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Nonstationary Blocking in Telephone Traffic

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Blocking is considered for an N-trunk group of exponential servers with Poisson-offered load whose rate parameter varies with time. The infinite trunk case is solved by means of a rapidly convergent series of Poisson-Charlier polynomials. This solution is used to obtain practical approximations of blocking probability, transition probabilities, and recovery function for general time-variable offered load in the finite trunk-group case. An integral equation is derived satisfied by the blocking probability in the general case. In the situation of constant offered load, two additional methods are derived for providing easily computable approximations; one based on the integral equation, the other based on an approximate inversion of the Laplace transform. To aid in the latter approximation, bounds on the roots of Poisson-Charlier polynomials are obtained; in particular, an approximation is obtained for the dominant root. The inversion of the integral equation is studied with the purpose of providing the basis for future investigations of errors of approximation. Curves are provided for a number of examples permitting comparison of exact and approximate solutions.

I. INTRODUCTION

The main purpose of this paper is to present a discussion of the time behavior of blocking in a fully available N-trunk group for any initial state with exponential servers and with Poisson-offered load whose rate parameter $a(t)$ itself may be considered to vary with time; that is, the probability of j calls arriving in a time interval $(0, t)$ is assumed given by

$$\exp\left(-\int_0^t a(u)du\right)\frac{\left[\int_0^t a(u)du\right]^j}{j!}, \qquad j = 0, 1, 2, \cdots. \tag{1}$$

The service rate is taken equal to 1, so that $a(t)$ is measured in erlangs.

The problem of blocking with time-variable offered load was considered by Palm[1] for finite trunk groups and by Khintchine[2] for infinite trunk groups. The impetus for this is the need felt for more accurate computation of blocking probabilities and correlation information[3] than can be obtained by quasi-stationary analyses, that is, by the use of equilibrium formulas in which the offered load parameter is replaced by its instantaneous value. Lack of statistical equilibrium renders this approach inaccurate. The time-variable aspect of the input stream should be carefully distinguished from other statistical descriptions such as peakedness,[4] since the effects on the system are separately identifiable. It has been reported, for example, that offered load and peakedness determinations from carried usage, peg count attempts offered to a group, and overflows are misled by the time variability of the offered load.

Palm had proposed an interesting method of accounting for the time variability of the offered load, i.e., his "slow variations" model. In this model, it was assumed that the actual ordering in time of $a(t)$, that is, the functional dependence of $a(t)$ on t could be ignored if $a(t)$ varied slowly. He replaced $a(t)$ by a random variable with an incomplete gamma function distribution.[1] Thus, traffic functions such as blocking may be obtained from their equilibrium values by averaging over the appropriate gamma distribution. This model, however, requires further elaboration in view of the investigations of Iversen,[3] who showed that the Palm approach does not correctly model the empirical data collected in the extensive Holbaek measurements of Danish telephone traffic. Iversen found that the correct time variation of the traffic could not be ignored.

The trunk provisioning procedure whereby one uses the average offered load over a busy hour to achieve a required grade of service results, in some cases that were considered, in only a small underestimate of the required number of trunks as calculated by the methods of this paper. Since the standard method is convenient, this may be viewed as substantiation of the approach.

Essential for the methods of this paper is a Volterra integral equation derived in Section II satisfied by the blocking probability, $P_N(t, N)$, experienced by a load of $a(t)$ erlangs offered to an N-trunk group. Exact analytical solution of this equation is not useful, but numerical methods may be advantageously used. An important feature of the equation, nonetheless, is that it permits studying errors of approximation and, in one instance (Appendix D), was directly used in the construction of an approximate operator for studying the transient response in the case of constant $a(t)$. Appendix A presents an explicit representation of $P_N(t, N)$ for general $a(t)$ by means of an infinite Neumann expansion. Inequalities for $P_N(t, N)$ and truncation error estimates for the Neumann series are also given.

The infinite trunk group case forms the basis of the approximations developed in Section IV that are applicable to the general case of time variable $a(t)$. Although this case was solved by Khintchine,[2] Section III presents new representations in terms of rapidly convergent Poisson-Charlier expansions. Truncation error estimates are obtained, and the rate at which the state probabilities approach the Poisson form is assessed. To aid in the use of Poisson-Charlier polynomials, Appendix B provides a short discussion of their properties, especially providing convenient means of expanding a function into a Poisson-Charlier series. Since the state probabilities of the infinite trunk group system are often close to Poisson, this form of representation is very useful. The Poisson-Charlier expansion expresses the deviation of a function from the Poisson form. Further, in Section IV, the Poisson-Charlier polynomials are used to express the transition probabilities in explicit, closed form.

The approximations of Section IV are applicable to time variable $a(t)$, and are developed from the infinite trunk group solution by renormalization appropriate to the finite trunk group. To facilitate the use of the approximations, closed expressions are obtained for the relevant infinite trunk group solutions. This approximation procedure gives rise to the useful notion of a "modified offered load." One of the approximations obtained was, in fact, already obtained by Palm.[1] This approximation is particularly interesting because it uses the Erlang loss function, $B(N, a)$, for which rapid methods of computation are available.[5,6] A special case of the approximations for transition probabilities is that for the recovery function, which is important in the discussion of correlation properties[7] and, hence, in the determination of variances of traffic parameter estimators.

The constant offered load case is studied in Appendix C. Although the solution for the state probabilities is known,[8] the integral equation formulation appears to be new. Certain advantages are obtainable from this formulation. The errors of approximations to the state probabilities satisfy the same integral equation but with a different inhomogeneous term; thus, the more general integral equation is studied, leading to methods for investigating the quality of approximation. For this purpose, a natural Banach space (uniform norm over $[0, \infty]$) is introduced, in which the integral operator is bounded and has a bounded inverse. Of course, the known Laplace transform of the transition probabilities is immediately obtained as a corollary. The integral equation is also used in Appendix D as a tool for the construction of an approximate solution (the scaling method) corresponding to an arbitrary initial state. Appendix C also presents several bounds on the required roots of Poisson-Charlier polynomials by obtaining general bounds on the largest and smallest members of sets of positive numbers subject to isoperimetric constraints; the results are then specialized to the Poisson-Charlier polynomials. One of the bounds for the dominant root is explicit and easily calculable. Its accuracy appears to be good (see Fig. 8).

Appendix D provides two approximations for the case of constant offered load. The scaling approximation, whose genesis was suggested by S. Horing, is constructed by obtaining an approximate time invariant of the transition blocking probability from the initially empty state. The subsequent generalization of this approximation to arbitrary initial states is then obtained by means of the integral equation. The Laplace transform approximation is constructed by an adaptation of the Widder formula[9] for the inversion of the Laplace transform. It requires the determination of the dominant root; but, depending on the needs, it may be made arbitrarily accurate. It has the interesting property that, under certain conditions, it provides bounds for the exact solution.

A discussion of some results and graphical illustrations is given in Section V. In testing the quality of the modified offered load approximations, high change rates of $a(t)$ were chosen, in fact, much higher than would occur in practice. The errors of approximation increase with increasing rate of change of $a(t)$, hence, the examples chosen indicate much higher errors than one would expect to encounter in the practical application of these methods.

II. INTEGRAL EQUATION FOR BLOCKING

It is the object of this section to establish Theorems 1 and 2, which provide integral representations of the binomial moments (13) of the probability distribution (2) of the number of busy trunks, and corollary 1 of Theorem 2, in which an integral equation is given for the probability that all trunks are busy at a given time.

Let $\xi = \xi(t, N)$ be the number of trunks busy at time t in an N-trunk group, and $P_j = P_j(t, N)$ the corresponding probability,

$$P[\xi(t, N) = j] = P_j(t, N). \tag{2}$$

The probability generating function $g(t, \zeta, N)$ is given by

$$g = E\zeta^{\xi(t,N)} = \sum_{j=0}^{N} P_j(t, N)\zeta^j. \tag{3}$$

At the point $t + dt$, one has

$$g(t + dt, \zeta, N) = g + \partial g, \qquad \xi(t + dt, N) = \xi + d\xi \tag{4}$$

and, hence,

$$g + \partial g = E_\xi\{\zeta^\xi E[\zeta^{d\xi} \mid \xi]\}, \tag{5}$$

in which $E[\zeta^{d\xi} \mid \xi]$ is the conditional expectation of $\zeta^{d\xi}$ given ξ, and E_ξ is the expectation over the probability distribution of ξ. The boundary at $j = N$ necessitates a further analysis of (5). One has

$$g + \partial g = (1 - P_N)E_\xi\{\zeta^\xi E[\zeta^{d\xi} \mid \xi < N]\} + P_N E_\xi\{\zeta^\xi E[\zeta^{d\xi} \mid \xi = N]\}. \tag{6}$$

The probability distribution of $d\xi$ is

$$\begin{aligned} &P[d\xi = -1 \mid \xi = j, 0 \leqq j < N] = jdt, \\ &P[d\xi = 0 \mid \xi = j, 0 \leqq j < N] = 1 - (a + j)dt, \\ &P[d\xi = 1 \mid \xi = j, 0 \leqq j < N] = adt, \\ &P[d\xi = 0 \mid \xi = N] = 1 - Ndt, \\ &P[d\xi = -1 \mid \xi = N] = Ndt; \end{aligned} \tag{7}$$

hence,

$$\begin{aligned} &E[\zeta^{d\xi} \mid \xi = j, 0 \leqq j < N] = 1 + (a\zeta - a - j + j\zeta^{-1})dt, \\ &E[\zeta^{d\xi} \mid \xi = N] = 1 - N(1 - \zeta^{-1})dt. \end{aligned} \tag{8}$$

From (6) and (8), one has

$$\frac{\partial g}{\partial t} = (1 - P_N)E_\xi[\zeta^\xi(a\zeta - a - \xi + \xi\zeta^{-1}) \mid \xi < N] - P_N E_\xi[\xi\zeta^{\xi-1}(\zeta - 1) \mid \xi = N], \tag{9}$$

$$\frac{\partial g}{\partial t} = (1 - P_N)a(\zeta - 1)E_\xi[\zeta^\xi \mid \xi < N] - (\zeta - 1)E_\xi[\xi\zeta^{\xi-1}];$$

hence,

$$\frac{\partial g}{\partial t} + (\zeta - 1)\frac{\partial g}{\partial \zeta} = a(\zeta - 1)g - a(\zeta - 1)\zeta^N P_N. \tag{10}$$

The infinite trunk group does not require the analysis of (6) nor the boundary conditions ($\xi = N$) of (7), hence (9) becomes

$$\frac{\partial g}{\partial t} = a(\zeta - 1)g - (\zeta - 1)E_\xi[\xi\zeta^{\xi-1}]. \tag{11}$$

Thus, the corresponding equation for the infinite trunk group is

$$\frac{\partial g}{\partial t} + (\zeta - 1)\frac{\partial g}{\partial \zeta} = a(\zeta - 1)g. \tag{12}$$

