THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 54, No. 1, January 1975
Printed in U.S.A.

Two Derivations of the Time-Dependent Coupled-Power Equations

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(Manuscript received April 12, 1974)

In this paper, the time-dependent coupled-power equations originally derived by Marcuse from intuitive arguments are rederived two ways, one using the coupled-line equations with perturbation theory and the other using the Kronecker product approach of Rowe and Young.†

I. INTRODUCTION

Suppose a multimode fiber guide with random mode coupling is excited by an optical source at one end at time $t = 0$.

This input excitation will produce, at time t, a response of optical power in each mode v at a position z from the input end given by $s_v(t, z)$.

Marcuse[2] has suggested from intuitive arguments that $s_v(t, z)$ should satisfy the following differential equation

$$\frac{\partial}{\partial z}\langle s_v(t, z)\rangle + \frac{1}{C_v}\frac{\partial}{\partial t}\langle s_v(t, z)\rangle = \sum_{\mu} |K_{\mu v}|^2 F\langle s_\mu(t, z)\rangle - \left(\sum_{\mu} |K_{\mu v}|^2 F\right)\langle s_v(t, z)\rangle, \quad (1)$$

where $K_{\mu v}$ is the coupling coefficient in the coupled-line equations (described below) between modes μ and v, F is the spectral height of the mechanical perturbation in the fiber geometry responsible for the coupling, $\langle s_\mu(t, z)\rangle$ is the average‡ power in mode μ at position z, and C_v is the group velocity in mode v.

In this paper, we derive eq. (1) from the coupled-line equations first using perturbation theory and then using the Kronecker product approach of Rowe and Young.[3]

The importance of this work is to show that the intuitive eq. (1) does in fact follow directly from (*i*) the coupled-line equations, (*ii*)

† Subsequent to the writing of this paper, it became known to the author that many of the results included in the paper were independently and concurrently derived by R. Steinberg of Columbia University.[1]

‡ Average for an ensemble of guides with similar gross properties and similar excitation.

voltage linearity of the guide (from Maxwell's equations), and (*iii*) ensemble averaging over the random coupling perturbation. It is encouraging to note that skillfully framed power-flow arguments such as those used by Marcuse lead to the same results as the more cumbersome approaches that start from the coupled-line equations.

It should be pointed out again that, however it is derived, eq. (1) describes the average of the flow of power for an ensemble of guides. How the flow of power in a particular guide compares to the average flow is still an open subject.[4]

II. ANALYSIS

We start with the coupled-line equations that describe the z evolution of $\mathcal{A}_v(\omega, z)$—the complex amplitude of the voltage in a mode v at position z resulting from a single Fourier component of the optical excitation at frequency ω at position $z = 0$. (We can use Fourier components since the guide is linear in voltage.)

$$\frac{\partial A_v(\omega, z)}{\partial z} = \sum_{\mu} A_\mu(\omega, z) K_{v\mu} f(z) \exp\left[i(\beta_v - \beta_\mu)z\right], \qquad (2)$$

where $\mathcal{A}_v(\omega, z)$ (the complex amplitude in mode v) $= A_v(\omega, z) \times \exp\left[-i\beta_v(\omega)z\right]$,

β_v = propagation constant for mode v (which is a function of ω), and

$K_{\mu v} f(z)$ = coupling coefficient between modes μ and v ($K^*_{\mu v} = -K_{v\mu}$).

If the input *power* excitation is a function of time, then the *average power* response at position z is given by the *average* of the square of the complex envelope, $a_v(z, t)$, at position z, i.e.,

$$\langle s_v(z, t)\rangle = \langle |a_v(z, t)|^2\rangle = \int \langle \mathcal{A}_v(\omega + \sigma, z)\mathcal{A}^*_v(\omega, z)\rangle \exp\left[i(\omega + \sigma)t\right] \times \exp(-i\omega t) d\omega d(\omega + \sigma), \qquad (3)$$

where the complex envelope $a_v(t, z)$ is the Fourier transform of $\mathcal{A}_v(\sigma, z)$.

