convenient treatment of line components present in special cases, and simple specialization of the present FSK results to prior PSK results.[1]

III. NOTATION AND STATISTICAL MODEL

We introduce the vector-matrix notation of the prior PSK study.[1] Since that study allowed correlated modulation parameters s_k while the present work is largely restricted to independent s_k, only a portion of Section III of Ref. 1 need be summarized here.

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§ Compare (5) of Ref. 1.

We write (3) as

$$f_d(t) = \sum_{k=-\infty}^{\infty} \underline{\mathbf{a}_k} \cdot \mathbf{h}(t - kT)], \tag{14}$$ [§]

where

$$\underline{\mathbf{a}_k} = \mathbf{a}_k]' \equiv [a_k^{(1)} \; a_k^{(2)} \; \cdots \; a_k^{(M)}], \tag{15}$$

$$\underline{\mathbf{h}(t)} = \mathbf{h}(t)]' \equiv [h_1(t) \; h_2(t) \; \cdots \; h_M(t)]. \tag{16}$$

For a given k (i.e., for a given time slot) one of the a_k's is unity and the rest are zero:

$$a_k^{(s_k)} = 1; \quad a_k^{(i)} = 0, \quad i \neq s_k. \tag{17}$$

Thus, $\mathbf{a}_k$ is a unit basis vector, i.e., $\mathbf{a}_k$ has one component unity and all other components zero.

The modulation process s_k is assumed stationary, as in the prior PSK study,[1] but here we make the stronger assumption of independence in most of what follows. Define the first-order probability

$$w_i \equiv \Pr \{s_k = i\} \tag{18}$$

as the probability that the ith signaling waveform is transmitted in the kth time slot. w_i is independent of k by stationarity. By independence,

$$\Pr \{s_k = i, s_l = j\} = w_i w_j, \quad k \neq l. \tag{19}$$

Then

$$w_i = \Pr \left\{ \mathbf{a}_k = [\underset{1}{0} \quad \underset{2}{0} \quad \underset{\cdots}{\cdots} \quad \underset{i-1}{0} \quad \underset{i}{1} \quad \underset{i+1}{0} \quad \underset{\cdots}{\cdots} \quad \underset{M}{0}] \right\}. \tag{20}$$

Normalization of the total probability requires

$$\sum_{i=1}^{M} w_i = 1. \tag{21}$$

We use the following vector notation for the probabilities:

$$\underline{\mathbf{w}} = \mathbf{w}]' \equiv [w_1 \quad w_2 \; \cdots \; w_M]. \tag{22}$$

---

§ The following notational conventions are adopted throughout:

(*i*) Boldface quantities denote matrices.
(*ii*) Row and column vectors are distinguished by the additional notation ⌴ and], respectively.
(*iii*) Ordinary matrix multiplication is indicated by ·, Kronecker matrix products by × (see second footnote, page 908, Ref. 1, for properties of Kronecker products used throughout the present paper).
(*iv*) The transpose of a matrix is indicated by ′.
(*v*) The Hermitian transpose of a matrix is indicated by †.
(*vi*) Multiple Kronecker products are indicated by $\Pi_\times$ (see footnote, page 911, Ref. 1) and the Kronecker power is indicated by an integer exponent enclosed in square brackets.

Further define a vector (here of dimension M) with all elements unity as

$$\underline{\mathbf{1}} = \mathbf{1}]' \equiv [1 \quad 1 \cdots 1]. \tag{23}$$

Then (21) may be written

$$\underline{\mathbf{1}} \cdot \mathbf{w}] = \underline{\mathbf{w}} \cdot \mathbf{1}] = 1. \tag{24}$$

Finally, for convenience later, define the diagonal matrix

$$\mathbf{w}_d \equiv \begin{bmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_M \end{bmatrix}. \tag{25}$$

Then, from (22) and (23),

$$\mathbf{w}_d \cdot \mathbf{1}] = \mathbf{w}], \quad \underline{\mathbf{1}} \cdot \mathbf{w}_d = \underline{\mathbf{w}}. \tag{26}$$

Note that

$$\langle \underline{\mathbf{a}_k} \rangle = \underline{\mathbf{w}}, \tag{27}$$

$$\langle \mathbf{a}_k] \cdot \underline{\mathbf{a}_k} \rangle = \mathbf{w}_d. \tag{28}$$

IV. FSK AS A BASEBAND PULSE TRAIN

We seek an expression of $v(t)$, given by (4), (9), (12), and (14), of the form

$$v(t) = \sum_{k=-\infty}^{\infty} \underline{\mathbf{c}_k} \cdot \mathbf{r}(t - kT)] \tag{29}$$

for signaling pulses of finite duration. Assume the $h_i(t)$ of (3) are strictly time-limited to an interval KT, as follows:

$$\mathbf{h}(t)] = \mathbf{0}], \quad t \leqq L_K, \quad t > U_K. \tag{30}§$$

$$L_K \equiv \begin{cases} -\dfrac{K-1}{2} T, & K \text{ odd.} \\ -\dfrac{K}{2} T, & K \text{ even.} \end{cases}$$
$$U_K \equiv \begin{cases} \dfrac{K+1}{2} T, & K \text{ odd.} \\ \dfrac{K}{2} T, & K \text{ even.} \end{cases} \tag{31}$$

L_K and U_K are respectively the lower and upper limits of the pulses. Figure 1 shows portions of $f_d(t)$ for four different maximum signal-pulse durations; the terms $k = -1, 0, 1, 2$ of (14) are shown, and for

§ $\underline{\mathbf{0}} = \mathbf{0}]'$ is a vector of appropriate dimension (here M) with all elements zero.

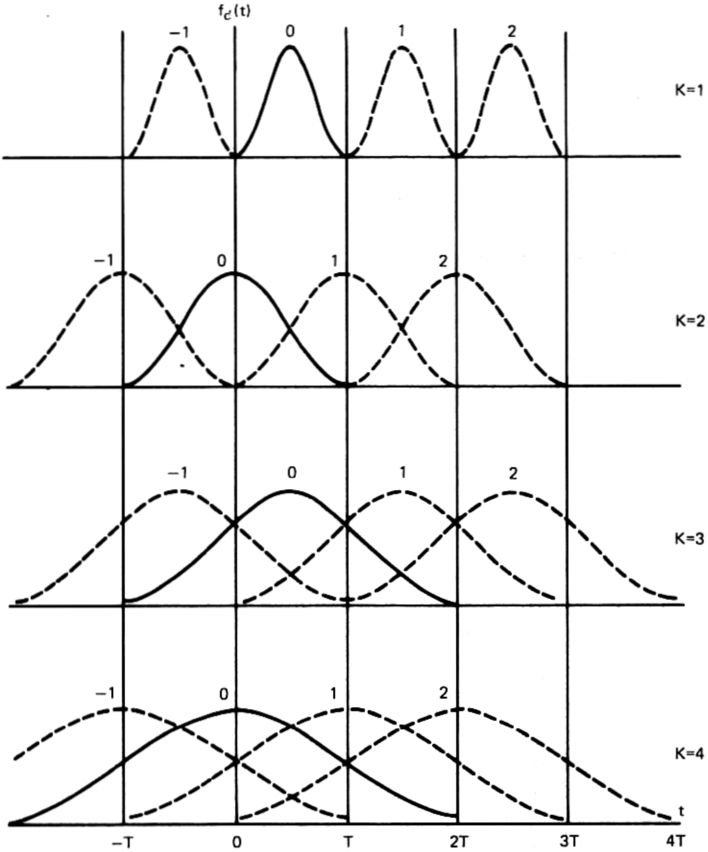

Fig. 1—Frequency modulation for different signal pulse durations. Index k is shown near peak of each pulse. Also, for simplicity, same signal pulse is shown for each k. T = time slot duration or signaling period. KT = maximum signal-pulse duration. Note different pulse center location for odd and even K.

convenience $\mathbf{a}_k$ has been taken the same for each of these time slots. The pulses are positioned along the time slots such that the limits of each signal pulse lie on the boundary between adjacent time slots (i.e., $t = \text{integer} \cdot T$); this results in different definitions for L_K, U_K for K even and odd. Since symmetric pulses have been chosen for illustration in Fig. 1, their maxima are centered in the time slots for K odd, and lie on the time-slot boundaries for K even. Discontinuities are permitted at the pulse edges (and elsewhere), but are not present in the example of Fig. 1 (and would not normally be present in a mathematical model of a physical system); discontinuities at the time-slot boundaries are restricted by the inequalities at the right of (30). Examine the (0, T]

time slot in Fig. 1 as typical; then the number of pulses contributing to $f_d(t)$ at every instant equals K.[§]

It remains for us to express the pulse shapes $\mathbf{r}(t)$ and coefficients $\mathbf{c}_k$ of (29) in terms of the signal pulses $\mathbf{h}(t)$ and coefficients $\mathbf{a}_k$ of (14). We give separate treatments for the cases $K = 1$ and $K = 2$, and extend these results to general K. The treatment is an extension of that for the PSK case, given in Section IV of Reference 1.

4.1 Nonoverlapping pulses: $K = 1$

The top portion of Fig. 1 shows digital frequency modulation for which the signal pulses in different time slots never overlap; in this case from (30) to (31),

$$\mathbf{h}(t)] = 0], \quad t \leqq 0, \quad t > T. \tag{32}$$

Define

$$\underline{\mathbf{q}(t)} \equiv \begin{cases} \left[\exp\left(j\int_0^t h_1(\mu)d\mu\right) \ \exp\left(j\int_0^t h_2(\mu)d\mu\right) \cdots \exp\left(j\int_0^t h_M(\mu)d\mu\right)\right], & 0 < t \leqq T. \\ \underline{\mathbf{0}}, & t \leqq 0, \quad t > T. \end{cases} \tag{33}$$

Equations (4), (9), (12), (14), and (33) yield

$$\phi(0) = 0; \quad v(0) = 1. \tag{34}$$

$$v(t) = \sum_{k=-\infty}^{\infty} S_k \underline{\mathbf{a}_k} \cdot \mathbf{q}(t - kT)], \tag{35}$$

where

$$S_k = \begin{cases} \prod_{i=k+1}^{0} \underline{\mathbf{a}_{i-1}} \cdot \mathbf{q}^*(T)], & k < 0. \\ 1, & k = 0. \\ \prod_{i=1}^{k} \underline{\mathbf{a}_{i-1}} \cdot \mathbf{q}(T)], & k > 0. \end{cases} \tag{36}$$

Comparing (35) and (29), the parameters of the latter are given as follows for nonoverlapping signal pulses:

$$\left.\begin{aligned} \underline{\mathbf{c}_k} = S_k \underline{\mathbf{b}_k}, \quad \underline{\mathbf{b}_k} \equiv \underline{\mathbf{a}_k}; \\ \mathbf{r}(t)] = \mathbf{q}(t)]; \end{aligned}\right\} \quad K = 1. \tag{37}$$

S_k in (37) is, of course, given by (36).