The binomial moments $\beta_s(t, N)$ are defined by

$$\beta_s(t, N) = \sum_{j=s}^{\infty} P_j(t, N)\binom{j}{s}, \tag{13}$$

in which $\binom{j}{s}$ designates the binomial function defined by

$$\binom{j}{s} = \frac{j(j-1)\cdots(j-s+1)}{s!}, \qquad s \geqq 1, \tag{14}$$
$$\binom{j}{0} = 1.$$

The binomial moment generating function $l(t, w, N) = \sum_{s=0}^{\infty} \beta_s(t, N)w^s$ is given by

$$l(t, w, N) = g(t, 1 + w, N). \tag{15}$$

Thus, the differential equation satisfied by l is

$$\frac{\partial l}{\partial t} + w\frac{\partial l}{\partial w} = awl - aw(1 + w)^N P_N. \tag{16}$$

The corresponding equation for the infinite trunk group is

$$\frac{\partial l}{\partial t} + w\frac{\partial l}{\partial w} = awl. \tag{17}$$

Equation (16) is a linear partial differential equation that can be solved by the following device (method of characteristics). Let θ be a new, independent variable and set

$$\begin{aligned} l &= l(\theta), \\ w &= w(\theta), \\ t &= t(\theta). \end{aligned} \tag{18}$$

Then, comparison of

$$\frac{dl}{d\theta} = \frac{\partial l}{\partial t}\frac{dt}{d\theta} + \frac{\partial l}{\partial w}\frac{dw}{d\theta} \tag{19}$$

with (16) yields the set of equations

$$\begin{aligned} \frac{dl}{d\theta} &= aw - aw(1 + w)^N P_N, \\ \frac{dw}{d\theta} &= w, \\ \frac{dt}{d\theta} &= 1, \end{aligned} \tag{20}$$

whose solution for l is then obtained. To exhibit the solution conveniently, let

$$\alpha(t, \tau) = e^{-t}\int_{\tau}^{t} e^s a(s)ds, \tag{21}$$

and

$$\Lambda = \Lambda(t) = \alpha(t, 0) = e^{-t} * a(t), \tag{22}$$

where $*$ designates convolution product over $(0, t)$. The solution of (16) takes the form

$$l(t, w, N) = l(0, we^{-t}, N)e^{\Lambda w} - w\int_0^t e^{\alpha w+\tau-t}[1 + we^{\tau-t}]^N a(\tau)P_N(\tau, N)d\tau. \tag{23}$$

Similarly, the solution of (17) is

$$l(t, w, \infty) = l(0, we^{-t}, \infty)e^{\Lambda w}. \tag{24}$$

To obtain the binomial moments themselves, it is convenient to introduce the Volterra operator K_s,

$$K_s f = \int_0^t K_s(t, \tau)f(\tau)d\tau, \tag{25}$$

defined by the kernel

$$K_s(t, \tau) = \sum_{j=0}^{s-1} \frac{\alpha^j}{j!}\binom{N}{s-1-j} e^{-(s-j)(t-\tau)}a(\tau). \tag{26}$$

Since the Laguerre polynomial $L_n^{(\alpha)}(-x)$[10] is given by

$$L_n^{(\alpha)}(-x) = \sum_{j=0}^{n} \frac{x^j}{j!}\binom{n+\alpha}{n-j}, \tag{27}$$

the kernel $K_s(t, \tau)$ may be written more compactly; thus,

$$K_s(t, \tau) = a(\tau)e^{-s(t-\tau)}L_{s-1}^{(N-s+1)}(-\alpha e^{t-\tau}). \tag{28}$$

The following theorems may now be stated.

Theorem 1: The binomial moments, $\beta_s(t, \infty)$, for the infinite trunk group are given by

$$\beta_s(t, \infty) = \sum_{j=0}^{s} \beta_{s-j}(0, \infty)e^{-(s-j)t}\frac{\Lambda^j}{j!}.$$

Proof: The coefficient of w^s in the expansion of the right-hand side of (24) yields the result.

Theorem 2: The binomial moments, $\beta_s(t, N)$, for the finite trunk group satisfy

$$\beta_s(t, N) = \beta_s(t, \infty) - K_s P_N.$$

Proof: The coefficient of w^s in the expansion of the right-hand side of (23) provides the required result.

Corollary 1: The probability, $P_N(t, N)$, that all trunks are busy satisfies the integral equation

$$P_N(t, N) = \beta_N(t, \infty) - K_N P_N,$$

in which

$$\beta_N(t, \infty) = \sum_{j=0}^{N} \beta_{N-j}(0, N) e^{-(N-j)t} \frac{\Lambda^j}{j!}$$

$$K_N(t, \tau) = a(\tau) e^{-N(t-\tau)} L_{n-1}^{(1)}(-\alpha e^{t-\tau}).$$

Proof: For the finite trunk group (13) shows that

$$\beta_N(t, N) = P_N(t, N).$$

Hence, the integral equation follows from Theorem 2. The explicit expressions for $\beta_N(t, \infty)$ and $K_N(t, \tau)$ are obtained from Theorem 1 and (28), respectively.

The special case of all trunks free initially leads to a somewhat simpler integral equation for $P_N(t, N)$. This is given in the following corollary.

Corollary 2: When all trunks are initially free, $P_N(t, N)$ satisfies

$$P_N(t, N) = \frac{\Lambda^N}{N!} - K_N P_N.$$

Proof: The initial probability distribution $P_j(0, N)$ has the form

$$\begin{aligned} P_j(0, N) &= 1 \qquad j = 0, \\ &= 0 \qquad j > 0. \end{aligned} \tag{29}$$

Hence, the binomial moments satisfy

$$\begin{aligned} \beta_s(0, N) &= 1 \qquad s = 0, \\ &= 0 \qquad s > 0. \end{aligned} \tag{30}$$

The equation for $\beta_N(t, \infty)$ given in the first corollary now yields

$$\beta_N(t, \infty) = \frac{\Lambda^N}{N!}. \tag{31}$$

The result of the corollary follows.

The probability $P_N(t, N)$ corresponding to all trunks initially busy is called the recovery function; it satisfies the following integral equation.

Corollary 3: When all trunks are initially busy, one has

$$P_N(t, N) = e^{-Nt} L_N(-e^t \Lambda) - K_N P_N.$$

Proof: We have

$$P_j(0, N) = 0 \qquad 0 \leqq j < N,$$
$$= 1 \qquad j = N. \tag{32}$$

Hence, the required binomial moments are

$$\beta_s(0, N) = \binom{N}{s}. \tag{33}$$

The result follows from the equation for $\beta_N(t, \infty)$ and from (27).

A noteworthy case occurs when a is constant, then the integral equation for $P_N(t, N)$ becomes of convolution type. Thus, let

$$K_N(t) = ae^{-Nt}L_{N-1}^{(1)}[-a(e^t - 1)]. \tag{34}$$

Then one has Corollary 4.

Corollary 4: When the offered load a is constant, $P_N(t, N)$ satisfies

$$P_N(t, N) = \beta_N(t, \infty) - K_N * P_N.$$

Proof: Use of (21) and $K_N(t, \tau)$ as given in Corollary 1 yields the result.

For constant a, an equilibrium distribution $P_j(\infty, N)$ exists.[2] Let

$$S_N(a) = \sum_{k=0}^{N} \frac{a^k}{k!}. \tag{35}$$

Then

$$P_j(\infty, N) = \frac{a^j}{j!\,S_N(a)}, \qquad 0 \leqq j \leqq N. \tag{36}$$

The notation $B(N, a)$ is used for the blocking probability $P_N(\infty, N)$ and is referred to as Erlang's loss formula.[6] Corresponding to the equilibrium distribution, one has the binomial moments $\beta_s(0, N)$ and, hence, the moments for the infinite trunk group given in Theorem 1. These moments will be denoted by $\beta_s^e(t, a)$. It may be noted that

$$\lim_{t \to \infty} \beta_s^e(t, a) = \frac{a^s}{s!}. \tag{37}$$

The integral of $K_N(t)$ that is useful in error analyses may now be easily obtained.

Theorem 3: When the offered load is constant, we have

$$\int_0^t K_N(\tau)d\tau = \frac{\beta_N^e(t, a)}{B(N, a)} - 1,$$

$$\int_0^\infty K_N(\tau)d\tau = S_N(a) - 1,$$

in which $K_N(\tau)$ is given in (34).

Proof: Since $B(N, a)$ is the equilibrium solution of the integral equation of Corollary 4, and since $\beta_N^e(t, a)$ corresponds to the equilibrium state, the solution of the integral equation is constant and equal to $B(N, a)$; thus,

$$B(N, a) + K_N * B(N, a) = \beta_N^e(t, a).$$

This immediately implies the first equation of the theorem. The second follows on considering $t \rightarrow \infty$, and using (36) and (37).

It may be useful to observe that the positive character of the general operator K_N in Corollary 1 immediately supplies the inequalities

$$\beta_N(t, \infty) - K_N \beta_N(t, \infty) < P_N(t, N) < \beta_N(t, \infty). \tag{38}$$

A Neumann-series solution of the integral equation of Corollary 1 is discussed in Appendix A. Higher-order inequalities of type (38) are also given.

III. THE INFINITE TRUNK SOLUTION

It will be convenient to express the solution for $P_x(t, \infty)$ in terms of Poisson-Charlier polynomials[11] whose relevant properties are discussed in Appendix B. The probability distribution of the number of busy trunks for the infinite trunk case was considered by Khintchine[2] and, for constant offered load, by Karlin and McGregor.[12] Theorem 4 presents a rapidly convergent form of the solution in terms of Poisson-Charlier polynomials valid for any initial state. This solution will be the main tool for the construction of approximations to distributions in the finite trunk case.

From (15), let $l(0, w, \infty)$ be given by

$$l(0, w, \infty) = \sum_{j=0}^{\infty} \beta_j(0, \infty) w^j, \tag{39}$$

then the binomial moment generating function for the infinite trunk case is, from (24),

$$l(t, w, \infty) = e^{\Lambda w} \sum_{j=0}^{\infty} \beta_j(0, \infty) e^{-jt} w^j. \tag{40}$$

The mean, μ, of this distribution is the coefficient of w; hence,

$$\mu = \Lambda + \mu_0 e^{-t}, \tag{41}$$

in which μ_0 is the mean of the initial distribution. One may now state Theorem 4.

Theorem 4: The probability distribution, $P_x(t, \infty)$, of the number of busy trunks in an infinite trunk group has the convergent representation

$$P_x(t, \infty) = \psi(x, \mu) \sum_{j=0}^{\infty} e^{-jt} d_j G_j(x, \mu),$$

$$d_j = \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \mu^\nu \beta_{j-\nu}(0, \infty),$$

provided that

$$l(0, z - 1, \infty) = \sum_{j=0}^{\infty} \beta_j(0, \infty)(z - 1)^j = \sum_{j=0}^{\infty} P_j(0, \infty) z^j$$

converges for $|z| < r (r > 2)$.