We shall next derive eq. (1) by obtaining a differential equation for $(\partial/\partial z)\langle s_v(t, z)\rangle$.

2.1 Perturbation theory approach

Following techniques used previously by Marcuse,[2] we first write

$$\frac{\partial}{\partial z}\langle A_v(\omega + \sigma, z) A^*_v(\omega, z)\rangle = \left\langle \left[\frac{\partial}{\partial z} A_v(\omega + \sigma, z)\right] A^*_v(\omega, z)\right\rangle + \left\langle A_v(\omega + \sigma, z)\left[\frac{\partial}{\partial z} A^*_v(\omega, z)\right]\right\rangle. \qquad (4)$$

Using (2) in (4) we obtain

$$\frac{\partial}{\partial z} \langle A_v(\omega + \sigma, z) A_v^*(\omega, z)\rangle$$
$$= \langle \sum_\mu A_\mu(\omega + \sigma, z) K_{v\mu} f(z)$$
$$\cdot \exp \{i[\beta_v(\omega + \sigma) - \beta_\mu(\omega + \sigma)]z\} A_v^*(\omega, z)\rangle$$
$$+ \langle \sum_\mu A_v(\omega + \sigma, z) A_\mu^*(\omega, z) K_{v\mu}^* f(z) \exp \{-i[\beta_v(\omega) - \beta_\mu(\omega)]z\}\rangle. \quad (5)$$

Next we recognize that the perturbation solution to (2) is given by

$$A_v(z) = A_v(z') + \sum_1^N K_{v\mu} A_\mu(z') \int_{z'}^{z} f(x) \exp [+i(\beta_v - \beta_\mu)x] dx. \quad (6)$$

We shall now substitute (6) into (5) with the following approximations: We assume $f(x)$ is independent of $A_v(z')A_\mu^*(z')$ for $x > z'$. We keep only terms that are second order in $K_{v\mu}$, because first-order terms would include the factor $f(x)$ to first order and, since $f(x)$ has been assumed independent of $A_v(z')A_\mu(z')$ for $x > z'$, the expectation of these terms would vanish [$f(x)$ has zero mean].

We obtain

$$\frac{\partial}{\partial z} \langle A_v(\omega + \sigma, z) A_v^*(\omega, z)\rangle$$
$$= \sum_\mu \sum_\delta \langle A_\delta(\omega + \sigma, z') A_v^*(\omega, z')\rangle K_{\mu\delta} K_{v\mu} \int_{z'}^{z} \langle f(x) f(z)\rangle$$
$$\cdot \exp i[\beta_\mu(\omega + \sigma) - \beta_\delta(\omega + \sigma)](x - z) dx$$
$$\cdot \exp \{i[\beta_v(\omega + \sigma) - \beta_\delta(\omega + \sigma)]z\}$$
$$+ \sum_\mu \sum_\delta \langle A_\mu(\omega + \sigma, z') A_\delta^*(\omega, z')\rangle K_{v\mu} K_{v\delta}^* \int_{z'}^{z} \langle f(x) f(z)\rangle$$
$$\cdot \exp \{-i[\beta_v(\omega) - \beta_\delta(\omega)](x - z)\} dx$$
$$\cdot \exp \{i[\beta_v(\omega + \sigma) - \beta_\mu(\omega + \sigma) - \beta_v(\omega) + \beta_\delta(\omega)]z\}, \quad (7)$$

plus two similar terms where ω and $\omega + \sigma$ are interchanged and the conjugate is taken.