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§ The same conventions were used in Fig. 1, Ref. 1, for the baseband modulation pulses in digital PSK.

4.2 Overlapping pulses: $K = 2$

This case is illustrated in the second portion of Fig. 1. In the $(0, T]$ time slot, the $k = 0, 1$ pulses contribute. We have from (30) to (31)

$$\mathbf{h}(t)] = 0], \quad t \leqq -T, \quad t > T. \tag{38}$$

Define

$$\underline{\mathbf{q}(t)} \equiv \begin{cases} \left[\exp\left(j \int_{-T}^{t} h_1(\mu)d\mu\right) \exp\left(j \int_{-T}^{t} h_2(\mu)d\mu\right) \cdots \exp\left(j \int_{-T}^{t} h_M(\mu)d\mu\right)\right], & -T < t \leqq T. \\ \underline{\mathbf{0}}, & t \leqq -T, \quad t > T. \end{cases} \tag{39§}$$

Equations (4), (9), (12), (14), and (39) yield

$$\phi(0) = \int_{-T}^{0} h_{s_0}(\mu)d\mu. \tag{40}$$

$$v(t) = \sum_{k=-\infty}^{\infty} S_k\{\underline{\mathbf{a}_k}\cdot\mathbf{q}(t - kT)]\}\{\underline{\mathbf{a}_{k+1}}\cdot\mathbf{q}(t - (k+1)T)]\}, \tag{41}$$

where S_k remains as given in (36), the same as for the prior $K = 1$ case. Proceeding exactly as in (51) of Ref. 1, (41) above yields (29) with the following parameters, when no more than two signal pulses overlap:

$$\left.\begin{aligned} \underline{\mathbf{c}_k} &= S_k\underline{\mathbf{b}_k}, \quad \underline{\mathbf{b}_k} \equiv \underline{\mathbf{a}_k} \times \underline{\mathbf{a}_{k+1}}; \\ \mathbf{r}(t)] &= \mathbf{q}(t)] \times \mathbf{q}(t - T)]; \end{aligned}\right\} \quad K = 2. \tag{42}$$

S_k is given by (36), and $\times$ denotes the Kronecker product [see footnote to (14)]. The term $\mathbf{b}_k$, like $\mathbf{a}_k$, is a unit basis vector, i.e., it has one element unity and the remaining $M^2 - 1$ elements zero.[1] Note from (42) and (39) that

$$\mathbf{r}(t)] = \mathbf{0}], \quad t \leqq 0, \quad t > T. \tag{43}$$

Binary FSK offers a simple example of these results, governed by the same relations between $\mathbf{a}_k$ and $\mathbf{b}_k$ as for binary PSK given in (57) of Ref. 1.

4.3 Overlapping pulses: general K

The general case follows by straightforward extension of the above; see (58) to (62) of Ref. 1. Figure 1 illustrates the frequency modulation for $K = 3, 4$. The modulation pulse restrictions are given by (30) and

§ Comparing (39) with (33), note that the definition of $\mathbf{q}(t)$ is different for different K; $\mathbf{q}(t) \neq 0$ over the same interval in which $\mathbf{h}(t)$ may be nonzero.

(31). Define

$$
\underline{\mathbf{q}(t)} \equiv \begin{cases} \left[\exp\left(j \int_{L_K}^{t} h_1(\mu)d\mu\right) \ \exp\left(j \int_{L_K}^{t} h_2(\mu)d\mu\right) \right. \\ \qquad \left. \cdots \exp\left(j \int_{L_K}^{t} h_M(\mu)d\mu\right)\right], & L_K < t \leqq U_K. \\ \underline{\mathbf{0}}, & t \leqq L_K, \quad t > U_K. \end{cases} \tag{44}
$$

$\phi(0)$ of (12) is

$$
\phi(0) = \begin{cases} \displaystyle\sum_{k=-(K-1)/2}^{(K-3)/2} \int_{L_K}^{-kT} h_{s_k}(\mu)d\mu, & K \text{ odd}, \quad K > 1. \\ \displaystyle\sum_{k=-(K-2)/2}^{(K-2)/2} \int_{L_K}^{-kT} h_{s_k}(\mu)d\mu, & K \text{ even}, \quad K > 0. \end{cases} \tag{45}
$$

The parameters of (29) are

$$
\underline{\mathbf{c}_k} = S_k \underline{\mathbf{b}_k}; \quad \underline{\mathbf{b}_k} = \begin{cases} \displaystyle\prod_{i=-(K-1)/2}^{(K-1)/2}{}_{\times} \ \underline{\mathbf{a}_{k+i}}, & K \text{ odd}. \\ \displaystyle\prod_{i=-(K-2)/2}^{K/2}{}_{\times} \ \underline{\mathbf{a}_{k+i}}, & K \text{ even}. \end{cases} \tag{46}
$$

$$
\mathbf{r}(t)] = \begin{cases} \displaystyle\prod_{i=-(K-1)/2}^{(K-1)/2}{}_{\times} \ \mathbf{q}(t - iT)], & K \text{ odd}. \\ \displaystyle\prod_{i=-(K-2)/2}^{K/2}{}_{\times} \ \mathbf{q}(t - iT)], & K \text{ even}. \end{cases} \tag{47}
$$

$\prod_{\times}$ denotes a multiple Kronecker product [see footnote to (14)]. S_k is given by

$$
S_k = \begin{cases} \displaystyle\prod_{i=k+1}^{0} \underline{\mathbf{a}_{i-(K+1)/2}} \cdot \mathbf{q}^*(U_K)], & k < 0; \\ 1, & k = 0; \quad K \text{ odd}. \\ \displaystyle\prod_{i=1}^{k} \underline{\mathbf{a}_{i-(K+1)/2}} \cdot \mathbf{q}(U_K)], & k > 0. \end{cases} \tag{48}
$$

$$
S_k = \begin{cases} \displaystyle\prod_{i=k+1}^{0} \underline{\mathbf{a}_{i-(K/2)}} \cdot \mathbf{q}^*(U_K)], & k < 0; \\ 1, & k = 0; \quad K \text{ even}. \\ \displaystyle\prod_{i=1}^{k} \underline{\mathbf{a}_{i-(K/2)}} \cdot \mathbf{q}(U_K)], & k > 0. \end{cases} \tag{49}
$$

Note that

$$\mathbf{r}(t)] = \mathbf{0}], \quad t \leqq 0, \quad t > T. \tag{50}$$

4.4 Discussion

It is instructive to obtain the prior PSK results[1] by specializing the present FSK results. Define the phase shift produced by each signaling pulse as

$$g_i(t) \equiv \int_{L_K}^{t} h_i(\mu)d\mu, \quad i = 1, 2, \cdots, M \tag{51}$$

or in vector notation

$$\mathbf{g}(t)] \equiv \int_{L_K}^{t} \mathbf{h}(\mu)]d\mu, \tag{52}$$

and substitute into (44)§ (or into (33) or (39) for special cases $K = 1, 2$). Then the present results of (29), (31), and Section 4.3 are similar to the former PSK results of (43) and Section 4.3 of Ref. 1, except for the factor S_k; in particular, the equations for $\mathbf{q}(t)$, $\mathbf{b}_k$, and $\mathbf{r}(t)$ have an identical form. The additional factor S_k present in the FSK case accounts for the total phase shift introduced by each of the signaling pulses.

To specialize the present FSK results to the PSK case, we require the total area of each of the present modulation pulses to be zero. Thus, (51) and (52) become

$$g_i(U_K) \equiv \int_{L_K}^{U_K} h_i(t)dt = 0, \quad i = 1, 2, \cdots, M \tag{53§§}$$

or

$$\mathbf{g}(U_K)] \equiv \int_{L_K}^{U_K} \mathbf{h}(t)]dt = 0]. \tag{54§§}$$

Substituting in (44),

$$\mathbf{q}(U_K)] = \mathbf{1}]. \tag{55}$$

Therefore, (17) yields

$$\underline{\mathbf{a}_i}\cdot\mathbf{q}(U_K)] = 1. \tag{56}$$

Substituting in (48) to (49),

$$S_k = 1, \quad \text{all } k, K, \tag{57}$$

completing the specialization of the present FSK results to the PSK case.[1]

In the general FSK case, the results of Sections 4.1 to 4.3 above reduce the FSK problem to determining the spectrum of (29). This is ac-

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§ Only the range $L_k < t \leqq U_k$ is relevant in (51), (52), since $\mathbf{q}(t) \equiv \mathbf{0}$ outside this range.

§§ Equations (30) to (31) and (53), (54) render (51) and (52) zero for $t \leqq L_K$, $t \geqq U_K$, satisfying (58) of Ref. 1.

complished by deleting (149), (153), and (163) to (171) of Ref. 1, Appendix B.[§] Then the spectral density of (29) above is given by setting $\mathbf{b} \to \mathbf{c}$ in (150) to (152) and (160) to (162) of Ref. 1 as follows:

$$P_v(f) = \frac{1}{T} \underline{\mathbf{R}(f)} \cdot \tilde{\mathbf{P}}_{\mathbf{c}}(fT) \cdot \mathbf{R}^*(f)]. \tag{58}$$

$$\tilde{\mathbf{P}}_{\mathbf{c}}(f) = \sum_{n=-\infty}^{\infty} e^{-j2\pi fn} \tilde{\mathbf{\Phi}}_{\mathbf{c}}(n) \tag{59}$$

[§§]

$$\tilde{\mathbf{\Phi}}_{\mathbf{c}}(n) = \langle \mathbf{c}_{k+n}] \cdot \underline{\mathbf{c}_k^*} \rangle \tag{60}$$

$$\mathbf{R}(f)] = \underline{\mathbf{R}(f)'} = \int_{-\infty}^{\infty} e^{-j2\pi ft} \mathbf{r}(t)] dt. \tag{61}$$

In these relations $\mathbf{R}(f)$ is the Fourier transform of $\mathbf{r}(t)$ of (37), (42), or (47), depending on K (i.e., the amount of pulse overlap). $\tilde{\mathbf{\Phi}}_{\mathbf{c}}(n)$ is determined from (36) and (37), (36) and (42), or (46) and (48) or (49) in Section VI for $K = 1, 2$, and general K, respectively.

