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Optimum Timing Phase for an Infinite Equalizer

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A digital equalizer for data transmission linearly combines a sequence of samples $[b(\tau + kt),\ k = 0, \pm 1, \cdots]$ *of the received data wave to mitigate the effects of intersymbol interference and noise. A natural question is, How will the performance of such a system depend on the timing phase* τ, $0 \leqq \tau \leqq T$?

We examine this problem in considerable detail for an infinite equalizer using a mean-square measure of performance. Excess bandwidth results in a significant difference in performance between the best and worst timing phase. Under practical noise conditions, it is estimated that the excess bandwidth must be down to 1 or 2 percent before the timing effect becomes insignificant. With a 10-percent roll-off, a 3-dB penalty can be incurred by choosing a bad timing epoch.

Our main result is that under conditions likely to be encountered on channels similar to voiceband telephone channels, the optimum sampling instants will be accurately approximated by the consecutive maxima and minima of the sine wave that result when an alternating sequence of positive and negative pulses (dotting sequence) is transmitted. There are no additional local minima of the minimum mean-square error as the timing phase is varied.

I. INTRODUCTION

We consider a noisy, random, baseband pulse train,

$$b(t) = \sum a_n x(t - nT) + n(t), \tag{1}$$

where $x(t)$ is the channel (or channel front-end filter) impulse response and the a_n are independent binary data which take values ± 1 with equal probability. The additive zero mean gaussian noise process is denoted by $n(t)$. Under very ideal conditions, the pulse $x(t)$ would be of the form $(\sin \pi t/T)/(\pi t/T)$, and sampling (1) at time $t = kT$ would yield the quantity $a_k + n(kT)$. Under more realistic conditions, the brick-wall shape of the above pulse is difficult to approximate in practice, and a smoother characteristic in the frequency domain is taken as the ideal. Using some excess bandwidth (i.e., the Fourier transform extends beyond π/T rad/s) still, in principle, allows the ideal set of samples expressed by

$$x(0) = 1, \quad x(kT) = 0, \quad k \neq 0;$$

but this ideal set is never exactly attained in practice because of unknown channel distortion. One possible measure of distortion could be

$$\frac{\sum_{k \neq 0} x^2(kT)}{x^2(0)}.$$

Actually, since there is now nothing special about choosing $t = 0$ as the sampling instant for $x(t)$, an even more appropriate measure would be

$$\min_{\tau} \frac{\sum_{k \neq 0} x^2(\tau + kT)}{x^2(\tau)}.$$

The above measure allows us to choose a best sampling epoch τ before declaring how badly the signal has been distorted. However, any direct consideration of the above type of criterion seems to result in considerable mathematical difficulty. Also, in practice, one is more interested in the situation where the set of digital samples $\{x(\tau + kT)\}$, or rather their noisy versions $\{b(\tau + kT)\}$, are linearly combined by an equalizer attempting to undo the channel distortion. It is the distortion at the output of the equalizer that is of practical interest. Remarkably enough, it is this distortion measured at the output of an infinite equalizer that is most easily studied with regard to its properties as a function of timing phase. A discussion of this dependence follows. We begin by making more precise some of our general remarks.

The received signal $b(t)$ is to be equalized by passing it through a $(2N + 1)$ tap transversal filter to yield for the signal portion of the output a waveform

$$q(t) = \sum a_n h(t - nT), \tag{2}$$

where

$$h(t) = \sum_{-N}^{N} c_n x(t - nT). \tag{3}$$

The coefficients c_n are the tap weights of the equalizer and are adjusted to optimize some measure of performance. Since decisions are based on the sampled output $q(t)$, one usually discusses the problem by fixing a sampling epoch τ and denotes the sampled values of $x(t)$ and $h(t)$ by x_k and h_k, respectively. That is,

$$\begin{aligned} x(\tau + kT) &\equiv x_k \\ h(\tau + kT) &\equiv h_k = \sum_{n=-N}^{N} c_n x_{k-n} \end{aligned} \qquad k = \cdots -1, 0, 1, \cdots. \tag{4}$$