In (7) we have again assumed independence between $f(x)$ and $A_\mu(\omega, z')$ for $x > z'$. Now we assume that terms that are rapidly varying in z can be neglected, i.e., we neglect terms proportional to $\exp\{i[\beta_\mu(\omega) - \beta_v(\omega)]z\}$ unless $\mu = v$. We obtain for small $z - z'$ [so that $\langle A(\omega + \sigma, z) A^*(\omega, z)\rangle \approx \langle A(\omega + \sigma, z') A^*(\omega, z')\rangle$].

$$\frac{\partial}{\partial z} \langle A_v(\omega + \sigma, z) A_v^*(\omega, z)\rangle = - \sum_\mu \langle A_v(\omega + \sigma, z) A_v^*(\omega, z)\rangle |K_{\mu v}|^2 F$$
$$+ \sum_\mu \langle A_\mu(\omega + \sigma, z) A_\mu^*(\omega, z)\rangle |K_{\mu v}|^2 F$$
$$\cdot \exp \{i[\beta_v(\omega + \sigma) - \beta_\mu(\omega + \sigma) - \beta_v(\omega) + \beta_\mu(\omega)]z\}, \quad (8)$$

where F is the spectral height of the random process $f(x)$, i.e.,

$$F \cong 2 \int_{z'-z}^{0} \langle f(x) f(z) \rangle \cdot \exp \{ i [\beta_\mu(\omega + \sigma) - \beta_v(\omega + \sigma)](x - z) \} d(x - z) \quad (9)$$

is assumed independent of $[\beta_u(\omega + \sigma) - \beta_v(\omega + \sigma)]$, which implies $\langle f(x) f(z) \rangle \approx F \delta(x - z)$.

Next we assume that each mode has a well-defined group velocity within the frequency band of interest, that is,

$$\beta_v(\omega + \sigma) - \beta_v(\omega) \cong \frac{\sigma}{C_v}. \quad (10)$$

Substituting (10) into (8) we obtain, using (2),

$$\begin{aligned} \frac{\partial}{\partial z} \langle A_v(\omega + \sigma, z) A_v^*(\omega, z) \rangle &= \frac{\partial}{\partial z} (\langle \mathcal{A}_v(\omega + \sigma, z) \mathcal{A}_v^*(\omega, z) \rangle \exp(i\sigma z / C_v)) \\ &= - \sum_\mu |K_{\mu v}|^2 F \langle \mathcal{A}_v(\omega + \sigma, z) \mathcal{A}_v^*(\omega, z) \rangle \exp(i\sigma z / C_v) \\ &\quad + \sum_\mu |K_{\mu v}|^2 F \langle \mathcal{A}_\mu(\omega + \sigma, z) \mathcal{A}_\mu^*(\omega, z) \rangle \exp(i\sigma z / C_v). \end{aligned} \quad (11)$$

But

$$\begin{aligned} \frac{\partial}{\partial z} (\langle \mathcal{A}_v(\omega + \sigma, z) \mathcal{A}_v^*(\omega, z) \rangle \exp(i\sigma z / C_v)) &= \exp(i\sigma z / C_v) \left[\frac{\partial}{\partial z} \langle \mathcal{A}_v(\omega + \sigma, z) \mathcal{A}_v^*(\omega, z) \rangle \right. \\ &\quad \left. + \frac{i\sigma}{C_v} \langle \mathcal{A}_v(\omega + \sigma, z) \mathcal{A}_v^*(\omega, z) \rangle \right]. \end{aligned} \quad (12)$$

Substituting (12) into (11) and Fourier transforming as in (3), we obtain

$$\begin{aligned} \frac{\partial}{\partial z} \langle s_v(t, z) \rangle + \frac{1}{C_v} \frac{\partial}{\partial t} \langle s_v(t, z) \rangle &= - \sum_\mu |K_{\mu v}|^2 F \langle s_v(t, z) \rangle + \sum_\mu |K_{\mu v}|^2 F \langle s_\mu(t, z) \rangle, \end{aligned} \quad (13)$$

which is the same as eq. (1).