Proof: Use of (40) in (101) which the choice $\lambda = \mu$ as given in (41) yields

$$C(w) = e^{-\mu_0 w e^{-t}} \sum_{j=0}^{\infty} \beta_j(0, \infty) e^{-jt} w^j. \tag{42}$$

Hence,

$$c_j = e^{-jt} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \mu_0^\nu \beta_{j-\nu}(0, \infty) \tag{43}$$

and, from (95),

$$P_x(t, \infty) = \psi(x, \mu) \sum_{j=0}^{\infty} c_j G_j(x, \mu). \tag{44}$$

Let

$$d_j = \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \mu_0^\nu \beta_{j-\nu}(0, \infty), \tag{45}$$

then

$$c_j = e^{-jt} d_j, \tag{46}$$

and the formula of the theorem follows. A theorem of Uspensky[13] states that the general representation of (95) is valid in the sense that the series converges to $F(x)$ if the series $\sum_{x=0}^{\infty} F(x) z^x$ has radius of convergence greater than 2. Since

$$l(t, z - 1, \infty) = e^{-\Lambda + \Lambda z} l[0, (z - 1)e^{-t}, \infty], \tag{47}$$

the radius of convergence of $l(t, z - 1, \infty)$ is greater than 2 by the condition on $l(0, z - 1, \infty)$ stated in the theorem. Hence, by Uspensky's theorem, the representation of (44) is valid for all $t \geqq 0$.

A truncation error estimate for the series representation of Theorem 4 is given in Theorem 5.

Theorem 5:

$$|P_x(t, \infty) - \psi(x, \mu) - \psi(x, \mu) \sum_{j=2}^{k} e^{-jt} d_j G_j(x, \mu)|$$
$$\leqq \left(\frac{2}{w}\right)^{k+1} \frac{e^{-(k+1)t}}{1 - (2/w)e^{-t}} e^{\mu_0 w} l(0, w, \infty),$$
$$k \geqq 1, \qquad R > w > 0, \qquad t > \ln \frac{2}{w},$$

in which R is the radius of convergence of $l(0, w, \infty)$.

Proof: Since

$$|\psi(x, \mu) G_j(x, \mu)| = |\psi^{(j)}(x, \mu)| \leqq 2^j, \qquad j \geqq 0, \tag{48}$$

we have

$$\sum_{j>k} e^{-jt} |d_j| \, |\psi(x, \mu) G_j(x, \mu)| \leqq \sum_{j>k} e^{-jt} 2^j |d_j|. \tag{49}$$

Also, from (45),

$$|d_j| \leqq \sum_{j=0}^{\nu} \frac{\mu_0^\nu}{\nu!} \beta_{j-\nu}(0, \infty) \leqq e^{\mu_0 w} l(0, w, \infty) w^{-j}. \tag{50}$$

Thus,

$$\sum_{j>k} e^{-jt} |d_j| \, |\psi(x, \mu) G_j(x, \mu)| \leqq e^{\mu_0 w} l(0, w, \infty) \sum_{j>k} e^{-jt} 2^j w^{-j} \tag{51}$$

$$\leqq e^{\mu_0 w} l(0, w, \infty) \left(\frac{2}{w}\right)^{k+1} \frac{e^{-(k+1)t}}{1 - (2/w)e^{-t}}. \tag{52}$$

The conditions of the theorem ensure the convergence of $l(0, w, \infty)$ and of the series of (51).

The corollaries below follow directly under the conditions of Theorem 5.

Corollary 1:

$$P_x(t, \infty) = \psi(x, \mu) \sum_{j=0}^{k} e^{-jt} d_j G_j(x, \mu) + 0(e^{-(k+1)t}).$$

Corollary 2:

$$|P_x(t, \infty) - \psi(x, \mu)| \leqq \frac{4}{w^2} \frac{e^{-2t}}{1 - (2/w)e^{-t}} e^{\mu_0 w} l(0, w, \infty).$$

Corollary 3:

$$P_x(t, \infty) = \psi(x, \mu) + 0(e^{-2t}).$$

Thus, the distribution quickly becomes nearly Poisson with the time variable parameter μ, regardless of the initial distribution. In fact, if the initial distribution is Poisson, one might anticipate $P_x(t, \infty)$ to remain Poisson for all $t \geqq 0$. This is asserted by Theorem 6.

Theorem 6: *If $P_x(0, \infty)$ is Poisson with parameter μ_0, then*

$$P_x(t, \infty) = \psi(x, \mu).$$

Proof: The binomial moments, $\beta_s(0, \infty)$, are

$$\beta_s(0, \infty) = \frac{\mu_0^s}{s!}, \qquad s \geqq 0. \tag{53}$$

It follows, from (45), that

$$d_j = \frac{\mu_0^j}{j!} \sum_{\nu=0}^{j} \binom{j}{\nu} (-1)^\nu = 0, \qquad j > 0. \tag{54}$$

The result is now obtained from Theorem 3.

IV. APPROXIMATIONS

The Neumann series (75) is an explicit solution of the integral equation of blocking given in Corollary 1, Theorem 2. For constant offered load a, the resolvent kernel solution (108) and Theorem 11 are also available; however, especially when N is large, these solutions are not convenient. It is therefore important to have approximations that lend themselves to computation for large N in a sufficiently convenient manner. Three such approximations have been developed, namely: the "modified offered load" approximation that is useful and fairly accurate in the general case, that is, for time variable offered load, the "scaling" approximation, and the "Laplace transform" approximation, which are applicable only to transient phenomena under constant offered load. The scaling approximation is also fairly accurate and does not require factorization of polynomials. The Laplace transform approximation consists in fact of an infinite set of approximations of arbitrarily high accuracy. It usually requires finding a single root—the so-called dominant root—of an appropriate polynomial. The scaling and Laplace transform approximations are discussed in Appendix D. Appendix C provides approximations for the required dominant root. The modified offered load approximation is presented below.

Let $P_{i,x}(t, N)$ be the probability that the N-trunk group started from state i at time 0 and proceeded to state x at time t, then $P_{i,x}(t, \infty)$ may be computed from Theorem 4 using

$$\beta_j(0, \infty) = \binom{i}{j}. \tag{55}$$

An approximation, $\bar{P}_{i,x}(t, N)$, for $P_{i,x}(t, N)$ is given by

$$\bar{P}_{i,x}(t, N) = \frac{P_{i,x}(t, \infty)}{\sum_{\nu=0}^{N} P_{i,\nu}(t, \infty)}. \tag{56}$$

This approximation is suggested by the following considerations. One has

$$\lim_{N\to\infty} \bar{P}_{i,x}(t, N) = P_{i,x}(t, \infty). \tag{57}$$

Hence, the approximation should be accurate even for time-varying offered load when N is large. Furthermore, when a is constant, since $\lim_{t\to\infty} \mu = a$, one has

$$\lim_{t\to\infty} \bar{P}_{i,x}(t, N) = \frac{a^x}{x!\,S_N(a)}, \tag{58}$$

which, as previously indicated, is the exact equilibrium distribution; hence, the approximation should be accurate when t is large even for time-varying $a(t)$, provided $\dot{a}(t)$ is small. Since, by the law of total probability,

$$P_x(t, N) = \sum_{i=0}^{N} P_i(0, N) P_{i,x}(t, N), \tag{59}$$

one can construct an approximation, $\bar{P}_x(t, N)$, to $P_x(t, N)$ by use of $\bar{P}_{i,x}(t, N)$; thus,

$$\bar{P}_x(t, N) = \sum_{i=0}^{N} P_i(0, N) \bar{P}_{i,x}(t, N). \tag{60}$$

To facilitate the use of (60), Theorem 7 expresses $P_{i,x}(t, \infty)$ in finite form.

Theorem 7:

$$P_{i,x}(t, \infty) = (1 - e^{-t})^i e^{-\Lambda} \frac{(-\Lambda)^x}{x!} G_x[i, -(e^t - 1)\Lambda].$$

Proof: From Theorem 4 and (55), we have

$$P_{i,x}(t, \infty) = \psi(x, \mu) \sum_{j=0}^{\infty} e^{-jt} d_j G_j(x, \mu), \tag{61}$$

$$d_j = \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} i^\nu \binom{i}{j-\nu}. \tag{62}$$

Comparison of (62) with (90) shows that

$$d_j = \frac{i^j}{j!} G_j(i, i). \tag{63}$$

Let

$$C_j = \frac{(ie^{-t})^j}{j!} G_j(i, i), \tag{64}$$

then

$$C(w) = \sum_{j=0}^{\infty} \frac{(ie^{-t})^j}{j!} G_j(i, i)w^j, \tag{65}$$

which, by comparison with (89), may be rewritten

$$C(w) = e^{-ie^{-t}w}(1 + e^{-t}w)^i. \tag{66}$$

Let

$$B(w) = e^{\mu w}C(w) = e^{\Lambda w}(1 + e^{-t}w)^i \tag{67}$$

and

$$g(z) = B(z - 1) = e^{-\Lambda}e^{\Lambda z}(1 - e^{-t} + e^{-t}z)^i. \tag{68}$$

Then

$$P_{i,x}(t, \infty) = (1 - e^{-t})^i(e^t - 1)^{-x}e^{-\Lambda} \sum_{\nu=0}^{x} \binom{i}{x - \nu} \frac{[(e^t - 1)\Lambda]^\nu}{\nu!}. \tag{69}$$

Hence, comparison of (69) with (90) finally yields

$$P_{i,x}(t, \infty) = (1 - e^{-t})^i e^{-\Lambda} \frac{(-\Lambda)^x}{x!} G_x[i, -(e^t - 1)\Lambda]. \tag{70}$$

Immediate corollaries are the following:

Corollary 1:

$$P_x(t, \infty) = e^{-\Lambda} \frac{(-\Lambda)^x}{x!} \sum_{i=0}^{\infty} P_i(0, \infty)(1 - e^{-t})^i G_x[i, -(e^t - 1)\Lambda].$$

Corollary 2:

$$\bar{P}_{i,x}(t, N) = \frac{G_x[i, -(e^t - 1)\Lambda]}{\sum_{\nu=0}^{N} (x!/\nu!)(-\Lambda)^{\nu-x}G_\nu[i, -(e^t - 1)\Lambda]}.$$

Of particular interest are the functions $P_{0,N}(t, N)$ and $P_{N,N}(t, N)$; the first describes the progression of the system from initially empty to blocked, and the second describes the recovery of the system from an initially blocked condition to the blocked condition again. The latter function is called the "recovery function."[7] The following formulas are obtained from Corollary 2.

$$\bar{P}_{0,N}(t, N) = B(N, \mu). \tag{71}$$

The general principle of approximation employed, namely, the renormalization of an appropriate solution for the infinite trunk group case, allows one to state, by use of Theorem 6, that, whenever one starts from an approximately Poisson initial state, an approximation to $P_N(t, N)$ is

$$P_N(t, N) = B(N, \mu). \tag{72}$$

This approximation was known to C. Palm.[1] The parameter μ is regarded as a modified offered load.