We assume throughout that the tap weights are chosen to minimize the expression

$$\mathcal{E}^2 = \sum_{n=-\infty}^{\infty}{}' h_n^2 + (h_0 - 1)^2 + \sigma_0^2, \tag{5}$$

where σ_0^2 is the sampled noise variance at the output of the equalizer, and the prime on the summation indicates deletion of the $n = 0$ term. Letting $R_n = R_{-n}$ be the (discrete) correlation function of the sampled noise at the input to the equalizer, we see that

$$\sigma_0^2 = \sum_{j,k=-N}^{N} c_j R_{j-k} c_k \tag{6}$$

and

$$\mathcal{E}^2 = \sum_{j,k=-N}^{N} c_j (R_{j-k} + A_{j-k}) c_k - 2 \sum_{n=-N}^{N} c_n x_{-n} + 1, \tag{7}$$

where the channel correlation matrix,

$$A_{n-m} = \sum_{k=-\infty}^{\infty} x_{k-n} x_{k-m}, \tag{8}$$

has been introduced. The optimum tap settings $(c_{\text{opt}})_n$ are determined by differentiating $\mathcal{E}^2$ with respect to the tap gains. This yields a linear system of equations for the best tap gains which, in an obvious matrix-vector notation, reads

$$(A + R) c_{\text{opt}} = x. \tag{9}$$

Using this in (7), we then obtain for the minimum mean-square error

$$\mathcal{E}^2_{\min} = 1 - h_0. \tag{10}$$

Finally, we may state our problem. The above discussion of known results was in terms of a fixed timing epoch τ. Here we shall determine for some special but important cases the value of τ, which will minimize $\mathcal{E}^2_{\min}$, as defined above. Instead of resorting to a direct numerical inversion of (9), we introduce the approximation of treating the equalizer as infinite. This approximation aids the analytical inversion of (9) and thereby directly leads to useful insights concerning the proper timing epoch. It will be seen that in the $N = \infty$ limit, timing recovery is only a problem when the received pulse $x(t)$ has a bandwidth greater than $1/2T$ Hz, a result that could have been guessed from the sampling theorem. With the practical situation of excess bandwidth, there is the possibility that a badly chosen timing phase will considerably degrade performance. Only when there is very little excess bandwidth, say of the order of one percent, is the timing unimportant in practice. Our main result can be best stated in terms of the pure sine wave that appears at the channel output (sampler input) when the sequence $a_n = (-1)^n$ (dotting sequence) is transmitted. The optimum sampling instants are, under certain conditions, the consecutive maxima and minima of this wave. Furthermore, if the timing phase is varied, no extraneous local minima of the minimum mean-square error (regarded as a function of timing phase) will appear.

II. MATHEMATICAL DETERMINATION OF OPTIMUM TIMING PHASE

On comparing (8) and (9), we see that the infinite matrix equation is in the nature of a convolution and, therefore, is best solved by transform methods. Thus, given a sequence $\{x_n\}$, we define a transform

$$X_{\text{eq}}(w) = T \sum_{-\infty}^{\infty} x_n \exp(-iwnT), \quad |w| \leq \frac{\pi}{T}, \tag{11}$$

with the x_k recoverable from $X_{\text{eq}}(w)$ in the obvious way:

$$x_k = \frac{1}{2\pi} \int_{-(\pi/T)}^{\pi/T} X_{\text{eq}}(w) e^{iwkT} dw. \tag{12}$$

This formula shows that, from the impulse-response point of view, $X_{\text{eq}}(w)$ is, in fact, the Nyquist equivalent spectrum of $x(t)$. That is, if $X(w)$ is the Fourier transform of $x(t)$, then

$$X_{\text{eq}}(w) = \sum_{k=-\infty}^{\infty} X\left(w - k\frac{2\pi}{T}\right), \quad |w| \leq \frac{\pi}{T}. \tag{13}$$