2.2 Kronecker product approach

Rowe and Young[3] assume two modes and write (in our notation)

$$\begin{aligned} \frac{\partial A_1(\omega, z)}{\partial z} &= K_{12} f(z) \exp [i(\beta_1 - \beta_2) z] A_2(\omega, z) \\ \frac{\partial A_2(\omega, z)}{\partial z} &= K_{21} f(z) \exp [i(\beta_2 - \beta_1) z] A_1(\omega, z), \end{aligned} \quad (14)$$

where

$$K_{12} = K_{21} = iK.$$

They then assume that $f(z)$ is the derivative of an independent increments process satisfying

$$\langle f(z)f(z')\rangle = F\delta(z - z') \text{ (Dirac delta).}$$

They divide the fiber into intervals (sections) of length Δ and assume that all the coupling takes place at the right end of each interval. They obtain the following difference equation:

$$\begin{aligned} A_1(\omega, l\Delta) &= \cos(Kc_l)A_1[\omega, (l-1)\Delta] + i\sin(Kc_l)A_2[\omega, (l-1)\Delta] \\ &\qquad \cdot \exp[i(\beta_1 - \beta_2)l\Delta] \\ A_2(\omega, l\Delta) &= i\sin(Kc_l)A_1[\omega, (l-1)\Delta]\exp[i(\beta_2 - \beta_1)l\Delta] \\ &\qquad + \cos(Kc_l)A_2[\omega, (l-1)\Delta], \end{aligned} \tag{15}$$

where

$$c_l = \int_{\text{interval } l} f(z)dz.$$

From (15) we obtain

$$\begin{aligned} &\langle A_1(\omega+\sigma, l\Delta)A_1^*(\omega, l\Delta)\rangle \\ &\quad = \langle\cos^2(Kc_l)\rangle\langle A_1[\omega+\sigma, (l-1)\Delta]A_1^*[\omega, (l-1)\Delta]\rangle \\ &\quad + \langle\sin^2(Kc_l)\rangle\langle A_2[\omega+\sigma, (l-1)\Delta]A_2^*[\omega, (l-1)\Delta]\rangle \\ &\qquad \cdot \exp\{i[\beta_1(\omega+\sigma) - \beta_1(\omega) - \beta_2(\omega+\sigma) + \beta_2(\omega)]z\}. \end{aligned} \tag{16}$$

In (16) we used the fact that c_l is independent of $A_1[\omega, (l-1)\Delta]$ and $A_2[\omega, (l-1)\Delta]$ because $f(z)$ has been assumed to be the derivative of an independent increments process. We also used the fact that terms like $\langle\sin(Kc_l)\cos(Kc_l)\rangle$ equal zero because c_l is a symmetrical random variable.

In the limit as $\Delta \to 0$ (intervals $\{l\}$ get small) we have

$$\langle\cos^2(Kc_l)\rangle \to 1 - FK^2\Delta \quad \langle\sin^2 Kc_l\rangle \to FK^2\Delta.$$

We obtain from (16)

$$\begin{aligned} &\frac{\partial}{\partial z}\langle A_1(\omega+\sigma, z)A_1^*(\omega, z)\rangle \\ &\quad = -FK^2\langle A_1(\omega+\sigma, z)A_1^*(\omega, z)\rangle + FK^2\langle A_2(\omega+\sigma, z)A_2^*(\omega, z)\rangle \\ &\qquad \cdot \exp\{i[\beta_1(\omega+\sigma) - \beta_1(\omega) - \beta_2(\omega+\sigma) + \beta_2(\omega)]z\}, \\ &\frac{\partial}{\partial z}\langle A_2(\omega+\sigma, z)A_2^*(\omega, z)\rangle \\ &\quad = -FK^2\langle A_2(\omega+\sigma)A_2^*(\omega)\rangle + FK^2\langle A_1(\omega+\sigma)A_1^*(\omega)\rangle \\ &\qquad \cdot \exp\{i[\beta_2(\omega+\sigma) - \beta_2(\omega) - \beta_1(\omega+\sigma) + \beta_1(\omega)]z\}. \end{aligned}$$

This is the same as eq. (8) with $K_{12} = K_{21} = iK$ and $K_{11} = K_{22} = 0$. Thus, the coupled-power equations follow from (17) by Fourier transforming and making use of approximation (10).

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