V. FSK WITH LINE COMPONENTS

From (21), (22), and (44),

$$|\underline{\mathbf{w}} \cdot \mathbf{q}(U_K)]| \leqq 1. \tag{62}$$

[§§§]

We show in the present section that equality in (62) corresponds to the presence of line components in the FSK spectrum. Conversely, if the inequality of (62) is satisfied, we see in Section VI that the FSK spectrum contains no line components.

Assume throughout the remainder of the present section that

$$|\underline{\mathbf{w}} \cdot \mathbf{q}(U_K)]| = 1. \tag{63}$$

This yields

$$\int_{L_K}^{U_K} h_i(t) dt = 2\pi f_l + \text{integer} \cdot 2\pi, \quad i = 1, 2, \cdots, M, \tag{64}$$

where $2\pi f_l$ is defined as the common area (mod 2π) of all of the FSK signaling pulses in the line case. For definiteness we take

$$-\tfrac{1}{2} < f_l \leqq \tfrac{1}{2}. \tag{65}$$

Equivalently,

$$\mathbf{q}(U_K)] = e^{j2\pi f_l} \mathbf{1}], \quad -\tfrac{1}{2} < f_l \leqq \tfrac{1}{2}. \tag{66}$$

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§ These deleted portions were relevant to the study of line spectral components of PSK.[1] The line spectral components of FSK are treated separately in Section V.

§§ See footnote, page 905, Ref. 1.

§§§ This follows because $|q_i(t)| = 1$.

Now substitute (66) into (48) and (49); we have

$$S_k = e^{jk2\pi f_l}, \quad \text{all } k,\ K. \tag{67}$$

Thus, when (62) is satisfied

$$v(t) = \sum_{k=-\infty}^{\infty} e^{jk2\pi f_l}\, \underline{\mathbf{b}_k}\cdot r(t - kT)]. \tag{68}$$

The relation (68) is the same as the PSK result of (43) of Ref. 1 except for the factor $e^{jk2\pi f_l}$; as noted following (52), $\mathbf{b}_k$ and $\mathbf{r}(t)$ have the same form in FSK and PSK. Consequently, when (63) is satisfied, the FSK spectrum may be obtained by simple transformation of the PSK results of Ref. 1. One way this may be done is to rewrite (68) as

$$v(t)e^{-j2\pi(f_l/T)t} = \sum_{k=-\infty}^{\infty} \underline{\mathbf{b}_k}\cdot (e^{-j2\pi(f_l/T)(t-kT)}\mathbf{r}(t - kT)]). \tag{69}$$

Comparing with (43) of Ref. 1, all the PSK results of Ref. 1 apply directly to the present case by making the substitutions

$$\begin{aligned} v(t) &\rightarrow v(t)e^{-j2\pi(f_l/T)t}, \\ \mathbf{r}(t)] &\rightarrow e^{-j2\pi(f_l/T)t}\,\mathbf{r}(t)]. \end{aligned} \tag{70}$$

Therefore, when (68) is satisfied, and consequently (68) holds, the FSK wave has line spectra. We separate the line and continuous components as

$$v(t) = v_l(t) + v_c(t). \tag{71}$$

The line component is given from (70), and (114)[§] of Ref. 1, as

$$\begin{aligned} v_l(t) &= \frac{1}{T}\underline{\mathbf{w}}^{[K]}\cdot \sum_{n=-\infty}^{\infty} \mathbf{R}\left(\frac{n + f_l}{T}\right)\Big]\, e^{j2\pi((n+f_l)/T)t}, \\ P_{v_l}(f) &= \frac{1}{T^2}\,|\mathbf{w}^{[K]}\cdot\mathbf{R}(f)]|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n + f_l}{T}\right). \end{aligned} \tag{72}$$ [§§]

The spectral density of the continuous component is found from (70), and (69),[§§§] (96), or (116) of Ref. 1 as follows:

$$P_{v_c}(f) = \frac{1}{T}\underline{\mathbf{R}(f)}\cdot\{\quad\}\cdot\mathbf{R}^*(f)]. \tag{73}$$

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§ Or from (66) and (95) of Ref. 1 for the special cases $k = 1, 2$.

§§ See footnote to (14); an exponent enclosed in square brackets denotes the Kronecker power, i.e., the Kronecker product of a matrix (or vector) with itself the indicated number of times.

§§§ This is equation (69) of Ref. 1 specialized to independent signal pulses, i.e., with (32) used for all $n \neq 0$, yielding (72) or (73), of Ref. 1. If this restriction is not imposed, and (70) above is used in the general form of (69) of Ref. 1, we obtain the spectrum of digital FSK with line spectra, with nonoverlapping signal pulses having arbitrary correlation (rather than being independent, as in the remainder of the present paper). This is the only correlated case that can be readily treated by the present methods.

$$\{\quad\} = \mathbf{w}_d - \mathbf{w}]\cdot\underline{\mathbf{w}}, \quad K = 1. \tag{74}$$

$$\begin{aligned}\{\quad\} = \mathbf{w}_d^{[2]} &- \mathbf{w}]^{[2]}\cdot\underline{\mathbf{w}}^{[2]} \\ &+ (\underline{\mathbf{w}} \times \mathbf{w}_d \times \mathbf{w}] - \mathbf{w}]^{[2]}\cdot\underline{\mathbf{w}}^{[2]})e^{-j2\pi(fT-f_l)} \\ &+ (\mathbf{w}] \times \mathbf{w}_d \times \underline{\mathbf{w}} - \mathbf{w}]^{[2]}\cdot\underline{\mathbf{w}}^{[2]})e^{+j2\pi(fT-f_l)}, \quad K = 2. \end{aligned} \tag{75§}$$

$$\{\quad\} = \begin{array}{l}\text{expression in } \{\quad\} \text{ in (116) of} \\ \text{Ref. 1 with } fT \to fT - f_l,\end{array} \quad \text{general } K. \tag{76}$$

The condition of (63), that has here been shown sufficient, is shown in Section VI also to be necessary for line components to be present in FSK spectra. This condition has a simple physical interpretation. Every signal pulse must introduce the same total phase change in the modulated carrier in order to have line components. This phase change has been denoted as $2\pi f_l$; the line components appear at frequencies

$$\pm\left(f_c + \frac{f_l + n}{T}\right), \quad n = \cdots, -1, 0, 1, \cdots. \tag{77}$$

When $f_l = 0$, (73) to (76) show that the FSK spectra are identical to the prior PSK results.[1] However the wave in this case is not necessarily a PSK wave. The stronger condition of (53) or (54) is required to have a PSK wave; this condition demands the net phase shift introduced by every signaling pulse to be zero. A wave with $f_l = 0$, but one or more signal pulses with net phase change equal, for example, to $\pm 2\pi$, will have a spectrum given by the PSK formula, but will not be a PSK wave.

VI. FSK WITH NO LINE COMPONENTS

Assume throughout this section that the inequality of (62) is satisfied:

$$|\underline{\mathbf{w}}\cdot\mathbf{q}(U_K)]| < 1. \tag{78}$$

We demonstrate that under these conditions the FSK spectrum contains no lines. The properties of Kronecker products given in Ref. 1 in the second footnote, p. 908, and in the first footnote, p. 915, are used throughout without further comment.

6.1 FSK spectrum with no line components: nonoverlapping pulses, K = 1

From (37) and (60),

$$\tilde{\Phi}_c(n) = \langle (S_{k+n}S_k^*)(\mathbf{a}_{k+n}]\cdot\underline{\mathbf{a}_k})\rangle. \tag{79}$$

From (36),

$$S_{k+n}S_k^* = \sum_{i=0}^{n-1} \underline{\mathbf{a}_{k+i}}\cdot\mathbf{q}(T)], \quad n > 0. \tag{80}$$

§ See footnote to eq. (72).

Let us first consider $\tilde{\mathbf{\Phi}}_c(0)$. Since $S_k S_k^* = |S_k|^2 = 1$, for all k, we have from (79) and (28)

$$\tilde{\mathbf{\Phi}}_c(0) = \mathbf{w}_d. \tag{81}$$

Next, (79) and (80) yield

$$\tilde{\mathbf{\Phi}}_c(n) = \left\{ \prod_{i=1}^{n-1} \underline{a_{k+i}\cdot} \mathbf{q}(T)] \right\} \{\underline{\mathbf{q}(T)}\cdot \mathbf{a}_k]\} \{\mathbf{a}_{k+n}]\cdot \underline{\mathbf{a}_k}\}, \quad n \geqq 1, \tag{82}$$

where we have split off the first factor of (80). For $n = 1$, the first { }, containing the $\prod$, is dropped. The first { } is independent of the remainder of the expression by (19); using the fact that $\underline{\mathbf{q}(T)}\cdot \mathbf{a}_k]$ is a complex number or equivalently a 1×1 complex matrix, and using (27) and (28),

$$\begin{aligned} \tilde{\mathbf{\Phi}}_c(n) &= \{\underline{\mathbf{w}}\cdot \mathbf{q}(T)]\}^{n-1} \langle \mathbf{a}_{k+n}]\cdot \underline{\mathbf{q}(T)}\cdot \mathbf{a}_k]\cdot \underline{\mathbf{a}_k}\rangle \\ &= \{\underline{\mathbf{w}}\cdot \mathbf{q}(T)]\}^{n-1} \{\mathbf{w}]\cdot \underline{\mathbf{q}(T)}\cdot \mathbf{w}_d\}, \quad n \geqq 1, \end{aligned} \tag{83}$$

the last step following from the independence of $\mathbf{a}_{k+n}$ and $\mathbf{a}_k$.