In this transform language, the solution of (9) is*

$$C(w) = \frac{TX^*_{\text{eq}}(w)}{R_{\text{eq}}(w) + \frac{1}{T}|X_{\text{eq}}(w)|^2}. \tag{14}$$

In writing this, we have made use of (8) to write

$$A(w) = \frac{1}{T}|X_{\text{eq}}(w)|^2. \tag{15}$$

Equation (4) gives the relation

$$H(w) = \frac{1}{T} C(w) X_{\text{eq}}(w) = \frac{|X_{\text{eq}}(w)|^2}{R_{\text{eq}}(w) + \frac{1}{T}|X_{\text{eq}}(w)|^2} \tag{16}$$

* We will not always use the subscript "eq" on the transform (11) when there does not exist an associated impulse response in a natural way.

and, hence, from (10) and the recovery formula for h_0, we have

$$\mathcal{E}^2_{\min} = \frac{T}{2\pi} \int_{-(\pi/T)}^{\pi/T} \frac{R(w)}{R_{eq}(w) + \frac{1}{T} |X_{eq}(w)|^2} dw. \tag{17}$$

Let us now explicitly put timing recovery into this solution for $\mathcal{E}^2_{\min}$. In evaluating the above formula, one would naturally calculate $X(w)$, possibly by choosing $t = 0$ as the peak of the pulse $x(t)$, and calculate $X_{eq}(w)$ by (13). Any other timing epoch τ can be obtained by replacing $X(w)$ by $X(w) \exp(iw\tau)$. This corresponds to sampling the original $x(t)$ at $t = \tau$ instead of $t = 0$; in general, $X_{eq}(w)$ will depend on τ.

For further specifics, we limit our discussion to small but nonzero excess bandwidth, letting α denote the fraction of excess bandwidth. Thus, $(1 + \alpha)\pi/T$ is the total bandwidth (in rad/s) occupied by the baseband pulse, $\alpha < 1$. Further, we take the noise at the input to the equalizer to be independent from sample to sample and to have variance σ_i^2; thus,

$$R_{eq}(w) = T\sigma_i^2. \tag{18}$$

Equation (17) thus becomes

$$\begin{aligned}
\mathcal{E}^2_{\min} &= \sigma_i^2 \frac{T}{2\pi} \int_{-(\pi/T)}^{\pi/T} \frac{dw}{\sigma_i^2 + \frac{1}{T^2} |X_{eq}(w)|^2} \\
&= \sigma_i^2 \frac{T}{2\pi} \int_{-(\pi/T)(1-\alpha)}^{\pi/T(1-\alpha)} \frac{dw}{\sigma_i^2 + \frac{1}{T^2} |X_{eq}(w)|^2} \\
&\quad + 2\sigma_i^2 \frac{T}{2\pi} \int_{+(\pi/T)(1-\alpha)}^{\pi/T} \frac{dw}{\sigma_i^2 + \frac{1}{T^2} |X_{eq}(w)|^2}.
\end{aligned} \tag{19}$$

We note that the first term in the right-hand side of (19) is not affected by timing phase since the portion of $|X_{eq}(w)|$ for $|w| \leqq \pi/T(1 - \alpha)$ is not affected, whereas the second term, on account of the foldover from excess bandwidth, is affected.*

To obtain estimates regarding the magnitude of the effects of poor timing phase, we consider the case of an undisturbed pulse $x(t)$ having a transform $X(w)$ that is equal to T for $0 \leqq w \leqq (1 - \alpha)\pi/T$ and decreases linearly to zero for $(1 - \alpha)\pi/T \leqq w \leqq (1 + \alpha)\pi/T$, with $X(-w) = X(w)$. If this pulse is sampled at $t = 0$, the "main bang" has value unity, and there is no intersymbol interference. From (19), the minimum mean-square error is $\sigma_i^2(1 + \sigma_i^2) \approx \sigma_i^2$ for small noise.

* Of course, if there is no noise, the equalizer inverts $X_{eq}(w)$ (if the latter is not zero) without any penalty, and the question of timing is moot.