The recovery function approximation obtained from Corollary 2 is

$$\bar{P}_{N,N}(t, N) = \frac{G_N[N, -(e^t - 1)\Lambda]}{\sum_{\nu=0}^{N} (N!/\nu!)(-\Lambda)^{\nu-N} G_\nu[N, -(e^t - 1)\Lambda]}. \tag{73}$$

V. NUMERICAL EXAMPLES

For the purpose of providing some idea of the accuracy of the approximations developed in Section IV and Appendix D, curves were drawn up comparing exact and approximate solutions for a group of ten trunks. These curves illustrate nonstationary behavior. Figure 1 shows the scaling and modified offered load approximations for a step input problem in which $a = 7$ erlangs is offered to an initially empty group. The scaling approximation of (166) was used, and (71) was used for the modified offered load approximation. Apparently, for this situation, the scaling approximation is somewhat more accurate.

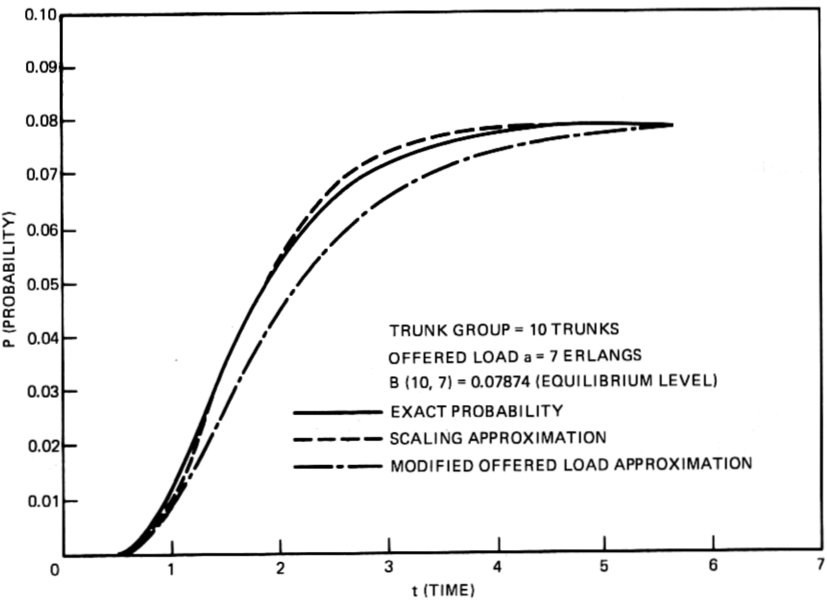

Fig. 1—All trunks empty initially—scaling and modified offered load approximations.

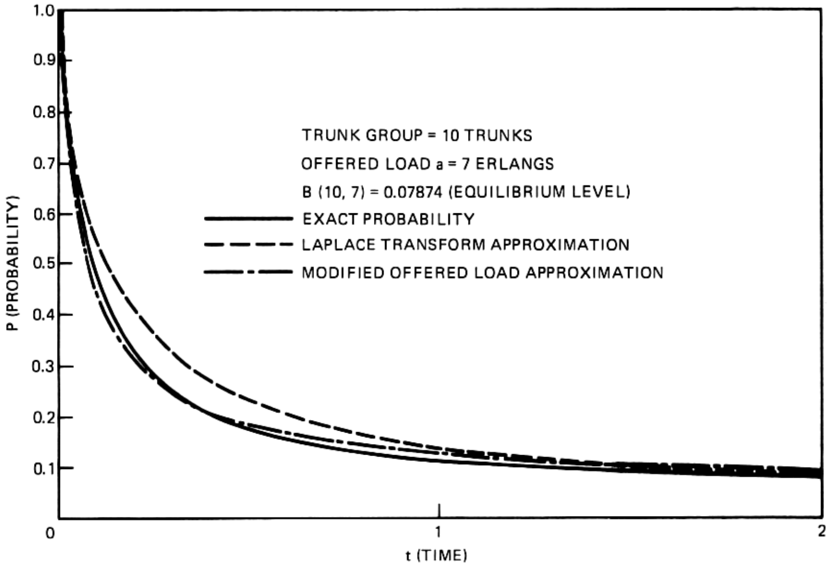

Fig. 2—Recovery function—Laplace transform and modified offered load approximations.

The recovery function approximations of (73) and (187) are compared to the exact solution of (185) in Fig. 2. The approximations are correct at the extremes $t = 0$, $t = \infty$, and track the exact curve reasonably well. The approximation of (73) is more accurate initially and is, of course, also applicable when the offered load is time-variable; how-

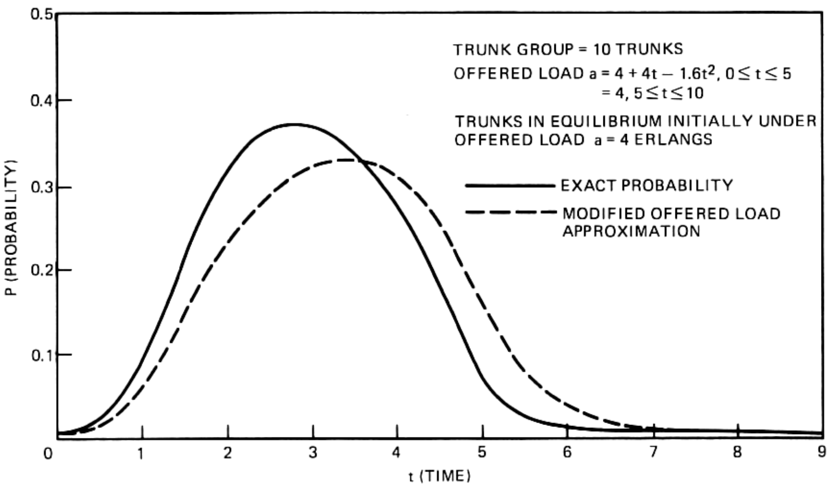

Fig. 3—Pulse response—modified offered load approximation.

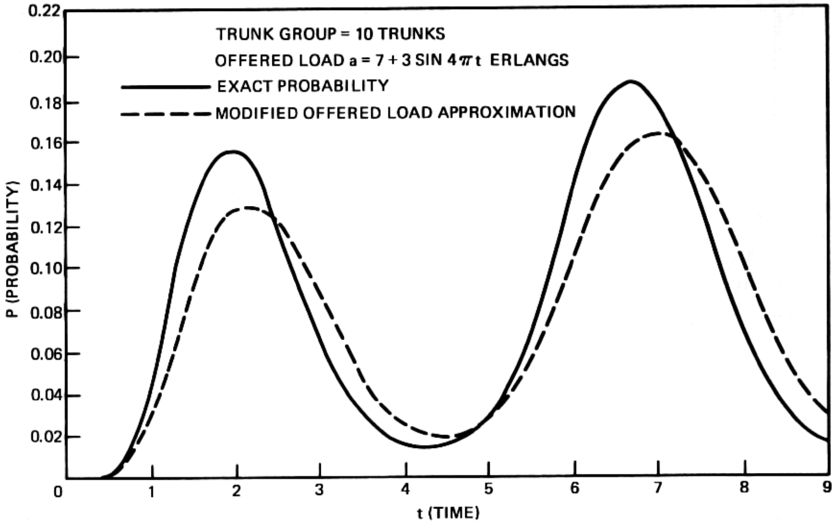

Fig. 4—All trunks empty initially—modified offered load approximation.

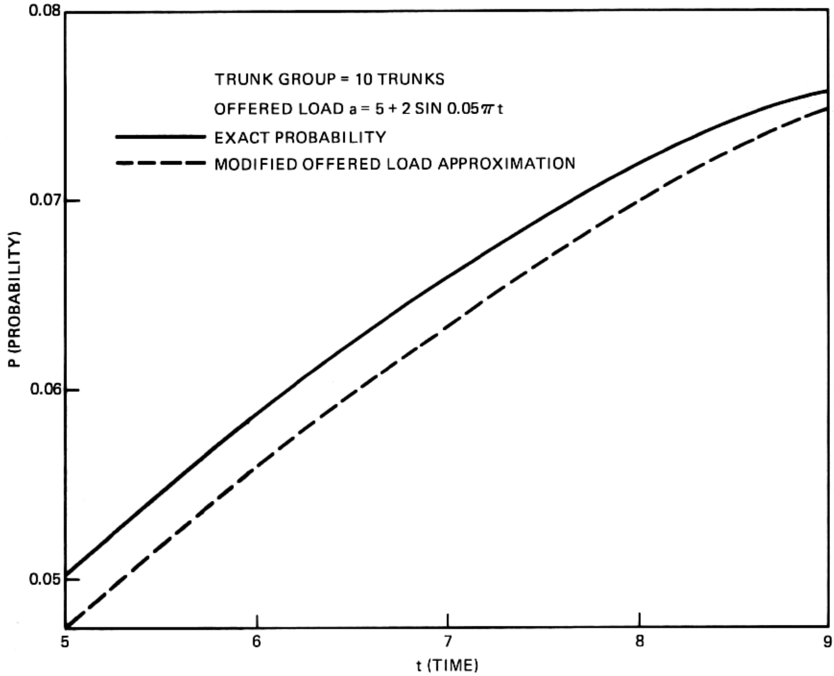

Fig. 5—All trunks empty initially—modified offered load approximation.

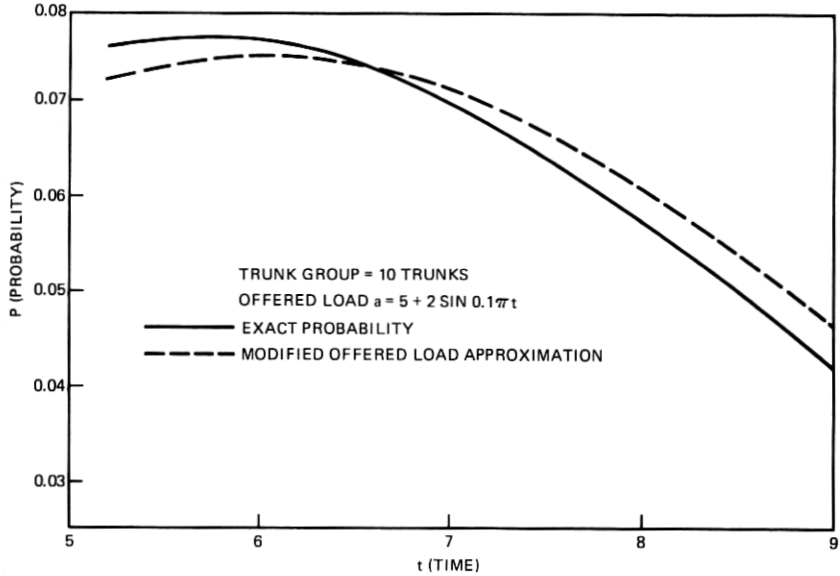

Fig. 6—All trunks empty initially—modified offered load approximation.