Finally, taking the Hermitian transpose§ of (79),

$$\tilde{\mathbf{\Phi}}_c^{\dagger}(n) = \langle (S_{k+n}^* S_k)(\mathbf{a}_k]\cdot \underline{\mathbf{a}_{k+n}})\rangle. \tag{84}$$

Alternatively, setting $n \to -n$ in (79),

$$\tilde{\mathbf{\Phi}}_c(-n) = \langle (S_k^* S_{k-n})(\mathbf{a}_{k-n}]\cdot \underline{\mathbf{a}_k})\rangle. \tag{85}$$

Since (79) has been shown independent of k, we may set $k \to k + n$ in (85), to yield upon comparison with (84)

$$\tilde{\mathbf{\Phi}}_c(-n) = \tilde{\mathbf{\Phi}}_c^{\dagger}(n). \tag{86}$$

From (58) to (61), (81), (83), and (86),

$$\tilde{\mathbf{P}}_c(fT) = \mathbf{A} + \mathbf{A}^{\dagger}, \tag{87}$$

$$\mathbf{A} = \tfrac{1}{2}\mathbf{w}_d + e^{-j2\pi fT}\mathbf{w}]\cdot \underline{\mathbf{q}(T)}\cdot \mathbf{w}_d \sum_{n=1}^{\infty} \{e^{-j2\pi fT}\, \underline{\mathbf{w}}\cdot \mathbf{q}(T)]\}^{n-1}. \tag{88}$$

Because of (78) the geometric series in (88) converges, and

$$\mathbf{A} = \frac{1}{2}\mathbf{w}_d + \frac{e^{-j2\pi fT}\, \mathbf{w}]\cdot \underline{\mathbf{q}(T)}\cdot \mathbf{w}_d}{1 - e^{-j2\pi fT}\, \underline{\mathbf{w}}\cdot \mathbf{q}(T)]}. \tag{89}$$

Finally, the spectrum of the complex FSK wave of (4) is

$$P_v(f) = \frac{1}{T} \underline{\mathbf{R}(f)}\cdot (\mathbf{A} + \mathbf{A}^{\dagger})\cdot \mathbf{R}^*(f)], \tag{90}$$

with $\mathbf{A}$ given by (89) for nonoverlapping pulses, $K = 1$.

§ See footnote to (14).

It is clear that the restriction of (78) renders $\mathbf{A}$ and, hence, $P_v(f)$ finite for all f; consequently, there can be no line components in the FSK spectrum. If we take the limit as $|\lfloor\mathbf{w}\rfloor\cdot\mathbf{q}(T)]| \rightarrow 1$, and substitute (66) into (88), (90) yields directly the appropriate results for the line case, i.e., (72) and (73) to (74) of Section V.

6.2 FSK spectrum with no line components: overlapping pulses, K = 2

$$\tilde{\mathbf{\Phi}}_c(n) = \langle (S_{k+n}S_k^*)(\mathbf{b}_k]\cdot\lfloor\mathbf{b}_k\rfloor)\rangle \tag{91}$$

with

$$\lfloor\mathbf{b}_k\rfloor \equiv \lfloor\mathbf{a}_k\rfloor \times \lfloor\mathbf{a}_{k+1}\rfloor. \tag{92}$$

We have noted in Section 4.4. that the present $\mathbf{b}_k$ for FSK has identical form to the prior $\mathbf{b}_k$ of Ref. 1 for PSK. In Section 4.2, we saw that S_k is identical for $K = 1$ and $K = 2$; therefore, (80) applies to the present case as well. Comparing the PSK analysis of Section VII of Ref. 1, we evaluate (91) above by inserting (80) above inside the $\langle \quad \rangle$ in (87) of Ref. 1.

For $n = 0$, (87) and (90) of Ref. 1 yield

$$\tilde{\mathbf{\Phi}}_c(0) = \mathbf{w}_d^{[2]}. \tag{93}$$

For $n = 1$, (87) and (91) of Ref. 1 yield

$$\begin{aligned}\tilde{\mathbf{\Phi}}_c(1) &= \langle\{\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{a}_k]\} \times \lfloor\mathbf{a}_k\rfloor \times \{\mathbf{a}_{k+1}] \times \lfloor\mathbf{a}_{k+1}\rfloor\} \times \mathbf{a}_{k+2}]\rangle \\ &= \langle\{\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{a}_k]\cdot\lfloor\mathbf{a}_k\rfloor\} \times \{\mathbf{a}_{k+1}]\cdot\lfloor\mathbf{a}_{k+1}\rfloor\} \times \mathbf{a}_{k+2}]\rangle.\end{aligned} \tag{94}$$

Since $\mathbf{a}_k$, $\mathbf{a}_{k+1}$, and $\mathbf{a}_{k+2}$ are independent, (27) and (28) yield

$$\tilde{\mathbf{\Phi}}_c(1) = \{\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{w}_d\} \times \mathbf{w}_d \times \mathbf{w}]. \tag{95}$$

Finally, substituting (80) inside the $\langle \quad \rangle$ of the third line of (87) of Ref. 1,

$$\begin{aligned}\tilde{\mathbf{\Phi}}_c(n) = \Bigg\langle \Bigg\{\prod_{i=2}^{n-1} \lfloor\mathbf{a}_{k+i}\rfloor\cdot\mathbf{q}(T)]\Bigg\} \{\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{a}_k]\}\{\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{a}_{k+i}]\}\{\mathbf{a}_{k+n}]\cdot\lfloor\mathbf{a}_k\rfloor\} \\ \times \{\mathbf{a}_{k+n+i}]\cdot\lfloor\mathbf{a}_{k+i}\rfloor\}\Bigg\rangle, \quad n \geqq 2,\end{aligned} \tag{96}$$

where we have split off the first two factors of (80). For $n = 2$, the first $\{ \quad \}$, containing the $\prod$, is dropped. Regarding the factors $\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{a}_k]$ and $\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{a}_{k+1}]$ as complex numbers or alternatively as 1×1 complex matrices, and using the independence of the different $\mathbf{a}_k$, (96) yields

$$\begin{aligned}\tilde{\mathbf{\Phi}}_c(n) &= \{\lfloor\mathbf{w}\rfloor\cdot\mathbf{q}(T)]\}^{n-2}\langle\{\mathbf{a}_{k+n}]\cdot\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{a}_k]\cdot\lfloor\mathbf{a}_k\rfloor\} \\ &\qquad \times \{\mathbf{a}_{k+n+i}]\cdot\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{a}_{k+i}]\cdot\lfloor\mathbf{a}_{k+i}\rfloor\}\rangle \\ &= \{\lfloor\mathbf{w}\rfloor\cdot\mathbf{q}(T)]\}^{n-2}\{\mathbf{w}]\cdot\lfloor\mathbf{q}(T)\rfloor\cdot\mathbf{w}_d\}^{[2]}, \quad n \geqq 2.\end{aligned} \tag{97}$$

The remainder of the analysis proceeds as in Section 6.1. The FSK spectrum for overlapping pulses with $K = 2$ is given by

$$P_v(f) = \frac{1}{T} \underline{\mathbf{R}(f)} \cdot (\mathbf{A} + \mathbf{A}^{\dagger}) \cdot \mathbf{R}^*(f)], \tag{98}$$

where

$$\mathbf{A} = \frac{1}{2}\,\mathbf{w}_d^{[2]} + e^{-j2\pi fT}\{\underline{\mathbf{q}(T)} \cdot \mathbf{w}_d\} \times \mathbf{w}_d \times \mathbf{w}] + \frac{\{e^{-j2\pi fT}\mathbf{w}] \cdot \underline{\mathbf{q}(T)} \cdot \mathbf{w}_d\}^{[2]}}{1 - e^{-j2\pi fT}\underline{\mathbf{w}} \cdot \mathbf{q}(T)]}. \tag{99}$$

$P_v(f)$ again contains no spectral lines by (78); the limit $\underline{\mathbf{w}} \cdot \mathbf{q}(T)] \rightarrow 1$ again yields the appropriate results of Section V for the line case.

6.3 FSK spectrum with no line components: overlapping pulses, general K

Similar analysis yields the generalization to any overlapping signal pulses. From (46) and (60),

$$\tilde{\mathbf{\Phi}}_c(n) = \langle (S_{k+n}S_k^*)(\mathbf{b}_k] \cdot \underline{\mathbf{b}_k}) \rangle \tag{100}$$

with $\mathbf{b}_k$ given by (46) and S_k by (48) to (49). Thus,

$$S_{k+n}S_k^* = \begin{cases} \prod_{j=1}^{n} \underline{\mathbf{a}_{k-((K+1)/2)+j}} \cdot \mathbf{q}\left(\frac{K+1}{2}\,T\right)\right], & k \text{ odd}, \\ \prod_{j=1}^{n} \underline{\mathbf{a}_{k-(K/2)+j}} \cdot \mathbf{q}\left(\frac{K}{2}\,T\right)\right], & k \text{ even}, \end{cases} \qquad n > 0. \tag{101}$$

Since the present $\mathbf{b}_k$ have identical form to $\mathbf{b}_k$ of Ref. 1, (100) is evaluated by substituting (101) inside the ⟨ ⟩ of (106) of Ref. 1, and making corresponding changes in the remainder of Section VII, Ref. 1. The factor

$$\prod_{j=1}^{n} \underline{\mathbf{a}_j} \cdot \mathbf{q}(U_K)] \tag{102}$$§

is inserted inside the ⟨ ⟩ of (107) of Ref. 1. In the following we recall that the factors of (102), $\underline{\mathbf{a}} \cdot \mathbf{q}] = \underline{\mathbf{q}} \cdot \mathbf{a}]$, may be regarded alternatively as complex numbers or as 1×1 complex matrices.