If, however, it is sampled at $\tau = T/2$, we have

$$|X_{\text{eq}}(w)|^2 = \begin{cases} T^2, & 0 \leqq w \leqq (1-\alpha)\dfrac{\pi}{T} \\ \dfrac{T^2}{\alpha^2\pi^2}\left(w - \dfrac{\pi}{T}\right)^2, & (1-\alpha)\dfrac{\pi}{T} \leqq w \leqq \dfrac{\pi}{T} \end{cases} \tag{20}$$

and, hence, with the infinite equalizer doing the best it can, we obtain from (19) for small noise

$$\begin{aligned} \mathcal{E}^2_{\min} &= (1-\alpha)\sigma_i^2 + \alpha\sigma_i \tan^{-1}\frac{1}{\sigma_i} \\ &\approx (1-\alpha)\sigma_i^2 + \frac{\pi}{2}\alpha\sigma_i. \end{aligned} \tag{21}$$

If for binary transmission we assume a 10 or 15 percent roll-off ($\alpha = 0.1$ or 0.15) and $\sigma_i^2 = 0.01$ to 0.04, we see that a degradation of from 2 dB to 4 dB might be encountered. Only if the roll-off is of the order of 1 or 2 percent is the effect of timing negligible for the value of noise power considered here.

We now concentrate further attention on the second term of (19). We introduce a variable

$$\mu = \frac{T}{\pi\alpha}\left[w - \frac{\pi}{T}\right] \tag{22}$$

for $\pi/T(1-\alpha) \leqq w \leqq \pi/T(1+\alpha)$; thus, $-1 \leqq \mu \leqq 1$. When w and μ are related by (22), we define a nonnegative amplitude function $R(w)$ and a real phase function $\varphi(\mu)$ by the relation

$$X(w) = R(\mu)e^{i\varphi(\mu)}, \quad R(\mu) \geqq 0. \tag{23}$$

Further, we introduce the even and odd parts of amplitude and phase about the Nyquist frequency $w = \pi/T(\mu = 0)$:

$$\begin{aligned} R(\mu) &= R_e(\mu) + R_0(\mu) \\ R_e(\mu) &= R_e(-\mu);\, R_0(\mu) = -R_0(-\mu); \end{aligned} \tag{24}$$

$$\begin{aligned} \varphi(\mu) &= \varphi_e(\mu) + \varphi_0(\mu) \\ \varphi_0(\mu) &= \varphi_e(-\mu);\, \varphi_0(\mu) = -\varphi_0(-\mu). \end{aligned} \tag{25}$$

Thus, for $-1 \leqq \mu \leqq 0$, we calculate

$$|X_{\text{eq}}(\mu)|^2 = 4[(R_e^2 - R_0^2)\cos^2\varphi_e + R_0^2], \tag{26}$$

where, for notational simplicity, the μ dependence of the function on the right has not been shown. Using (26), the second term S of (19) is written

$$S = \alpha\sigma_i^2 \int_{-1}^{0} \frac{d\mu}{\sigma_i^2 + \dfrac{4}{T^2}[(R_e^2 - R_0^2)\cos^2\varphi_e + R_0^2]}. \tag{27}$$

A number of remarks concerning (27) are in order.

(*i*) The odd component of phase distortion about the Nyquist frequency does not influence "foldover" or the timing recovery problem.

(*ii*) From the positivity of $R(\mu)$, we note that $(R_e^2 - R_0^2) \geqq 0$. We assume that $[R_e^2(\mu) - R_0^2(\mu)] > 0$ on some interval contained in $-1 \leqq \mu \leqq 0$.

(*iii*) Relative to an arbitrarily selected sampling instant, called zero, the timing phase τ enters φ_e through

$$\varphi_e = \frac{\pi}{T}\tau + \bar{\varphi}_e(\mu), \tag{28}$$

where $\bar{\varphi}_e(\mu)$ is the phase characteristic for $\tau = 0$. It is evident that the mean-square error is a periodic function of τ with period T. Physically, if τ is displaced by an integer (k) multiple of T, the "center tap" will be shifted k units down the equalizer.