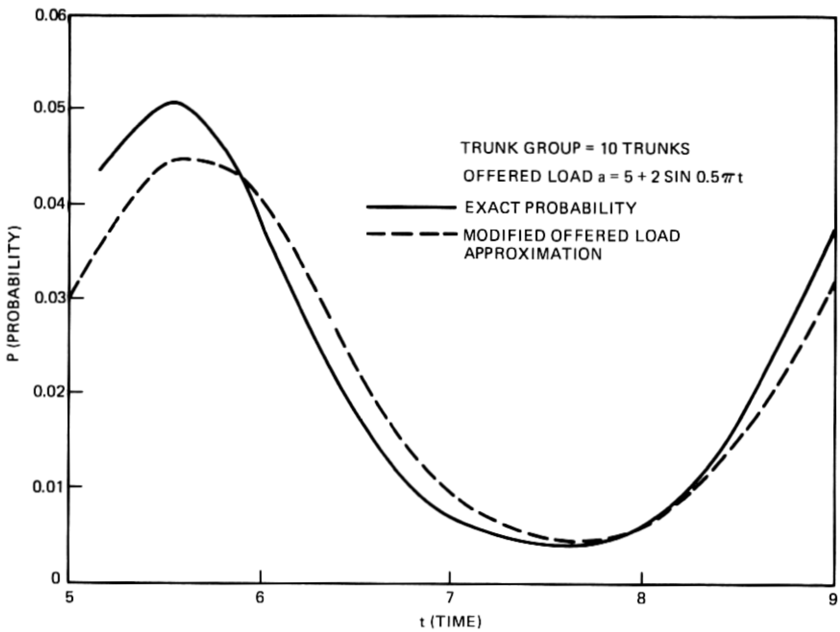

Fig. 7—All trunks empty initially—modified offered load approximation.

ever, the approximation of (187) is simply the case $n = 0$ of Theorem 16. Considerable enhancement of accuracy may, for example, be obtained by using $n = 1$ or even higher values of n.

The modified offered load approximation, (72), is compared to a pulse response in Fig. 3. It is seen that, despite the rapid variation of $a(t)$ (as high as 14 erlangs/call duration) and the large range of probability values, the approximation well imitates the course of the response.

Figures 4, 5, 6, and 7 illustrate the modified offered load approximation applied to sinusoidal inputs. In all cases, the trunk group was initially empty. Figure 4 shows the response, starting from $t = 0$, to $a = 7 + 3 \sin 4\pi t$. This may be considered to have wide excursions compared to the constant term 7, and rapid oscillations, i.e., the period, T, is 5. The exact curve is seen to be well imitated by the approximation.

One may consider the total error to consist of two components, an evanescent part arising from the specific initial state and a component

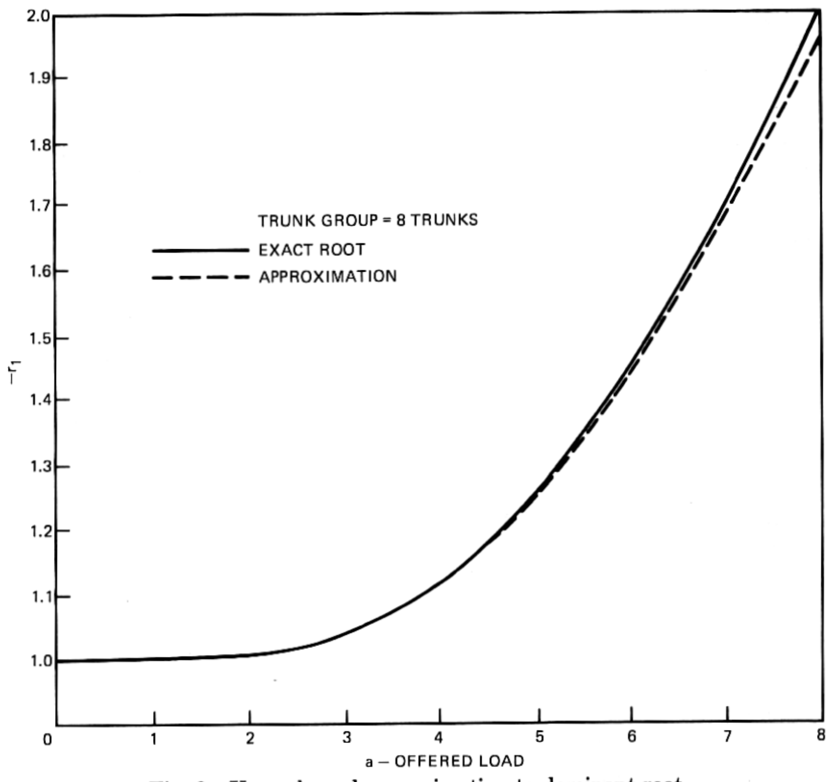

Fig. 8—Upper bound approximation to dominant root.

resulting from the rate of change of the offered load itself, that is, a function of $\dot{a}(t)$ which vanishes when $\dot{a}(t) \equiv 0$. Figures 5, 6, and 7 are intended to illustrate the latter component above; hence, the time scale starts at five. The periods of 40, 20, and 4, respectively, were chosen to reflect the effect of $\dot{a}(t)$ on the approximations. To provide clearer comparison, the probability scales of the graphs have been expanded.

The lower bound of Theorem 13 provides an upper bound on the dominant root. A comparison with the exact values for an eight-trunk group taken from Beneš[7] is given in Fig. 8.

VI. NEEDED INVESTIGATIONS

Much work remains to be done to provide a satisfying and viable tool for fully available trunk group analyses. One may mention error estimation of the approximations suggested in Section IV and Appendix D, the investigation of new approximations, such as studying the consequences of using a refined scaling approximation or an improved modified offered load, and the study of $(I + K_N)^{-1}$ in the Banach space X, for variable $a(t)$. This, in turn, would permit new approximations to be constructed and would provide improved means of error investigation. Relatively little is known about the behavior of the zeros of Poisson-Charlier polynomials, especially in the present context, as functions of a, N. It is hoped this paper will provide an impetus for further investigation.

VII. ACKNOWLEDGMENTS

It is my pleasure to acknowledge the helpful discussions of this material with S. Horing and the critical examination of the paper by L. J. Forys and R. P. Marzec. I wish to thank S. E. Miller and M. D. Vizgirda for the numerical calculations on which the comparative curves are based.

APPENDIX A

Neumann-Series Solution

The Neumann-series solution of the integral equation for blocking,

$$P_N(t, N) = \beta_N(t, \infty) - K_N P_N, \tag{74}$$

is

$$P_N(t, N) = \beta_N(t, \infty) - K_N\beta_N(t, \infty) + K_N^2\beta_N(t, \infty) - \cdots, \tag{75}$$

which, of course, is convergent for all t. The positivity of K_N implies the system of inequalities

$$\sum_{j=0}^{2k+1} (-1)^j K_N^j \beta_N(t, \infty) < P_N(t, N) < \sum_{j=0}^{2k} (-1)^j K_N^j \beta_N(t, \infty), \; k \geqq 0, \tag{76}$$

which generalize (38).

Truncation error estimates for (75) will now be obtained. The inequalities (76) yield

$$-K_N^{2k+1}\beta_N(t, \infty) < P_N(t, N) - \sum_{j=0}^{2k} (-1)^j K_N^j \beta_N(t, \infty) < 0, \tag{77}$$

$$0 < P_N(t, N) - \sum_{j=0}^{2k+1} (-1)^j K_N^j \beta_N(t, \infty) < K_N^{2k+1}\beta_N(t, \infty). \tag{78}$$

Hence, it is only necessary to bound $K_N^{2k+1}\beta_N(t, \infty)$. Let

$$a(t) \leqq \bar{a}, \tag{79}$$

then, from (27) and (34),

$$K_N(t, \tau; a) \leqq K_N(t - \tau; \bar{a}) \leqq Ae^{-(t-\tau)}, \tag{80}$$

$$A = \bar{a}L_{N-1}^{(1)}(-\bar{a}). \tag{81}$$

The dependence on a is explicitly shown in (80). One similarly obtains

$$\beta_N(t, \infty; a) \leqq \bar{\beta} = \sum_{j=0}^{N} \beta_{N-j}(0, N) \frac{\bar{a}^j}{j!}, \qquad t \geqq 0. \tag{82}$$

One may now state Theorem 8.

Theorem 8:

$$K_N^r \beta_N(t, \infty) \leqq \bar{\beta} \frac{(At)^r}{r!} e^{-t}.$$

Proof: One has

$$K_N(t; \bar{a}) \leqq Ae^{-t}. \tag{83}$$

Hence,

$$K_{N,r}(t; \bar{a}) \leqq A^r \frac{t^{r-1}}{(r-1)!} e^{-t}, \tag{84}$$

in which $K_{N,r}(t; \bar{a})$ is the r-fold convolution of $K_N(t; \bar{a})$ with itself. Convolving this with $\beta_N(t, \infty)$ finally yields

$$K_{N,r}(t; \bar{a}) * \beta_N(t, \infty) \leqq \bar{\beta} \frac{(At)^r}{r!} e^{-t}. \tag{85}$$

APPENDIX B

Poisson-Charlier Polynomials

Some properties of Poisson-Charlier polynomials[11] are developed in this appendix, especially with a view to convenient representation of functions by series expansion. Let

$$\psi(x, \lambda) = e^{-\lambda} \frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, \cdots. \tag{86}$$

Then the polynomials $G_j(x, \lambda)$ are defined by

$$\frac{d^j}{d\lambda^j} \psi(x, \lambda) = \psi^{(j)}(x, \lambda) = G_j(x, \lambda)\psi(x, \lambda). \tag{87}$$

The Taylor expansion

$$\psi(x, \lambda + \zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{j!} \psi^{(j)}(x, \lambda) \tag{88}$$

yields the generating function

$$e^{-\zeta} \left(1 + \frac{\zeta}{\lambda}\right)^x = \sum_{j=0}^{\infty} G_j(x, \lambda) \frac{\zeta^j}{j!}. \tag{89}$$

Thus, explicit formulas for $G_j(x, \lambda)$ are

$$G_j(x, \lambda) = \frac{j!}{\lambda^j} \sum_{\nu=0}^{j} (-1)^{\nu} \binom{x}{j - \nu} \frac{\lambda^{\nu}}{\nu!}$$

$$= \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{j}{\nu} \nu! \lambda^{-\nu} \binom{x}{\nu}. \tag{90}$$

The first few polynomials are

$$\begin{aligned} G_0(x, \lambda) &= 1, \\ G_1(x, \lambda) &= \frac{1}{\lambda} (x - \lambda), \\ G_2(x, \lambda) &= \frac{1}{\lambda^2} [x^2 - (2\lambda + 1)x + \lambda^2], \\ G_3(x, \lambda) &= \frac{1}{\lambda^3} [x^3 - 3(\lambda + 1)x^2 + (3\lambda^2 + 3\lambda + 2)x - \lambda^3]. \end{aligned} \tag{91}$$

A recurrence relation derived from (89) is

$$G_{j+1}(x, \lambda) = \frac{x - j - \lambda}{\lambda} G_j(x, \lambda) - \frac{j}{\lambda} G_{j-1}(x, \lambda). \tag{92}$$