In (109) of Ref. 1, the first factor of the second line is modified as

$$\underline{\langle \mathbf{a}_i \rangle} \rightarrow \underline{\mathbf{q}(U_K)} \cdot \langle \mathbf{a}_i] \cdot \underline{\mathbf{a}_i} \rangle = \underline{\mathbf{q}(U_K)} \cdot \mathbf{w}_d; \tag{103}$$

consequently,

$$\tilde{\mathbf{\Phi}}_c(1) = \{\underline{\mathbf{q}(U_K)} \cdot \mathbf{w}_d\} \times \mathbf{w}_d^{[K-1]} \times \mathbf{w}]. \tag{104}$$

In (110), Ref. 1, the first two factors of the fourth line are modified

§ U_K is given by eq. (31).

as in (103), yielding

$$\tilde{\boldsymbol{\Phi}}_c(2) = \{\underline{\mathbf{q}(U_K)}\cdot \mathbf{w}_d\}^{[2]} \times \mathbf{w}_d^{[K-2]} \times \mathbf{w}]^{[2]}. \tag{105}$$

By induction,

$$\tilde{\boldsymbol{\Phi}}_c(n) = \{\underline{\mathbf{q}(U_K)}\cdot \mathbf{w}_d\}^{[n]} \times \mathbf{w}_d^{[K-n]} \times \mathbf{w}]^{[n]}, \quad n \leqq K. \tag{106}$$

Next, for $n \geqq K$, inserting (102) inside the $\langle \quad \rangle$ of the second line of (107) of Ref. 1, associating the factors of (102) with corresponding factors of the second $\prod_\times$ of (107), Ref. 1, with $n - K$ factors of (102) left over, and then noting that all factors have different indices and, hence, are independent,

$$\begin{aligned} \tilde{\boldsymbol{\Phi}}_c(n) &= \mathbf{w}]^{[K]}\cdot\{\underline{\mathbf{q}(U_K)}\cdot \mathbf{w}_d\}^{[K]}\{\underline{\mathbf{q}(U_K)}\cdot \mathbf{w}]\}^{n-K} \\ &= \{\underline{\mathbf{w}}\cdot \mathbf{q}(U_K)]\}^{n-K}\{\mathbf{w}]\cdot\underline{\mathbf{q}(U_K)}\cdot \mathbf{w}_d\}^{[K]}, \quad n \geqq K. \end{aligned} \tag{107}$$

Finally, from (108) of Ref. 1 and (100) above,

$$\tilde{\boldsymbol{\Phi}}_c(0) = \mathbf{w}_d^{[K]}. \tag{108}$$

The FSK spectrum for overlapping pulses with general K is given by

$$P_v(f) = \frac{1}{T}\underline{\mathbf{R}(f)}\cdot(\mathbf{A} + \mathbf{A}^\dagger)\cdot\mathbf{R}^*(f)], \tag{109}$$

where

$$\begin{aligned} \mathbf{A} = \frac{1}{2}\mathbf{w}_d^{[K]} + \sum_{n=1}^{K-1} e^{-jn2\pi fT}\,\underline{\mathbf{q}(U_K)}\cdot \mathbf{w}_d^{[n]} \times \mathbf{w}_d^{[K-n]} \times \mathbf{w}]^{[n]} \\ + \frac{\{e^{-j2\pi fT}\mathbf{w}]\cdot\underline{\mathbf{q}(U_K)}\cdot \mathbf{w}_d\}^{[K]}}{1 - e^{-j2\pi fT}\underline{\mathbf{w}}\cdot\mathbf{q}(U_K)]}. \end{aligned} \tag{110}$$

The condition (78) again guarantees no spectral lines; as equality is approached in (78) the present results approach those of Section V for the line case. The condition (63) is, therefore, necessary and sufficient for $P_v(f)$ to have line components.

VII. ILLUSTRATIVE EXAMPLES

The computation of the digital FM spectral density from the above methods is straightforward. For a given set of baseband signaling pulse shapes and symbol probability distribution, we determine K, the overlap parameter, and the probability row vector $\underline{\mathbf{w}}$. We then evaluate $\mathbf{q}(U_K)$ and $\underline{\mathbf{w}}\cdot\mathbf{q}(U_K)]$.

If $|\underline{\mathbf{w}}\cdot\mathbf{q}(U_K)| < 1$, we know that there are no line components in $P_v(f)$. The continuous part of the spectrum is evaluated from the appropriate Hermitian form given in Section VI.

If $|\underline{\mathbf{w}}\cdot\mathbf{q}(U_K)]| = 1$, we know that there are line components in the spectrum, and also that

$$\mathbf{q}(U_K)] = e^{j2\pi f_l}\cdot\mathbf{1}], \quad |f_l| \leqq \tfrac{1}{2}. \tag{111}$$

We determine f_l, and then $P_{v_l}(f)$ and $P_{v_c}(f)$ from the methods given in Section V.

In the following examples, the digital computer is programmed to work directly with the Hermitian forms (both ordinary and Kronecker matrix multiplications are performed by the computer). In this way, complicated cases involving multilevel signal pulses overlapping several time slots may be simply treated.

The case of rectangular-pulse FSK modulation is treated in Ref. 3, and consequently will not be considered here.§

We make the following assumptions for convenience; none are essential.

(*i*) The number of frequency levels is a power of 2,

$$M = 2^N, \quad N \text{ an integer}. \tag{112}$$

(*ii*) The M baseband signaling pulses have a common shape;

$$g_k(t) = C_k g(t). \tag{113}$$

(*iii*) All signal pulses are equally likely;

$$\mathbf{w}] = \frac{1}{M}\,\mathbf{1}], \quad \mathbf{w}_d = \frac{1}{M}\,\mathbf{I}_M, \tag{114}$$

where $\mathbf{I}_M$ is the identity matrix of order M.

7.1 Raised-cosine nonoverlapping signal pulses: $K = 1$

If the pulses have a common raised-cosine shape,

$$\mathbf{g}(t)] = \begin{cases} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_M \end{bmatrix} \pi f_d \left[1 - \cos\dfrac{2\pi t}{T}\right], & 0 < t \leqq T, \quad f_d > 0, \\ 0], & \text{otherwise}, \end{cases} \tag{115}$$

where $\Delta_1, \Delta_2, \cdots, \Delta_M$ are the peak-frequency-deviation parameters of the FSK signals. We assume that $\Delta_1 = 1$, $\Delta_2 = -1$, $\Delta_3 = 3$, $\Delta_4 = -3$, $\cdots$, $\Delta_M = -(M-1)$.

Since $K = 1$,

$$R_k(f) = T \sum_{n=-\infty}^{\infty} J_n\left(\frac{f_d T \Delta_k}{2}\right) e^{-j\pi[f-(f_d\Delta_k/2)]T} \cdot \frac{\sin \pi\left(fT - \dfrac{f_d\Delta_k T}{2} + n\right)}{\pi\left(fT - \dfrac{f_d\Delta_k T}{2} + n\right)}, \tag{116}$$

where $J_n(x)$ is the Bessel function of the first kind and of order n.

§ This is the only discrete-frequency-modulation spectrum given in Ref. 3.

7.1.1 Raised-cosine signaling with no line spectrum: $K = 1$

Since

$$\underline{\mathbf{q}(U_1)} = \underline{e^{j\Delta_1 \pi f_d T} \; e^{j\Delta_2 \pi f_d T} \; \dots \; e^{j\Delta_M \pi f_d T}}, \tag{117}$$

line spectra are absent if and only if f_dT is not an integer. In this case,

$$P_v(f) = P_{v_c}(f) = \frac{1}{T} \underline{\mathbf{R}(f)} \cdot (\mathbf{A} + \mathbf{A}^\dagger) \cdot \underline{\mathbf{R}(f)}^\dagger, \tag{118}$$

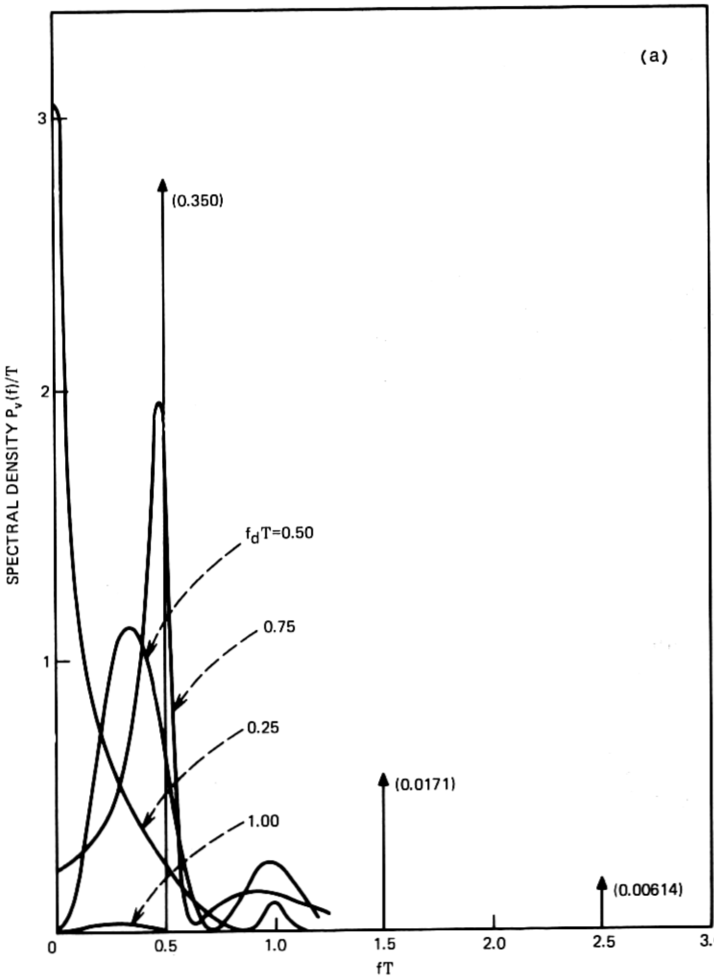

Fig. 2—Spectral density of binary FSK system with raised-cosine signaling and pulse duration T. $K = 1$. $2f_d$ is the spacing between two adjacent *a priori* chosen frequencies.