(*iv*) In many situations, $\bar{\varphi}_e$ can, to a first approximation, be treated as a constant. A recent survey[1] shows that an "average" telephone channel has an envelope delay that changes by 312 μs from 2850 Hz to 3000 Hz. Assuming this to be an excess bandwidth region and the change to be due entirely to a quadratic term in the phase curve yields a phase change of 8.4 degrees.*

We now come to the main point of this section, which is to recognize where the optimum timing phase is. This will be done with the simplifying assumption

$$\bar{\varphi}_e = \theta = \text{const.} \tag{29}$$

Since S is a smooth periodic function of τ, we look for a minimum by examining solutions of

$$\frac{\partial S}{\partial \tau} = 0, \tag{30}$$

that is,

$$\sin 2\varphi_e \int_{-1}^{0} \frac{(R_e^2 - R_0^2)d\mu}{\left\{\sigma_i^2 + \frac{4}{T^2}\left[(R_e^2 - R_0^2)\cos^2\varphi_e + R_0^2\right]\right\}^2} = 0. \tag{31}$$

From (*ii*) we deduce that the integral is strictly positive and, hence, the only solutions are those of

$$\sin 2\varphi_e = 0. \tag{32}$$

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* This is meant to be a rough estimate only. No attempt was made to separate out even and odd contributions to the change in phase. Low-end envelope delay is more severe and this separation should be carried out so as to obtain a not-too-pessimistic answer.

Thus, because of periodicity the only solutions τ of interest may be written

$$\frac{\pi}{T}\tau + \theta = 0 \tag{33a}$$

or

$$\frac{\pi}{T}\tau + \theta + \frac{\pi}{2} = 0. \tag{33b}$$

By taking second derivatives of S with respect to τ, we see that the first of these, namely (33a), corresponds to the minimum of S while the second solution is a maximum. In addition to giving a description of optimum timing phase, our discussion has also shown that for φ_e = const. there are no other local minima of $\mathcal{E}^2_{\min}$ as τ is varied.* Thus, a gradient search for the optimum τ is feasible. More precisely, we have shown this for the quadratic Nyquist I criterion (5), and have assumed that the search is slow in the sense that the equalizer "settles down" before the next step in τ is taken.

Equation (33a) provides a mathematical description of the optimum timing phase τ in terms of the phase θ of the channel at the Nyquist frequency π/T. It is useful to give a more physical interpretation of this result. Consider the transmission of the special sequence $a_n = (-1)^n$. From (1), we receive the signal

$$\sum_{n=\infty}^{\infty} (-1)^n x(t - nT),$$

which has as its Fourier transform

$$\begin{aligned}
X(w) &\sum_{n=\infty}^{\infty} (-1)^n \exp(-iwnT) \\
&= X(w) \sum_{k=\infty}^{\infty} \exp(-iw2kT) - X(w)\exp(-iwT) \sum_{k=-\infty}^{\infty} \exp(-iw2kT) \\
&= \frac{2\pi}{T} X(w) \sum_{-\infty}^{\infty} \delta\left(w - \frac{k\pi}{T}\right) - \frac{2\pi}{T} X(w)\exp(-iwT) \sum_{-\infty}^{\infty} \delta\left(w - \frac{k\pi}{T}\right) \\
&= \frac{4\pi}{T} X(w) \sum_{k \text{ odd}} \delta\left(w - \frac{k\pi}{T}\right).
\end{aligned}$$

Hence, if $X(w)$ is band-limited to a bandwidth less than $3\pi/T$, and if we write $X(\pi/L) = A \exp(i\theta)$, the received signal is

$$\frac{4}{T} A \cos\left(\frac{\pi t}{T} + \theta\right).$$

Comparing this with (32) and (33a), we see that the optimum timing

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* See the appendix for a more general condition under which the existence of a unique minimum is guaranteed.

phase is at the successive maxima and minima of the tone that results when a dotting sequence is transmitted.