The following inner product is defined for functions $f(x)$, $g(x)$ of the discrete variable x:

$$(f, g) = \sum_{x=0}^{\infty} \psi(x, \lambda) f(x) g(x). \tag{93}$$

The Poisson-Charlier polynomials are orthogonal with respect to this inner product[11]

$$\begin{aligned} (G_j, G_k) &= 0, \qquad j \neq k \\ (G_j, G_j) &= \frac{j!}{\lambda^j}. \end{aligned} \tag{94}$$

Accordingly, the coefficients c_j in the expansion of a function $f(x)$ in the form

$$f(x) = \psi(x, \lambda) \sum_{j=0}^{\infty} c_j G_j(x, \lambda) \tag{95}$$

are given by

$$c_j = \frac{\lambda^j}{j!} \sum_{x=0}^{\infty} G_j(x, \lambda) f(x). \tag{96}$$

For the purpose of the present investigation, a more convenient mode of determining c_j is achieved by obtaining their generating function; that is,

$$C(w) = \sum_{j=0}^{\infty} c_j w^j. \tag{97}$$

From (96), one has

$$C(w) = \sum_{x=0}^{\infty} f(x) \sum_{j=0}^{\infty} \frac{(\lambda w)^j}{j!} G_j(x, \lambda). \tag{98}$$

Hence, from (89),

$$C(w) = e^{-\lambda w} \sum_{x=0}^{\infty} f(x)(1 + w)^x. \tag{99}$$

Let

$$B(w) = \sum_{x=0}^{\infty} f(x)(1 + w)^x. \tag{100}$$

Then $B(w)$ is the binomial moment generating function of $f(x)$ and

$$C(w) = e^{-\lambda w} B(w). \tag{101}$$

From (101), the first few coefficients c_j are obtained in terms of the corresponding binomial moments β_j of $f(x)$; thus,

$$\begin{aligned} c_0 &= \beta_0, \\ c_1 &= \beta_1 - \lambda\beta_0, \\ c_2 &= \beta_2 - \lambda\beta_1 + \tfrac{1}{2}\lambda^2\beta_0, \\ c_3 &= \beta_3 - \lambda\beta_2 + \tfrac{1}{2}\lambda^2\beta_1 - \tfrac{1}{6}\lambda^3\beta_0. \end{aligned} \tag{102}$$

A useful choice of the parameter λ is suggested by (102), namely,

$$\lambda = \beta_1/\beta_0, \tag{103}$$

which yields $c_1 = 0$ and

$$\begin{aligned} c_2 &= \beta_2 - \frac{\beta_1^2}{2\beta_0}, \\ c_3 &= \beta_3 - \frac{\beta_1\beta_2}{\beta_0} + \frac{\beta_1^3}{2\beta_0^2}. \end{aligned} \tag{104}$$

For a probability distribution, one has $\beta_0 = 1$, and the choice (103) for λ implies that λ is equated to the mean of the distribution. In this case, one has

$$\begin{aligned} c_2 &= \tfrac{1}{2}(\sigma^2 - \mu), \\ c_3 &= \tfrac{1}{6}(\alpha - 3\sigma^2 + 2\mu), \end{aligned} \tag{105}$$

in which μ is the mean, σ^2 the variance, and α the third moment about the mean of the distribution.

APPENDIX C

Constant Offered Load—Dominant Roots

In this appendix, the integral equation for the constant a case, namely,

$$\begin{aligned} P_N(t, N) &= \beta_N(t, \infty) - K_N * P_N(t, N), \\ K_N(t) &= ae^{-Nt}L_{N-1}^{(1)}[-a(e^t - 1)], \\ \beta_N(t, \infty) &= e^{-Nt}\sum_{j=0}^{N} \beta_{N-j}(0, N)\,\frac{[a(e^t - 1)]^j}{j!}, \end{aligned} \tag{106}$$

is studied. In fact, the somewhat more general equation,

$$f(t) + K_N * f(t) = g(t), \tag{107}$$

is resolved. This presents a considerable advantage over the solution of (106), since the errors of approximations to $P_N(t, N)$ satisfy (107), and hence may be studied by means of Theorem 10 and its corollary. Solutions for blocking have been obtained in the literature,[7,8] but do not provide a means for error analysis. For the practical utilization of the solutions, bounds for the exponents occurring in the explicit representation of the resolvent kernel will also be obtained.

Integral equation theory[14] asserts that a resolvent kernel, $Q_N(t)$, exists with the property

$$f(t) = g(t) + Q_N * g(t). \tag{108}$$

Laplace transformation will be used to study (107) and (108).

Theorem 9: The Laplace transforms, $\tilde{K}_N(s)$, $\tilde{Q}_N(s)$, of $K_N(t)$, $Q_N(t)$, respectively, are

$$\tilde{K}_N(s) = \frac{(-1)^N a^N G_N(-s-1, a)}{(s+1)\cdots(s+N} - 1,$$

$$\tilde{Q}_N(s) = \frac{(s+1)\cdots(s+N) - (-1)^N a^N G_N(-s-1, a)}{(-1)^N a^N G_N(-s-1, a)}.$$

Proof: One has, from (27),

$$L_{N-1}^{(1)}(-X) = \sum_{j=0}^{N-1} \binom{N}{j+1} \frac{X^j}{j!}, \tag{109}$$

and hence,

$$K_N(t) = ae^{-Nt} \sum_{j=0}^{N-1} \binom{N}{j+1} \frac{a^j}{j!} (e^t - 1)^j. \tag{110}$$

Thus,

$$\tilde{K}_N(s) = a \sum_{j=0}^{N-1} \binom{N}{j+1} \frac{a^j}{j!} \int_0^\infty e^{-(N+s-j)t}(1 - e^{-t})^j dt. \tag{111}$$

Letting $X = e^{-t}$, one has

$$\tilde{K}_N(s) = a \sum_{j=0}^{N-1} \binom{N}{j+1} \frac{a^j}{j!} \int_0^1 X^{N+s-j-1}(1 - X)^j dx. \tag{112}$$

The integral in (112) is the beta function, $B(N + s - j,\ j + 1)$. Hence,

$$\tilde{K}_N(s) = a \sum_{j=0}^{N-1} \binom{N}{j+1} a^j \frac{\Gamma(N+s-j)}{\Gamma(N+s+1)}. \tag{113}$$

One has the following transformations:

$$\tilde{K}_N(s) = \sum_{j=1}^{N} \binom{N}{j} \frac{a^j}{(N+s)\cdots(N+s-j+1)}, \tag{114}$$

$$\tilde{K}_N(s) = \frac{1}{(s+1)\cdots(s+N)} \sum_{j=1}^{N} \binom{N}{j} a^j(s+1)\cdots(s+N-j), \tag{115}$$

$$\tilde{K}_N(s) = \frac{a^N}{(s+1)\cdots(s+N)} \sum_{j=1}^{N} \binom{N}{j} a^{-j}(s+1)\cdots(s+j), \tag{116}$$

$$\tilde{K}_N(s) = \frac{a^N}{(s+1)\cdots(s+N)} \sum_{j=1}^{N} (-1)^j \binom{N}{j} j!\, a^{-j} \binom{-s-1}{j}. \tag{117}$$

Thus, the required formula for $\tilde{K}_N(s)$ is obtained from (117) by comparison with (90). From (107), one has

$$\tilde{f} + \tilde{K}_N * \tilde{f} = \tilde{g}. \tag{118}$$

Hence,

$$\tilde{f} = \frac{\tilde{g}}{1 + \tilde{K}_N} = \tilde{g} - \frac{\tilde{K}_N}{1 + \tilde{K}_N}\tilde{g}; \tag{119}$$

thus,

$$\tilde{Q}_N(s) = -\frac{\tilde{K}_N(s)}{1 + \tilde{K}_N(s)}. \tag{120}$$

This, together with the formula for $\tilde{K}_N(s)$, yields the required expression for $\tilde{Q}_N(s)$.

Corollary:

$$\tilde{P}_{i,N}(s, N) = (-1)^{N-i}\frac{G_i(-s, a)}{sG_N(-s-1, a)}.$$

Proof: Use of Theorem 2, Corollary 4, Theorem 9, and eq. (178) with $\beta_j(0, N) = \binom{i}{j}$.

An expression equivalent to this corollary was given by Takács.[15]

It will be useful now to introduce the Banach space, X, of functions $f(t)$ that are bounded and measurable over $(0, \infty)$ and normed by

$$\|f\| = \sup_{t \geqq 0} |f(t)|. \tag{121}$$

One may now state Theorem 10.

Theorem 10: *The operators* $I + K_N$ *and* $(I + K_N)^{-1}$ *are bounded; further,*

$$\|I + K_N\| = S_N(a),$$
$$\|(I + K_N)^{-1}\| = 1 + \int_0^\infty |Q_N(t)|\,dt,$$

in which the operator norms are those induced by (121).

Proof: The quantity $\|I + K_N\|$ is obtained directly from Theorem 3 and the formula

$$\|I + K_N\| = 1 + \int_0^\infty K_N(t)dt. \tag{122}$$

Since the polynomials $G_N(x, a)$ are orthogonal over $(0, \infty)$, it follows that the zeros, $P_1, \cdots, P_N$ are distinct, real, and positive; hence, the zeros, $r_1, \cdots, r_N$, of $G_N(-s-1, a)$ are distinct, real, and less than minus one. The Paley-Wiener theorem[9] applied to $\tilde{Q}_N(s)$ now asserts that $I + K_N$ has a bounded inverse with norm C given by

$$C = \|(I + K_N)^{-1}\| = 1 + \int_0^\infty |Q_N(t)|\,dt. \tag{123}$$

An immediate corollary is the following.

Corollary: The integral equation

$$f + K_N * f = g, \qquad g \in X,$$

possesses a solution $f \in X$ satisfying

$$\|f\| \leqq C\|g\|,$$

$$\overline{\lim_{t\to\infty}}\, |f(t)| \leqq C\, \overline{\lim_{t\to\infty}}\, |g(t)|.$$

In particular, if $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} f(t) = 0$.

Proof: The result follows from

$$f = g + Q_N * g \tag{124}$$

and the definition of C.

The following theorem provides a representation of $Q_N(t)$ and an estimate of C.