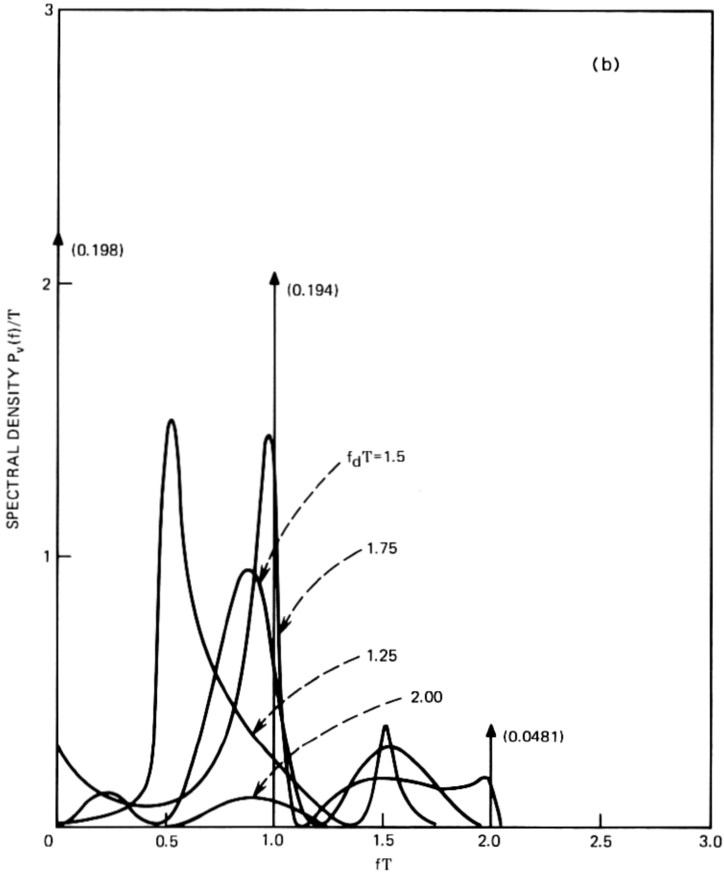

Fig. 2 (continued).

where $\mathbf{R}(f)]$ is given by (116) and

$$\mathbf{A} = \frac{1}{2M}\mathbf{I}_M + \frac{1}{M^2}\,\frac{e^{-j2\pi fT}\ \mathbf{1}]\cdot\underline{\mathbf{q}(U_1)}}{1 - e^{-j2\pi fT}\,\dfrac{1}{M}\,\dfrac{\sin M\pi f_dT}{\sin \pi f_dT}}. \tag{119}$$

For $M = 2$, 4, and 8 and various values of $k = f_dT$, $P_v(f)$ is plotted in Figs. 2, 3, and 4.

7.1.2 Raised-cosine signaling with line spectrum: $K = 1$

The FM spectral density $P_v(f)$ contains lines if and only if $|\underline{\mathbf{w}}\cdot\mathbf{q}(U_1)]| = 1$; that is, if and only if

$$f_dT = 1, 2, 3, \cdots. \tag{120}$$

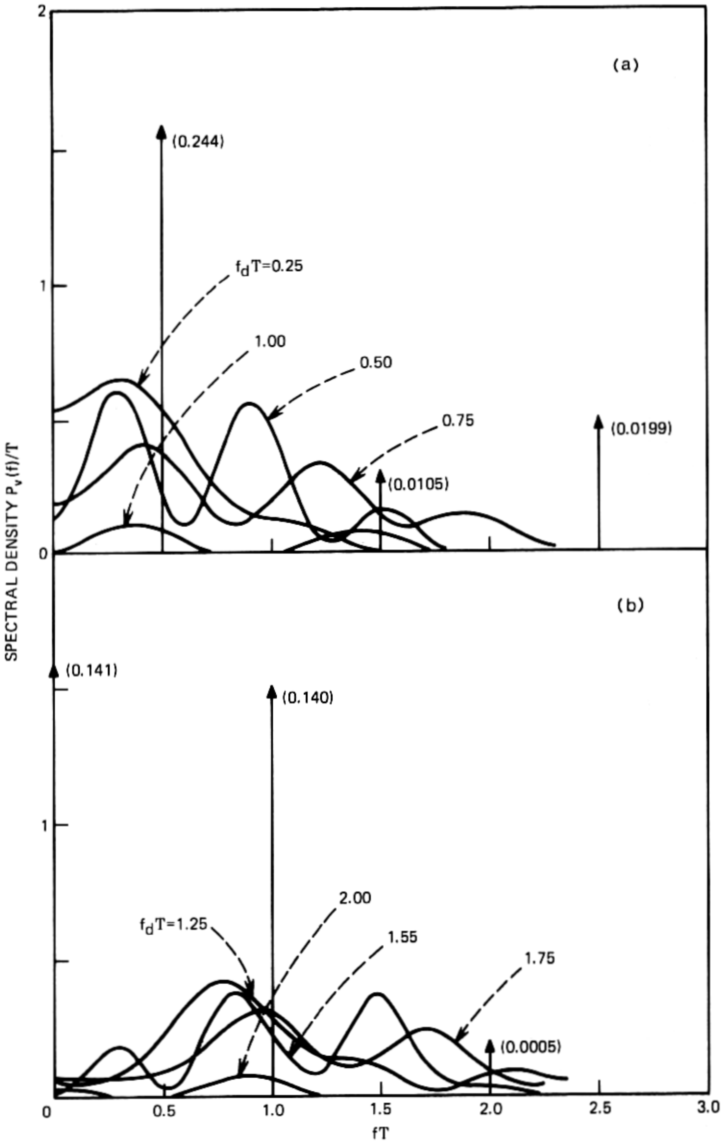

Fig. 3—Spectral density of quaternary FSK system with raised-cosine signaling and pulse duration T. $K = 1$. $2f_d$ is the spacing between two adjacent *a priori* chosen frequencies.

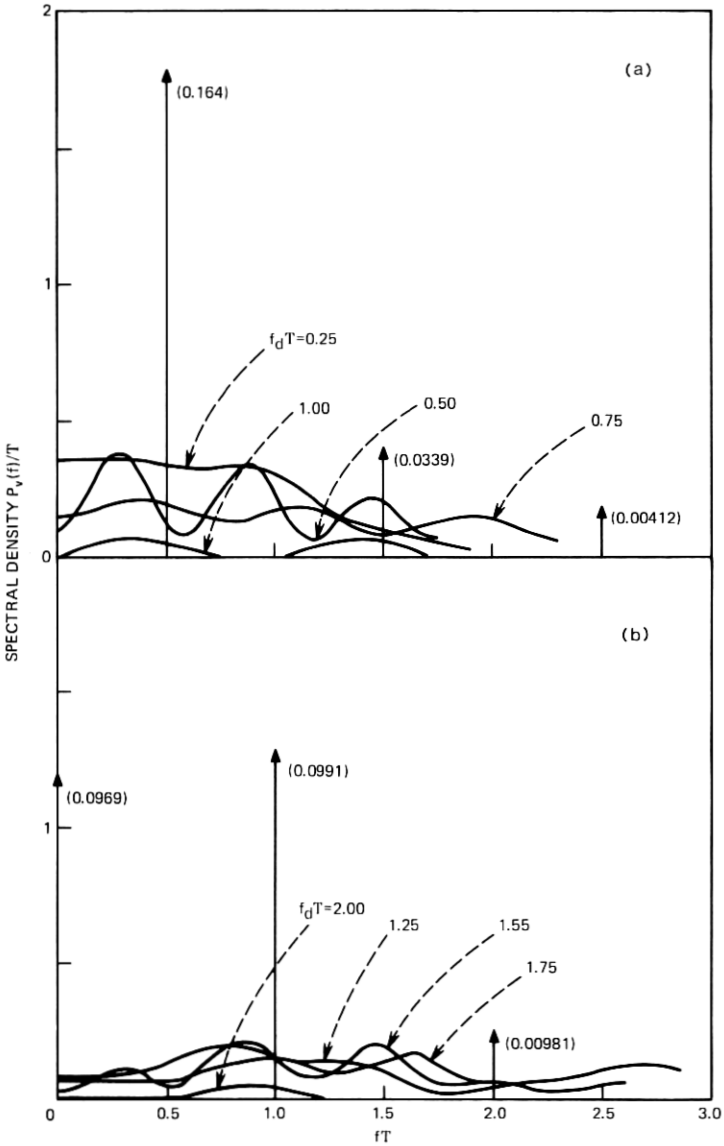

Fig. 4—Spectral density of octonary FSK system with raised-cosine signaling and pulse duration T. $K = 1.2f_d$ is the spacing between two adjacent *a priori* chosen frequencies.

In this case,

$$f_l = \tfrac{1}{2}, \quad f_d T = 1, 3, 5, \cdots, \tag{121}$$

$$f_l = 0, \quad f_d T = 2, 4, 6, \cdots. \tag{122}$$

From Section V,

$$P_v(f) = \frac{1}{T^2}\frac{1}{M^2}\left|\underline{1}\cdot \mathbf{R}(f)\right]\right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n+f_l}{T}\right) + \frac{1}{T}\underline{\mathbf{R}(f)}\cdot(\mathbf{B}+\mathbf{B}^\dagger)\cdot\underline{\mathbf{R}(f)}^\dagger, \tag{123}$$

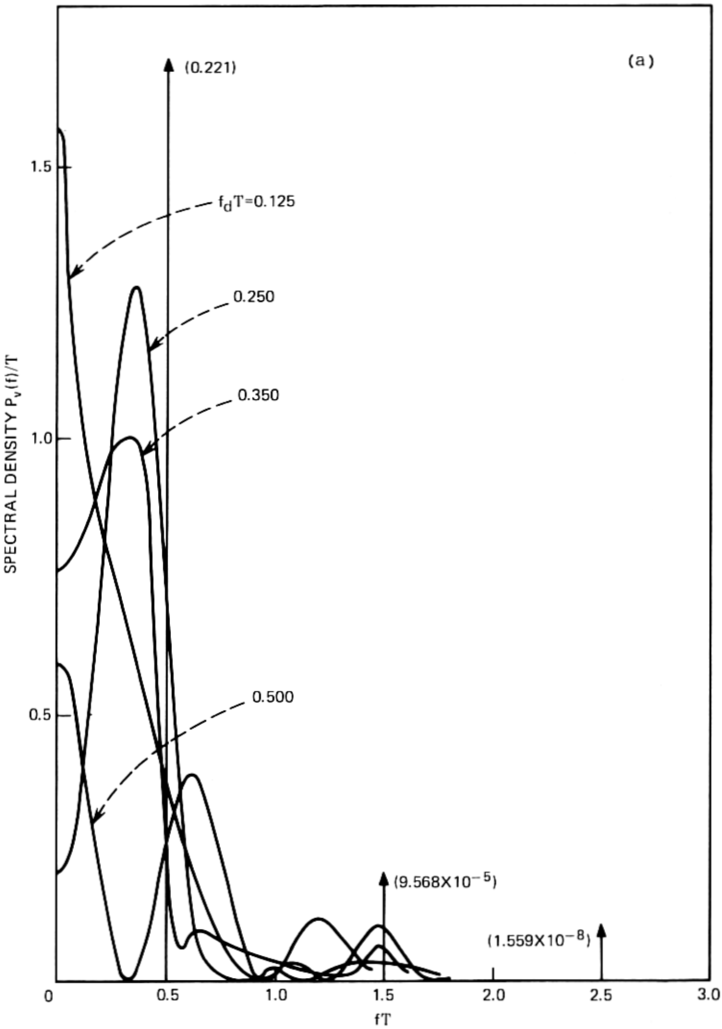

Fig. 5—Spectral density of binary FSK system with raised-cosine signaling and pulse duration $2T$. $K = 2.2 f_d$ is the spacing between two adjacent *a priori* chosen frequencies.