Finally, we note that corrections for a small, varying, even component of phase distortion about the Nyquist frequency are easy to make. If we define γ by

$$\frac{\pi}{T}\tau + \theta \equiv \gamma, \tag{34}$$

where, again, θ is the phase at the Nyquist frequency, and if we write [recalling (28)],

$$\varphi_e(\mu) = \gamma + \epsilon(\mu), \quad \epsilon(0) = 0,$$

where $\epsilon(\mu)$ is assumed small, then

$$\gamma^* = -\frac{\displaystyle\int_{-1}^{0} \frac{(R_e^2 - R_0^2)\epsilon(\mu)}{[\sigma_i^2 + (4/T^2)R_e^2]^2} d\mu}{\displaystyle\int_{-1}^{0} \frac{(R_e^2 - R_0^2)}{\{\sigma_i^2 + (4/T^2)R_e^2\}^2} d\mu}, \tag{35}$$

where γ^* is the approximate value of γ for optimum timing phase τ. We see γ^* is bounded in magnitude by max $\epsilon(\mu)$, but in general it will be a fraction of this. Assuming max $\epsilon(\mu)$ corresponds to 9 degrees, this rough bound suggests that a correction of at most 5 percent of a symbol interval would be required.

For a more refined estimate we take the following

$$R_e(\mu) = 1, \quad R_0(\mu) = \mu, \quad \text{and} \quad \epsilon(\mu) = \epsilon\mu^2. \tag{36}$$

Note that $R_e^2 - R_0^2 = (R_e - R_0)(R_e + R_0) = R(\mu)R(-\mu)$. Since $R(1)$ will vanish by the design of the signal, $R_e^2 - R_0^2$ will be zero at $\mu = -1$ and, thus, no multiplicative constant is included in $R_0(\mu)$. Using (36) in (35) yields

$$\gamma^* \approx -\epsilon \frac{\displaystyle\int_{-1}^{0} (1 - \mu^2)\mu^2 d\mu}{\displaystyle\int_{-1}^{0} (1 - \mu^2) d\mu} = -\frac{\epsilon}{5}. \tag{37}$$

Thus an estimate of $\frac{1}{5}$ of the even part of the phase variation over the roll-off band results for the correction term to (34). This would appear to be a negligible effect.

III. DISCUSSION FOR PASSBAND TRANSMISSION

We now generalize the foregoing discussion to treat a QAM passband signal format. We assume a transmitted signal

$$[\sum a_n p(t - nT)] \cos w_c t - [\sum b_n p(t - nT)] \sin w_c t, \tag{38}$$

where (a_n) and (b_n) are independent digits. Let the impulse response of the linear distortion medium $m(t)$ be written as

$$m(t) = 2F_1(t)\cos w_c t - 2F_2(t)\sin w_c t. \tag{39}$$

Further, denote the convolution of $p(t)$ with $F_1(t)$ and $F_2(t)$ by $x(t)$ and $y(t)$, i.e.,

$$p(t)*F_1(t) = x(t) \quad p(t)*F_2(t) = y(t). \tag{40}$$

Then the received signal is

$$r(t) = \cos w_c t[\sum a_n x(t-nT) - \sum b_n y(t-nT)] - \sin w_c t[\sum a_n y(t-nT) + \sum b_n x(t-nT)] + n(t), \tag{41}$$

where $n(t)$ represents additive gaussian noise. The waveform $r(t)$ is then linearly filtered by a passband equalizer, whose output can be described by replacing $x(t)$ by $g(t)$ and $y(t)$ by $h(t)$, where

$$g(t) = \sum_{-N}^{N} c_n x(t-nT) - \sum_{-N}^{N} d_n y(t-nT)$$

and (42)

$$h(t) = \sum_{-N}^{N} d_n x(t-nT) + \sum_{-N}^{N} c_n y(t-nT).$$

Further, if the input noise $n(t)$ has correlation $R(t)$, then the output noise has variance