Theorem 11:

$$Q_N(t) = \sum_{j=1}^{N} \frac{\prod_{\nu=1}^{N} (r_j + \nu)}{\prod_{\substack{\nu=1\\ \nu\neq j}}^{N} (r_j - r_\nu)} e^{r_j t}, \qquad C \leqq 1 + \sum_{j=1}^{N} \frac{\prod_{\nu=1}^{N} |r_j + \nu|}{|r_j| \prod_{\substack{\nu=1\\ \nu\neq j}}^{N} |r_j - r_\nu|}.$$

Proof: One has

$$(-1)^N a^N G_N(-s-1, a) = (s - r_1)\cdots(s - r_N) \tag{125}$$

and, hence, from Theorem (9),

$$\tilde{Q}_N(s) = \frac{(s+1)\cdots(s+N) - (s-r_1)\cdots(s-r_N)}{(s-r_1)\cdots(s-r_N)}. \tag{126}$$

The partial fraction expression for $\tilde{Q}_N(s)$ is

$$\tilde{Q}_N(s) = \sum_{j=1}^{N} \frac{\prod_{\nu=1}^{N} (r_j + \nu)}{\prod_{\substack{\nu=1\\ \nu\neq j}}^{N} (r_j - r_\nu)} \frac{1}{s - r_j}. \tag{127}$$

Thus,

$$Q_N(t) = \sum_{j=1}^{N} \frac{\prod_{\nu=1}^{N} (r_j + \nu)}{\prod_{\substack{\nu=1 \\ \nu \neq j}}^{N} (r_j - r_\nu)} e^{r_j t}. \tag{128}$$

Also, one has

$$|Q_N(t)| \leqq \sum_{\nu=1}^{N} \frac{\prod_{\nu=1}^{N} |r_j + \nu|}{\prod_{\substack{\nu=1 \\ \nu \neq j}}^{N} |r_j - r_\nu|} e^{r_j t}, \tag{129}$$

and, accordingly,

$$C = 1 + \int_0^\infty |Q_N(t)|\, dt \leqq 1 + \sum_{j=1}^{N} \frac{\prod_{\nu=1}^{N} |r_j + \nu|}{|r_j| \prod_{\substack{\nu=1 \\ \nu \neq j}}^{N} |r_j - r_\nu|}. \tag{130}$$

To make the results of Theorem 11 accessible for estimation, particularly in numerical applications, and for the Laplace transform approximation of Appendix D, bounds for the roots, r_j, will be obtained. The generating function, $g(z)$, for the equilibrium probability distribution, $P_j(\infty, N)$, of (36) may be written as

$$g(z) = \frac{S_N(az)}{S_N(a)}. \tag{131}$$

Thus, the mean, m, and variance, σ^2, are

$$m = a[1 - B(N, a)], \tag{132}$$

$$\sigma^2 = m - (N - m)aB(N, a). \tag{133}$$

We now have Theorem 12.

Theorem 12: *(Beneš)* $r_1 > -(m/\sigma^2)$.

To obtain further bounds, the following lemmas are needed.

Lemma 1: $\rho_N \geqq \cdots \geqq \rho_1 > 0$,

$$\rho_1 + \cdots + \rho_N = s_1, \qquad \rho_1^2 + \cdots + \rho_N^2 = s_2, \qquad \rho_1 \cdots \rho_N = D,$$

then

$$\rho \leqq \rho_1 \leqq \cdots \leqq \rho_N \leqq \frac{s_1 + \sqrt{(N-1)(Ns_2 - s_1^2)}}{N},$$

in which ρ *is the small positive root of*

$$\rho(s_1 - \rho)^{N-1} = D(N-1)^{N-1}.$$

These bounds are sharp.

Proof: The equations

$$\rho_1 + \cdots + \rho_N = s_1, \tag{134}$$

$$\rho_1^2 + \cdots + \rho_N^2 = s_2 \tag{135}$$

may be written

$$\rho_1 + \cdots + \rho_{N-1} = s_1 - \rho_N, \tag{136}$$

$$\rho_1^2 + \cdots + \rho_{N-1}^2 = s_2 - \rho_N^2. \tag{137}$$

Let

$$\rho_N^* = \max \rho_N \tag{138}$$

over all allowable sequences. Then the sum (137) is minimum when ρ_N is replaced by ρ_N^*. This occurs, however, only when

$$\rho_1 = \cdots = \rho_{N-1} = \frac{s_1 - \rho_N^*}{N-1}. \tag{139}$$

Thus, from (137),

$$(N-1)\left(\frac{s_1 - \rho_N^*}{N-1}\right)^2 = s_2 - \rho_N^{*2}. \tag{140}$$

The solution of (140) for ρ_N^* is

$$\rho_N^* = \frac{s_1 + \sqrt{(N-1)(Ns_2 - s_1^2)}}{N}. \tag{141}$$

This proves the upper bound of the lemma. The inequality is attained for the vector $(\rho_1, \cdots, \rho_N)$ defined by (139) and (141).

Similarly, one may write

$$\rho_2 + \cdots + \rho_N = s_1 - \rho_1, \tag{142}$$

$$\rho_2 \cdots \rho_N = D/\rho_1. \tag{143}$$

Let

$$\rho_1^* = \min \rho_1 \tag{144}$$

over all allowable sequences. Then the product (143) is maximum when ρ_1 is replaced by ρ_1^*. This occurs only when

$$\rho_2 = \cdots = \rho_N = \frac{s_1 - \rho_1^*}{N-1}. \tag{145}$$

Thus,

$$\left[\frac{s_1 - \rho_1^*}{N-1}\right]^{N-1} = D/\rho_1^* \tag{146}$$

and the equation of the lemma defining ρ has been established. The inequality is attained for the vector $(\rho_1, \cdots, \rho_N)$ defined by (145) and (146).

Lemma 2: $\rho_N \geqq \cdots \geqq \rho_1 > 0$,

$$s_{-1} = \frac{1}{\rho_1} + \cdots + \frac{1}{\rho_N}, \qquad D = \rho_1 \cdots \rho_N.$$

Then

$$\rho_N \geqq \cdots \geqq \rho_1 \geqq \rho,$$

in which ρ *is the small positive root of*

$$D\left(s_{-1} - \frac{1}{\rho}\right)^{N-1} = \rho(N-1)^{N-1}.$$

The bound is sharp.

Proof: One may write

$$\frac{1}{\rho_2} + \cdots + \frac{1}{\rho_N} = s_{-1} - \frac{1}{\rho_1}, \tag{147}$$

$$\rho_2 \cdots \rho_N = D/\rho_1. \tag{148}$$

Let

$$\rho_1^* = \min \rho_1 \tag{149}$$

over all allowable sequences. Then the product (148) is maximum when ρ_1 is replaced by ρ_1^*. This occurs when

$$\rho_2 = \cdots = \rho_N = \frac{N-1}{s_{-1} - \dfrac{1}{\rho_1^*}}. \tag{150}$$

Thus,

$$\left[\frac{N-1}{s_{-1} - \dfrac{1}{\rho_1^*}}\right]^{N-1} = D/\rho_1^* \tag{151}$$

and the equation of the lemma is established. The inequality is attained for the vector $(\rho_1, \cdots, \rho_N)$ defined by (150) and (151).

The application of the lemmas to the polynomials $G_N(x, a)$ is accomplished by identifying $\rho_1, \cdots, \rho_N$ with its zeros. For this purpose, the form of $G_N(x, a)$ given in (90) will be recast, by the help of the Stirling numbers of first kind,[11] into standard form; that is,

$$G_N(x, a) = \sum_{m=0}^{N} a_{N-m} x^m. \tag{152}$$

The Stirling numbers, S_j^m, are defined by

$$\prod_{\nu=0}^{j-1} (x - \nu) = \sum_{m=1}^{j} S_j^m x^m. \tag{153}$$

Thus, one has

$$G_N(x, a) = (-1)^N + \sum_{m=1}^{N} x^m \sum_{\nu=0}^{N-m} (-1)^\nu a^{\nu-N} \binom{N}{\nu} S_{N-\nu}^m. \tag{154}$$

The sums s_1, s_2 are accordingly given by

$$s_1 = \binom{N}{2} + aN, \tag{155}$$

$$s_2 = s_1^2 - 6\binom{N}{4} - (6a + 4)\binom{N}{3} - 2a^2\binom{N}{2}. \tag{156}$$

The reciprocal polynomial, that is, the polynomial whose zeros are $\rho_1^{-1}, \cdots, \rho_N^{-1}$ is given by

$$X^N G_N(x^{-1}, a) G_N(x, a) = \sum_{m=0}^{N} a_m x^m. \tag{157}$$

Hence, the analogous quantities s_{-1} and $s_{-2} = \rho_1^{-2} + \cdots + \rho_N^{-2}$ are given by

$$s_{-1} = \sum_{\nu=1}^{N} \frac{1}{\nu} N^{(\nu)} a^{-\nu}, \tag{158}$$

$$s_{-2} = s_{-1}^2 - 2 \sum_{\nu=2}^{N} \frac{1}{\nu} N^{(\nu)} a^{-\nu} \sum_{j=1}^{\nu-1} \frac{1}{j}, \tag{159}$$

in which

$$N^{(0)} = 1, \qquad N^{(\nu)} = N(N-1)\cdots(N-\nu+1), \qquad \nu > 0. \tag{160}$$

The upper bound of Lemma 1 now establishes Theorem 13.

Theorem 13:

$$\frac{N}{s_{-1} + \sqrt{(N-1)(Ns_{-2} - s_{-1}^2)}} \leqq \rho_1 < \cdots < \rho_N \leqq \frac{s_1 + \sqrt{(N-1)(Ns_2 - s_1^2)}}{N}.$$

Proof: The upper bound is immediate. The lower bound is obtained by applying the upper bound of Lemma 1 to the reciprocal equation.

A numerical illustration of Theorem 13 is provided by the zeros of $G_{10}(x, 7)$ used in obtaining the recovery function plotted in Fig. 2. They are 0.332811, 2.05847, 4.06653, 6.31227, 8.81308, 11.5197, 14.6407, 18.0255, 22.0872, and 27.1438. The lower and upper bounds given by the theorem are 0.32964 and 36.82292, respectively. It appears that the lower bound may well be usable as an approximation to ρ_1. The accuracy of this when used to approximate $-r_1 = 1 + \rho_1$ is illustrated in Fig. 8. The exact values of $-r_1$ are taken from Beneš.[7] This provides an upper bound for r_1 which, together with the available lower bounds, is useful for error investigations.

APPENDIX D

Approximations—Constant Offered Load

It was suggested by S. Horing that an appropriate scaling between $P_N(t, N)$ and $P_k(t, k)$ may exist; that is, a function $F[P_N(t, N), P_k(t, k)]$ may exist, which would be approximately independent of t and which may, therefore, permit the approximate determination of $P_N(t, N)$ in terms of $P_1(t, 1)$, for example, thus permitting large trunk groups to be studied in terms of the behavior of small ones. Since it is feasible to use (108) for small trunk groups, this would constitute an approximation of the solution of (106) for large trunk groups.

Consider the following

$$\frac{B(kl, a)}{B(l, a/k)^k} = \frac{l\,!^k k^{kl}}{(kl)\,!} \frac{S_l(a/k)^k}{S_{kl}(a)}. \tag{161}$$

The ratio $S_l(a/k)^k/S_{kl}(a)$ is approximately independent of a since $S_l(a/k)^k \cong S_{kl}(a) \cong e^a$ for k large. Thus,

$$\frac{B(kl, a)}{B(l, a/k)^k} \tag{162}$$

is approximately independent of a. It would seem, therefore, that the ratio (162) is approximately a time invariant of (106), especially for large t; that is, the function

$$\frac{P_{kl}(t, kl; a)}{P_l(t, l; a/k)^k} \tag{163}$$

is approximately equal to the ratio (162) when t is large; thus,

$$P_{kl}(t, l; a) \cong B(kl, a) \left[\frac{P_l(t, l; a/k)}{B(l, a/k)} \right]^k. \tag{164}$$

We have, from (107) and Theorem 11,

$$P_1(t, 1; a) = \frac{a}{a+1} \{1 - e^{-(a+1)t}\}, \tag{165}$$

when the trunk is initially empty. Hence, from (164) with $l = 1$,

$k = N$,

$$P_N(t, N) \cong B(N, a)\{1 - e^{-[(a/N)+1]t}\}^N \tag{166}$$

for the case when all N trunks are initially empty.