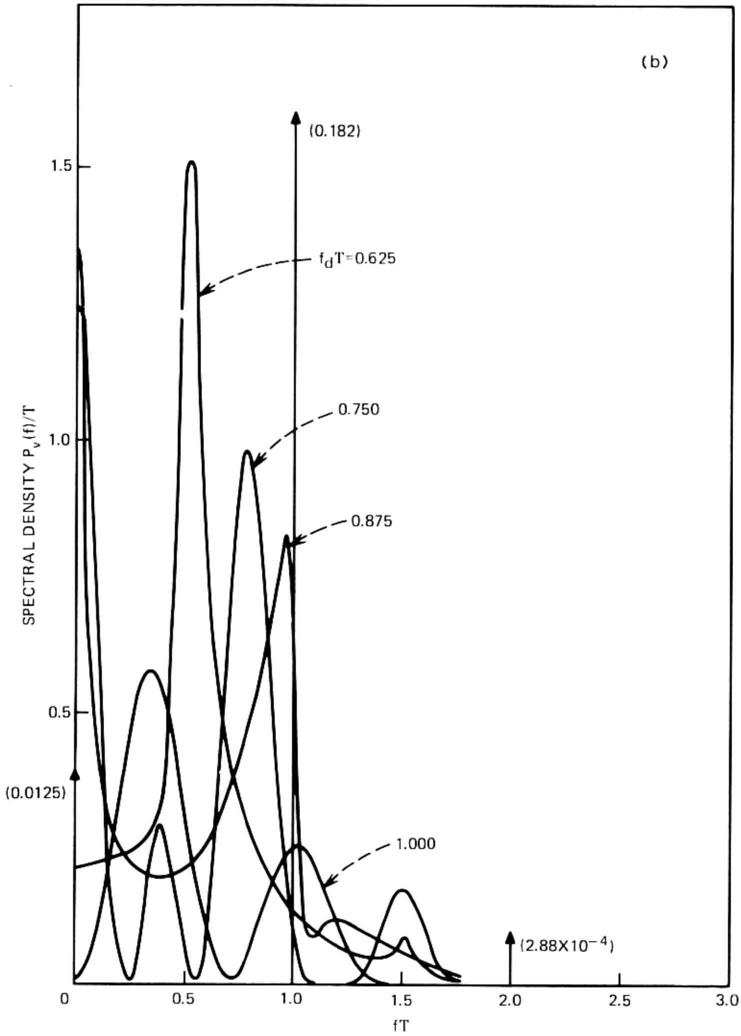

Fig. 5 (continued).

where[§]

$$\mathbf{B} + \mathbf{B}^{\dagger} = \frac{1}{M}\,\mathbf{I}_M - \frac{1}{M^2}\,\mathbf{1}]\cdot\underline{\mathbf{1}}. \tag{124}$$

For $M = 2$, 4, and 8 and for some integral values of f_dT, $P_v(f)$ is also plotted in Figs. 2, 3, and 4.

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§ Note that $\mathbf{B} + \mathbf{B}^{\dagger}$ for $K = 1$ is given in (74).

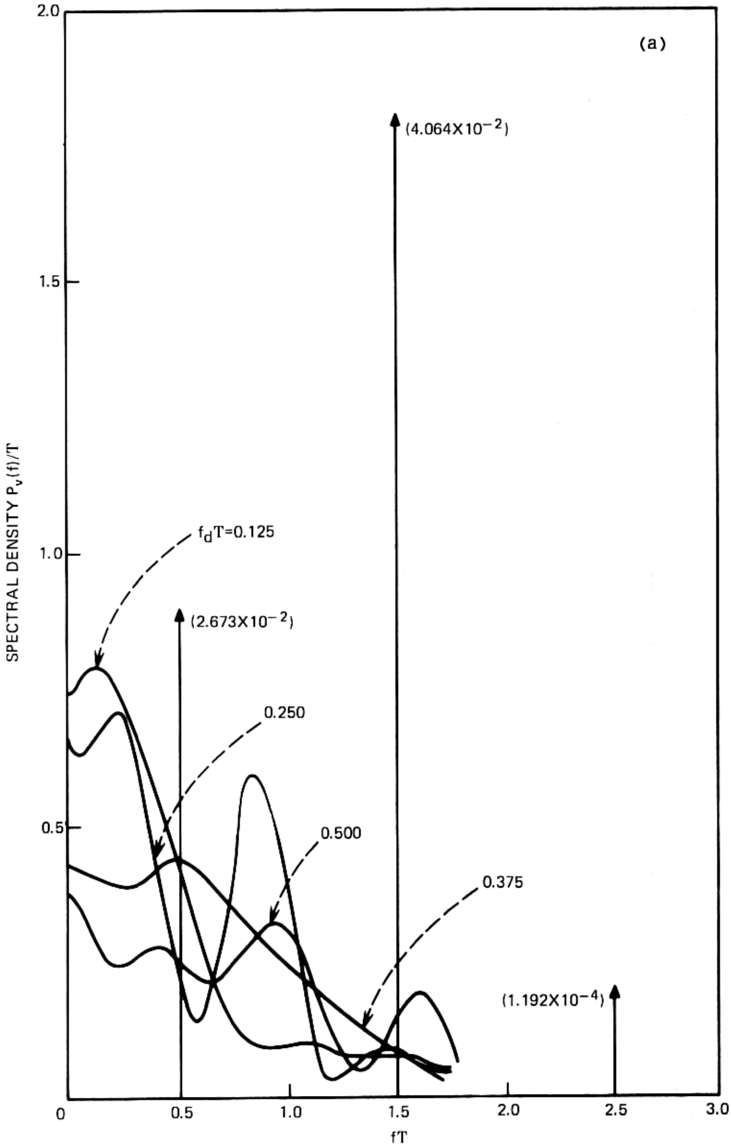

Fig. 6—Spectral density of quaternary FSK system with raised-cosine signaling and pulse duration $2T$. $K = 2.2f_d$ is the spacing between two adjacent *a priori* chosen frequencies.

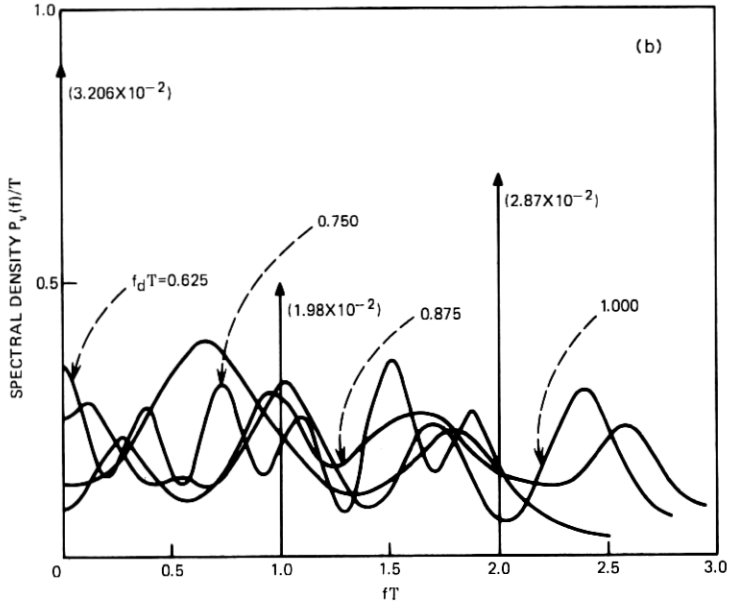

Fig. 6 (continued).

7.2 Raised cosine overlapping signal pulses: $K = 2$

If a raised-cosine signal pulse just fills up two time slots,

$$\mathbf{g}(t)] = \begin{cases} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_M \end{bmatrix} \pi f_d \left(1 + \cos \dfrac{\pi t}{T}\right), & -T < t \leqq T, \quad f_d > 0 \\ 0], & \text{otherwise}, \end{cases} \tag{125}$$

$K = 2$, and the spectral density may be calculated from Sections 4.2, V, or 6.2 according to whether

$$|\underline{\mathbf{w}}\cdot\mathbf{q}(U_2)]| = 1 \tag{126}$$

or

$$|\underline{\mathbf{w}}\cdot\mathbf{q}(U_2)| < 1. \tag{127}$$

If $\Delta_1 = 1$, $\Delta_2 = -1$, $\Delta_3 = 3$, $\Delta_4 = -3$, $\cdots$, $\Delta_{M-1} = (M-1)$, $\Delta_M = -(M-1)$, (116) can be shown to be satisfied if and only if $2f_dT$ is an integer. Further,

$$f_l = 0, \quad \text{if } f_dT = 1, 2, 3, \cdots, \tag{128}$$

and

$$f_l = \tfrac{1}{2}, \quad \text{if } f_dT = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \cdots. \tag{129}$$

In this case, note that

$$\llcorner\underline{\mathbf{w}}\lrcorner \cdot \mathbf{q}(T) \rrbracket = \frac{1}{M} \frac{\sin (2M\pi f_d T)}{\sin (2\pi f_d T)}. \tag{130}$$

In this case, the FSK spectrum contains line components, and the continuous and line spectra for $M = 2$ and 4 are shown in Figs. 5 and 6.

If $2f_dT$ is not an integer, the FSK spectrum does not contain any lines, and the continuous spectrum given by (97) to (98) for $M = 2$ and 4 is also plotted in Figs. 5 and 6.