$$\sigma_0^2 = c^+Rc + d^+Rd, \tag{43}$$

where $R_{ij} = R[(i-j)T]$. We adopt as our criterion

$$\mathcal{E}^2 = \sum_{i\neq 0} g_i^2 + (1-g_0)^2 + \sum_{i=-\infty}^{\infty} h_i^2 + \sigma_0^2 \tag{44}$$

and seek to minimize this quantity. Analogously with the baseband case we define two baseband autocorrelation functions [corresponding to $x(t)$ and $y(t)$ in (41)] by

$$A_{nm} = A_{n-m} = \sum_{j=-\infty}^{\infty} x_{j-n}x_{j-m}$$

and (45)

$$B_{nm} = B_{n-m} = \sum_{j=-\infty}^{\infty} y_{j-n}y_{j-m}.$$

Further, a cross-correlation matrix between $x(t)$ and $y(t)$, namely,

$$K_{nm} = K_{n-m} = -K_{mn} = \sum_j (x_{j-n}y_{j-m} - x_{j-m}y_{j-n}). \tag{46}$$

If, as before, we write $(x)_n = x_{-n}$, $(y)_n = y_{-n}$, then

$$\mathcal{E}^2 = c^+(A + B)c + d^+(A + B)d - 2c^+Kd - 2c\cdot x + 2d\cdot y + 1 + c^+Rc + d^+Rd. \quad (47)$$

If one defines the matrix $Q \equiv A + B + R$, the equations for the optimum taps are

$$\begin{aligned} \sum_m (Q_{n-m}c_m - K_{n-m}d_m) &= x_{-n} \\ \sum_m (Q_{n-m}d_m + K_{n-m}c_m) &= -y_{-n} \end{aligned} \quad (48)$$

and

$$\mathcal{E}^2_{\min} = 1 - (q_0)_{\text{opt}} = 1 - c_{\text{opt}}\cdot x + d_{\text{opt}}\cdot y. \quad (49)$$

We introduce the same transform as before, noting

$$Q(w) = \frac{|X_{\text{eq}}(w)|^2}{T} + \frac{|Y_{\text{eq}}(w)|^2}{T} + R(w) = Q(-w) \quad (50)$$

and

$$K(w) = \frac{1}{T}[X^*_{\text{eq}}(w)Y_{\text{eq}}(w) - X_{\text{eq}}(w)Y^*_{\text{eq}}(w)] = -K(-w). \quad (51)$$

Straightforward solution of (48) and (49) yields

$$\begin{aligned} \mathcal{E}^2_{\min} &= \frac{T}{2\pi}\int_{-(\pi/T)}^{\pi/T} \frac{Q(w)R(w)}{Q^2(w) + K^2(w)}\,dw \\ &= \frac{T}{2\pi}\int_{-(\pi/T)}^{\pi/T} \frac{[Q(w) - iK(w)]R(w)}{Q^2(w) + K^2(w)}\,dw \\ &= \frac{T}{2\pi}\int_{-(\pi/T)}^{\pi/T} \frac{R(w)}{Q(w) + iK(w)}\,dw, \end{aligned} \quad (52)$$

where the oddness of $K(w)$ [and evenness of $Q(w)$ and $R(w)$] has been used. If we formally introduce $S(w)$ by

$$S(w) = X(w) + iY(w), \quad (53)$$

and

$$S_{\text{eq}}(w) = X_{\text{eq}}(w) + iY_{\text{eq}}(w), \quad (54)$$

we see that

$$\begin{aligned} |S_{\text{eq}}(w)|^2 &= [X_{\text{eq}}(w) + iY_{\text{eq}}(w)][X^*_{\text{eq}}(w) - iY^*_{\text{eq}}(w)] \\ &= |X_{\text{eq}}(w)|^2 + |Y_{\text{eq}}(w)|^2 + iY_{\text{eq}}(w)X^*_{\text{eq}}(w) - iX_{\text{eq}}(w)Y^*_{\text{eq}}(w) \\ &= T[A(w) + B(w) + iK(w)], \end{aligned} \quad (55)$$

or, briefly,

$$Q + iK = \frac{|S_{\text{eq}}|^2}{T} + R. \quad (56)$$

Thus,

$$\mathcal{E}^2_{\min} = \frac{T}{2\pi}\int_{-(\pi/T)}^{\pi/T} \frac{R(w)}{\frac{|S_{eq}(w)|^2}{T} + R(w)}\,dw. \tag{57}$$