It is desired now to extend the approximate solution (166) for application to any initial condition. Define $q_N(t)$ by

$$q_N(t) = B(N, a)\{1 - e^{-[(a/N)+1]t}\}^N. \tag{167}$$

Then a convolution operator with kernel $L_N(t)$ will be constructed so that one has

$$q_N + L_N * Q_N = \frac{\Lambda^N}{N!}, \tag{168}$$

exactly. The operator L_N will then be taken to approximate the operator K_N of (106). Accordingly, an approximation $\bar{P}_N$ to P_N corresponding to initial states other than all trunks empty is defined by the equation

$$\bar{P}_N + L_N * \bar{P}_N = \beta_N. \tag{169}$$

Theorem 14: The Laplace transform, $\tilde{L}_N(s)$, of the approximating $L_N(t)$ is

$$1 + \tilde{L}_N = \left(\frac{a}{N} + 1\right) S_N(a) \frac{\Gamma(s)\Gamma\left(N + \frac{s}{(a/N)+1} + 1\right)}{\Gamma(s + N + 1)\Gamma\left(\frac{s}{(a/N)+1}\right)}$$

$$= \left(\frac{a}{N} + 1\right)^{-N} S_N(a) \prod_{\nu=1}^{N} \frac{s + \nu[(a/N) + 1]}{s + \nu}.$$

Proof: One has

$$\tilde{q}_N = B(N, a) \int_0^\infty e^{-st}\{1 - e^{-[(a/N)+1]t}\}^N dt. \tag{170}$$

Let $u = [(a/N) + 1]t$, then

$$\tilde{q}_N = \frac{B(N, a)}{(a/N) + 1} \int_0^\infty e^{-[s/(a/N+1)]u}(1 - e^{-u})^N du. \tag{171}$$

The substitution $x = e^{-u}$ yields

$$\tilde{q}_N = \frac{B(N, a)}{(a/N) + 1} \int_0^1 x^{[s/(a/N+1)]-1}(1 - x)^N dx; \tag{172}$$

thus,

$$\tilde{q}_N = \frac{B(N, a)}{(a/N) + 1} \frac{N!\,\Gamma\left[\frac{s}{(a/N)+1}\right]}{\Gamma\left[N + \frac{s}{(a/N)+1} + 1\right]}. \tag{173}$$

One also has

$$\int_0^\infty e^{-st}(1 - e^{-t})^N dt = \frac{N!\,\Gamma(s)}{\Gamma(N + s + 1)}. \tag{174}$$

The Laplace transform of each term of (168) now yields

$$\tilde{q}_N(1 + \tilde{L}) = \frac{a^N \Gamma(s)}{\Gamma(N + s + 1)}. \tag{175}$$

Use of (173) yields the result of the theorem.

It may be noted that L_N has an impulsive component at $t = 0$ whose value is

$$\left(\frac{a}{N} + 1\right)^{-N} S_N(a) - 1. \tag{176}$$

For large N, this is nearly zero. The effect of this is to create an error at $t = 0$; that is, $\bar{P}_N(0, N) \neq P_N(0, N)$ if $\beta_N(0, N) \neq 0$. The larger N is (for fixed a), the smaller the discrepancy.

Equation (169) is studied in Theorem 15.

Theorem 15: The solution of (169) is

$$\tilde{\bar{P}}_N = \frac{N!}{(a/N) + 1} B(N, a) \frac{\Gamma\left[\dfrac{s}{(a/N) + 1}\right]}{\Gamma(s)\Gamma\left[N + \dfrac{s}{(a/N) + 1} + 1\right]} \cdot \sum_{j=0}^{N} \beta_j(0, N) a^{-j} \Gamma(s + j),$$

$$\tilde{\bar{P}}_N = \sum_{j=0}^{N} \beta_j(0, N) a^{-j} \frac{\Gamma(D + j)}{\Gamma(D)} q_N(t), \qquad D \equiv \frac{d}{dt}.$$

Proof: From Corollary 1, Theorem 2, one has

$$\beta_N = \sum_{j=0}^{N} \beta_{N-j}(0, N) \frac{a^j}{j!} e^{-(N-j)t}(1 - e^{-t})^j; \tag{177}$$

hence,

$$\tilde{\beta}_N = \sum_{j=0}^{N} \beta_j(0, N) a^{N-j} \frac{\Gamma(s + j)}{\Gamma(s + N + 1)}. \tag{178}$$

Transforming the terms of (169) yields

$$\tilde{\bar{P}}_N(1 + \tilde{L}_N) = \tilde{\beta}_N, \tag{179}$$

and hence the result of the theorem is obtained from Theorem 14 and eq. (178). The inversion of the transform, $\tilde{\bar{P}}_N$, by use of differentiation follows on use of (173) in $\tilde{\bar{P}}_N$.

Another method of approximation useful for constant offered load is based on an approximate inversion of the Laplace transform. Let

$$\tilde{f}(s) = \int_0^\infty e^{-su} f(u)du, \qquad s > 0; \tag{180}$$

then

$$\frac{(-1)^n}{n!} s^{n+1} \tilde{f}^{(n)}(s) = \frac{s^{n+1}}{n!} \int_0^\infty e^{-su} u^n f(u)du. \tag{181}$$

The function $(s^{n+1}/n!)e^{-su}u^n$ is a probability density function for any $s > 0$, $n \geqq 0$ whose mean is $(n+1)/s$ and variance $(n+1)/s^2$. Letting $s = (n+1)/t$, the mean and variance are t and $t^2/(n+1)$, respectively; hence, Korovkin's theorem on sequences of positive functionals[16] establishes the Widder inversion formula[9]:

$$\lim_{n\to\infty} \frac{(-1)^n}{n!} s^{n+1} \tilde{f}^{(n)}(s)\bigg|_{s=(n+1)/t} = f(t). \tag{182}$$

The above discussion forms the basis for Theorem 16.

Theorem 16: For $\epsilon > 0$, let the transform of $e^{\epsilon t} f(t)$, namely, $\tilde{f}(s-\epsilon)$, exist for $s > 0$, and let $e^{\epsilon t} f(t)$ be convex in $t > 0$, then

$$f(t) \leqq e^{-\epsilon t} \frac{(-1)^n}{n!} s^{n+1} \tilde{f}^{(n)}(s-\epsilon)\bigg|_{s=(n+1)/t}, \qquad n \geqq 0, \qquad t > 0.$$

Proof: Jensen's inequality applied to (181) in the form

$$\frac{(-1)^n}{n!} s^{n+1} \tilde{f}^{(n)}(s-\epsilon) = \frac{s^{n+1}}{n!} \int_0^\infty e^{-su} u^n e^{\epsilon u} f(u)du \tag{183}$$

establishes the theorem. By virtue of (182), when similarly modified for the function $e^{\epsilon t} f(t)$, the dexter of the inequality always provides an approximation to $f(t)$ even when $e^{\epsilon t} f(t)$ is not convex.

Corollary: $\tilde{f}(s-\epsilon)$ exists for $s > 0$, and $e^{\epsilon t} f(t)$ is convex in $t > 0$ implies

$$f(t) \leqq \frac{1}{t} e^{-\epsilon t} \tilde{f}\left(\frac{1}{t} - \epsilon\right), \qquad t > 0.$$

Proof: The case $n = 0$ of Theorem 16.

If $\tilde{f}(s)$ should have a dominant pole, it is usually advantageous to choose ϵ equal to the negative of that pole.

The above corollary will now be applied to obtaining an inequality for the recovery function. The corollary to Theorem 9 shows that

$$\tilde{P}_{N,N}(s, N) = \frac{G_N(-s, a)}{sG_N(-s-1, a)}. \tag{184}$$

The inversion of $\tilde{P}_{N,N}(s, N)$ is readily accomplished; the result is[7]

$$P_{N,N}(t, N) = B(N, a) - \sum_{j=1}^{N} \frac{e^{r_j t}}{r_j} \prod_{i \neq j} \left(1 - \frac{1}{r_j - r_i}\right). \quad (185)$$

It follows from (185) that

$$e^{-r_1 t}[P_{N,N}(t, N) - B(N, a)] \quad (186)$$

is convex for $t > 0$ and that the corresponding Laplace transform exists for $s > 0$, hence, by the above corollary, one obtains

$$P_{N,N}(t, N) \leqq B(N, a) + e^{r_1 t}\left[\frac{G_N[-(1/t) - r_1, a]}{(1 + r_1 t)G_N[-(1/t) - r_1 - 1, a]} - \frac{B(N, a)}{1 + r_1 t}\right]. \quad (187)$$

REFERENCES

1. B. Wallström, "Congestion Studies in Telephone Systems with Overflow Facilities," Ericsson Technics, No. 3.
2. A. Y. Khintchine, *Mathematical Methods in the Theory of Queuing*, New York: Hafner, 1969.
3. V. B. Iversen, "Analysis of Traffic Processes Based on Data Obtained by the Scanning Method," Lyngby, Denmark: Technical University of Denmark.
4. R. I. Wilkinson, "Theories for Toll Traffic Engineering in the U.S.A.," B.S.T.J., *35*, No. 2 (March 1956), pp. 421–514.
5. D. L. Jagerman, "An Approximation Theorem of Central Limit Type," unpublished work.
6. D. L. Jagerman, "Some Properties of the Erlang Loss Function," B.S.T.J., *53*, No. 3 (March 1974), pp. 525–551.
7. V. E. Beneš, *Mathematical Theory of Connecting Networks and Telephone Traffic*, New York: Academic Press, 1965, Chapter 6.
8. J. Riordan, *Stochastic Service Systems*, New York: Wiley, 1962.
9. R. E. A. C. Paley and N. Wiener, "Fourier Transforms in the Complex Domain," Amer. Math. Soc., 1934.
10. G. Sansone, *Orthogonal Functions*, New York: Interscience, 1959, Chapter IV.
11. C. Jordon, *Calculus of Finite Differences*, New York: Chelsea, 1947, Chapter VIII.
12. S. Karlin and J. L. McGregor, "The Differential Equations of Birth-and-Death Processes and the Stieltjes Moment Problem," Trans. Amer. Math. Soc., *85*, 1957, pp. 489–546.
13. E. Schmidt, "Über die Charlier-Jordansche Entwicklung einer Willkürlichen Funktion nach der Poissonschen Funktion und ihren Ableitungen," Zeitschift für Ange. Mathematik und Mechanik, V. 13 (1933).
14. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, New York: Oxford, 1948.
15. L. Takács, *Introduction to the Theory of Queues*, New York: Oxford, 1962, Chapter 4.
16. P. P. Korovkin, "Linear Operators and Approximation Theory," New York: Gordon and Breach, 1960, p. 14.