Several observations can be made from Figs. 2 to 6. For both $K = 1$ and $K = 2$, discrete spectral lines appear as f_dT approaches the limiting value and the power in the lines is substantial for $K = 1$ and $M = 2$. Also note that power in the line components with $K = 2$ is smaller than the power in the lines with $K = 1$.

For the same value of f_dT and K, the principal portion of the spectrum of binary FSK is narrower than that of quaternary FSK. For $K = 1$, quaternary FSK spectrum is narrower than that of octonary FSK spectrum for the same value of f_dT.

Since lines can appear in the spectrum for a set of values of f_dT, the FSK spectrum is quite different from the PSK spectrum even when lines are present in the FSK spectrum.

VIII. SUMMARY AND CONCLUSIONS

Matrix methods are given to express the spectral density of a carrier, frequency-modulated by a random baseband pulse train, in a concise and computable form. Arbitrary pulse shapes may be used for M-ary digital signaling, and they may overlap over a finite number of signal intervals.

The spectral density is expressed as a compact Hermitian form suitable for numerical computation by a digital computer. The computer is readily programmed to perform matrix operations directly, rather than expanding the Hermitian form and evaluating the individual terms. In this way, quite complicated cases involving multilevel signal pulses overlapping several time slots may be treated simply.

Simple conditions in terms of the modulation parameters are given under which discrete spectral lines are present in the spectrum. A method is given to evaluate the power in the discrete spectral lines. The utility of the method is illustrated by giving several examples.

The present results are restricted to independent signal pulses, except for the very special case of nonoverlapping ($K = 1$) signal pulses with line spectra present [see footnote following (72)]. In Ref. 1 we saw that for overlapping signal pulses with $K = 2$ (i.e., signal pulses that occupy no more than two time slots), PSK with correlated signal pulses required the fourth-order statistics of the digital modulation process. FSK is more complicated; the correlated $K = 2$ case requires statistics of *all* orders for the modulation process, except in the line spectrum case. However, certain special cases with correlated signal pulses have been handled by extension of the present methods.

APPENDIX

Spectra of Complex and Real FSK Waves

From (130), (131) of Ref. 1, Appendix A, we see that (13) of the present paper holds true if

$$\overline{e^{j4\pi f_c t}\Phi_{vv^*}(t, \tau)} = 0, \tag{131}$$

where

$$\Phi_{vv^*}(t, \tau) = \langle v(t+\tau)v(t)\rangle = \langle e^{j[\phi(t+\tau)+\phi(t)]}\rangle. \tag{132}$$

From (3) and (9),

$$\Phi_{vv^*}(t, \tau) = \left\langle e^{j2\phi(0)} \exp\left(j \sum_{k=-\infty}^{\infty} \left[\int_{-kT}^{t+\tau-kT} + \int_{-kT}^{t-kT}\right] h_{s_k}(\mu) d\mu\right)\right\rangle. \tag{133}$$

If $\phi(0)$ is independent of the modulation parameters s_k, (133) becomes

$$\Phi_{vv^*}(t, \tau) = \langle e^{j2\phi(0)}\rangle \left\langle \exp\left(j \sum_{k=-\infty}^{\infty} \left[\int_{-kT}^{t+\tau-kT} + \int_{-kT}^{t-kT}\right] h_{s_k}(\mu) d\mu\right)\right\rangle. \tag{134}$$

Thus, if (11) holds, the first factor in (134) = 0, and (131) and, hence, (13) follow immediately.

If, instead, (10) holds, and imposing in addition the independence of the s_k, (134) becomes

$$\Phi_{vv^*}(t, \tau) = \prod_{k=-\infty}^{\infty} \left\langle \exp\left(j \left[\int_{-kT}^{t+\tau-kT} + \int_{-kT}^{t-kT}\right] h_{s_k}(\mu) d\mu\right)\right\rangle. \tag{135}$$

For large enough $|t|$ and finite pulse length, the integrals in the exponent fall into three classes, depending on k:

(*i*) Both limits on both integrals lie to one side of the signal pulses. The integrals equal 0 and the corresponding factor in the infinite product equals 1 and, hence, may be ignored.

(*ii*) The limits on integrals straddle the pulse, and the limits may consequently be replaced by $\int_{L_K}^{U_K}$, where the maximum pulse length is KT and L_K, U_K are given in (31).

(*iii*) The common lower limit, or one or both upper limits, lie within the pulse.

Therefore, for large enough $|t|$ (depending on the pulse length and on τ)

$$\Phi_{vv^*}(t+T,\tau) = \Phi_{vv^*}(t,\tau)\left\langle \exp\left(j2\int_{L_K}^{U_K} h_{s_k}(\mu)d\mu\right)\right\rangle, \quad t \text{ sufficiently positive.} \tag{136}$$

$$\Phi_{vv^*}(t-T,\tau) = \Phi_{vv^*}(t,\tau)\left\langle \exp\left(-j2\int_{L_K}^{U_K} h_{s_k}(\mu)d\mu\right)\right\rangle, \quad t \text{ sufficiently negative.} \tag{137}$$

Condition (12) may be substituted for condition (10) with only minor changes in the above discussion, and identical results (136) and (137). From (44),

$$\mathbf{q}(U_K)] = \begin{bmatrix} \exp\left(j\int_{L_K}^{U_K} h_1(\mu)d\mu\right) \\ \exp\left(j\int_{L_K}^{U_K} h_2(\mu)d\mu\right) \\ \vdots \\ \exp\left(j\int_{L_K}^{U_K} h_M(\mu)d\mu\right) \end{bmatrix}; \tag{138}$$

the integrals in the exponents are the areas of the signal pulses. We consider the two cases of Sections V and VI:

(*i*) All pulse areas are identical (mod 2π); line components are present in the spectrum.

(*ii*) The contrary; no line components are present.

In case (*i*), from (6)

$$\mathbf{q}(U_K)] = e^{j2\pi f_l}\cdot\mathbf{1}], \quad -\tfrac{1}{2} < f_l \leqq \tfrac{1}{2}; \tag{139}$$

$2\pi f_l$ is the common area (mod 2π) of all the signal pulses and $\mathbf{1}]$ is the unit column vector of (23). It therefore follows that the final factors of (136) and (137) satisfy the following:

$$\left\langle \exp\left(\pm j2\int_{L_K}^{U_K} h_{s_k}(\mu)d\mu\right)\right\rangle = e^{\pm j4\pi f_l}, \quad \text{line spectrum present.} \tag{140}$$

$$\left|\left\langle \exp\left(\pm j2\int_{L_K}^{U_K} h_{s_k}(\mu)d\mu\right)\right\rangle\right| < 1, \quad \text{line spectrum absent.} \tag{141}$$

With line spectra absent, (141), (136), and (137) show immediately that (131) holds for all f_c, and, hence, condition (10) or (12) guarantees the result of (13) for independent s_k without further restriction.

With line components present, (140), (136), and (137) show that $\Phi_{vv^*}(t,\tau)$ regarded as a function of t contains a periodic (line) component. Therefore, we write

$$\Phi_{vv^*}(t,\tau) = \Phi_{vv^*}(t,\tau)_l + \Phi_{vv^*}(t,\tau)_c, \tag{142}$$

where

$$\Phi_{vv^*}(t,\tau)_c = 0, \quad |t| \text{ sufficiently large,} \tag{143}$$

and

$$\Phi_{vv^*}(t,\tau)_l = e^{j4\pi f_l(t/T)} \sum_{n=-\infty}^{\infty} \varphi_n(\tau) e^{jn2\pi(t/T)}. \tag{144}$$ §

Only the line component (144) can possibly contribute to the average on the left-hand side of (131):

$$\overline{e^{j4\pi f_c t}\Phi_{vv^*}(t,\tau)}$$
$$= \sum_{n=-\infty}^{\infty} \varphi_n(\tau) \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} e^{j2\pi[2f_c+(2f_l+n)/T]t} dt$$
$$= \sum_{n=-\infty}^{\infty} \varphi_n(\tau) \lim_{A\to\infty} \frac{\sin 2\pi[2f_c+(2f_l+n)/T]A}{2\pi[2f_c+(2f_l+n)/T]A}. \tag{145}$$

The $\lim_{A\to\infty} \to 0$ if $2f_c + (2f_l + n)/T \neq 0$ for every integer n, thus satisfying (131). Hence, condition (10) or (12) and

$$2\left(f_c + \frac{f_l}{T}\right) \neq \frac{n}{T}, \quad n = 0, 1, 2, 3, \cdots, \tag{146}$$

guarantees the result of (13) for independent s_k. Since, by (72), the line spectral components of $v(t)$ occur at $(f_l + n)/T$, (146) is equivalent to requiring that the line components of $v(t)e^{j2\pi f_c t}$ and of $v^*(t)e^{-j2\pi f_c t}$ never coincide. Note that condition (5) of Ref. 1 is the special case of (146) above for $f_l = 0$.

Finally, (135) to (140) of Ref. 1, Appendix A apply also in the present case, and establish that (13) is a good approximation if f_c is high enough so that there is no significant overlap between the two terms of (13).

REFERENCES

1. V. K. Prabhu and H. E. Rowe, "Spectra of Digital PM by Matrix Methods," B.S.T.J., *53*, No. 5 (May–June 1974), pp. 899–935.
2. W. R. Bennett and S. O. Rice, "Spectral Density and Autocorrelation Functions Associated with Binary Frequency Shift Keying," B.S.T.J., *42*, No. 5 (September 1963), pp. 2355–2385.
3. R. R. Anderson and J. Salz, "Spectra of Digital FM," B.S.T.J., *44*, No. 6 (July–August 1965), pp. 1165–1189.
4. J. E. Mazo and J. Salz, "Spectra of Frequency Modulation with Random Waveforms," Information and Control, *9*, No. 4 (August 1966), pp. 414–422.
5. O. Shimbo, "General Formula for Power Spectra of Digital FM Signals," Proc. IEE (GB), *113*, No. 11 (November 1966), pp. 1783–1789.
6. F. A. Graybill, *Introduction to Matrices with Applications in Statistics*, Belmont, Calif.: Wadsworth, 1969, Section 8.8.
7. D. E. Cartier, "The Power Spectrum of PCM/FSK-AM/PM," IEEE Trans. on Comm., *COM-21*, No. 7 (July 1973), pp. 847–850.

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§ These φ_n are generalizations of those in (133) of Ref. 1, Appendix A.