This formula has a close analogy to the baseband formula when we realize that the overall channel (transmitting and distorting filter) has frequency characteristic

$$C(w) = \tfrac{1}{2}S(w - w_c) + \tfrac{1}{2}S^*(-w - w_c). \tag{58}$$

Thus, in terms of the overall channel, (57) has a straightforward interpretation. One uses a formula similar to the baseband version (17) except that now one forms an equivalent Nyquist characteristic using twice the positive frequency part of the passband channel and treats the carrier frequency as zero. It is essential to note that the transfer characteristic from which $S_{eq}(w)$ is formed no longer needs to have even amplitude and odd phase about "zero" as would be required in the baseband case. In particular, the envelope delays at the two band edges can have an effect on timing phase. In practical cases, the carrier frequency can be chosen to give equal envelope delays, thus minimizing the effects of odd phase about the band edges. However, in doing this, the amplitude effects may have worsened the situation. The main point to be emphasized here is the interplay of carrier placement and bit timing with an excess bandwidth system.

Regarding the description of optimum timing phase, let $\varphi^{\pm}$ be the channel phase at $w = w_c \pm \pi/T$, and assume these phases to be constant over their respective roll-off regions. Then, for any reasonable amplitude distortions, the unique best sampling instant τ is given by

$$\frac{\pi}{T}\tau + \tfrac{1}{2}(\varphi^+ - \varphi^-) = 0. \tag{59}$$

Under these conditions, we again have that there exists but one local minimum.

IV. ACKNOWLEDGMENT

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APPENDIX

Uniqueness of Local Minimum of Mean-Square Error for Small $\bar{\varphi}_e(x)$

The interesting portion of the mean-square error is given by (27), which is of the form

$$g(\gamma) \equiv \int_0^1 \frac{dx}{a(x) + b(x) \cos^2 [\gamma + \varphi(x)]}, \tag{60}$$

with $a(x)$ and $b(x)$ positive. If $\varphi(x)$ is always small, then any local minimum of (60) has to occur when γ is small. Thus, if $|\varphi(x)| < \pi/8$, we see that the derivative

$$g'(\gamma) = \int_0^1 \frac{b(x) \sin [2\gamma + 2\varphi(x)]}{\{a(x) + b(x) \cos^2 [\gamma + \varphi(x)]\}^2} dx \tag{61}$$

can only vanish if $|\gamma| < \pi/8$ or $|\pi - \gamma| < \pi/8$, for otherwise the integrand always has the same sign. Let us investigate the solution in $|\gamma| < \pi/8$ in more detail. Taking another derivative gives

$$g''(\gamma) = 2\int_0^1 dx \frac{b \cos (2\gamma + 2\varphi)}{[a + b \cos^2 (\gamma + \varphi)]^2} + 2\int_0^1 dx \frac{b^2 \sin^2 (2\gamma + 2\varphi)}{[a + b \cos^2 (\gamma + \varphi)]^3}, \tag{62}$$

which is positive. Hence, there can be only one solution of $g'(\gamma) = 0$ for $|\gamma| < \pi/8$ (we cannot have two minima without a maximum in between). Likewise, if $|\varphi(x)| < \pi/8$, there is precisely one maximum of $g(\gamma)$, and it occurs in the range $|\pi - \gamma| < \pi/8$.

REFERENCE

1. F. P. Duffy and T. W. Thatcher, "Analog Transmission Performance on the Switched Telecommunications Network," B.S.T.J., *50*, No. 4 (April 1971), pp. 1311–1347.