Copyright © 1973 American Telephone and Telegraph Company
THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 52, No. 9, November, 1973
Printed in U.S.A.

Peakedness of Traffic Carried by a Finite Trunk Group With Renewal Input

By H. HEFFES and J. M. HOLTZMAN

(Manuscript received May 17, 1973)

In trunking theory, peakedness is defined conventionally as the variance-to-mean ratio of a traffic load when carried on an infinite trunk group. For analysis of switching machine delays, it has proven useful to define a peakedness measure associated with the Carried Arrival Process (CAP), the stream of call arrivals carried on an incoming trunk group. The peakedness of the CAP is defined to be the conventional peakedness of a fictitious traffic-load process generated by associating with each carried arrival an independent exponentially distributed holding time with mean equal to the mean of calls actually carried on the trunk group.

The problem considered is the effect of trunk group congestion on the peakedness of the CAP for traffic consisting of renewal inputs offered on a blocked-calls-cleared basis to a finite trunk group with exponential holding times. The CAP is characterized as a semi-Markov process. This model leads to the determination of the peakedness of the CAP. Numerical results illustrate the reduction of peakedness, or smoothing, introduced by the congestion.

I. INTRODUCTION

This paper is concerned with characterizing the traffic offered to a switching machine, taking into account both the alternate routing that the traffic may have undergone and the smoothing of the traffic resulting from congestion on the trunk group incoming to the machine. In trunking theory, peakedness is defined conventionally as the variance-to-mean ratio of a traffic load carried on an infinite trunk group. It is well known that trunk group blocking of peaked traffic, such as overflow traffic, can be substantially larger than the blocking seen by Poisson traffic with the same intensity. Similarly, switching machine* delay and capacity can be quite sensitive to the peakedness

* Throughout this paper, when we refer to a switching machine we mean the common control devices in a switching machine.

of the incoming traffic.[1] To determine the peakedness of the traffic offered to a switching machine, we must take into account the smoothing effect of the incoming trunk groups. To this end, we consider the process of arrivals offered to a trunk group which are carried by that trunk group. We call this process the Carried Arrival Process, or CAP.

To illustrate the CAP, consider the alternate routing network shown in Fig. 1. Here traffic overflowing trunk group AB is then offered to trunk group AC [Fig. 1(c)]. Those calls finding free circuits on AC then appear at node C as requests for service. The CAP is illustrated in Fig. 1(d).

The basic model used in the analysis is shown in Fig. 2 where a renewal process is offered to a group of N trunks on a blocked-calls-cleared (BCC) basis. The renewal input allows us to consider overflow traffic offered to an incoming trunk group. The holding times on the trunks are assumed to be independent, identically distributed, exponential random variables with service rate μ.

For analysis of machine performance it has proven useful to define a measure of peakedness for the CAP, z_c, equal to the variance-to-mean ratio of the traffic load carried (number of trunks occupied) on an infinite trunk group to which the Carried Arrival Process has been offered. By definition, the holding times on the infinite trunk group

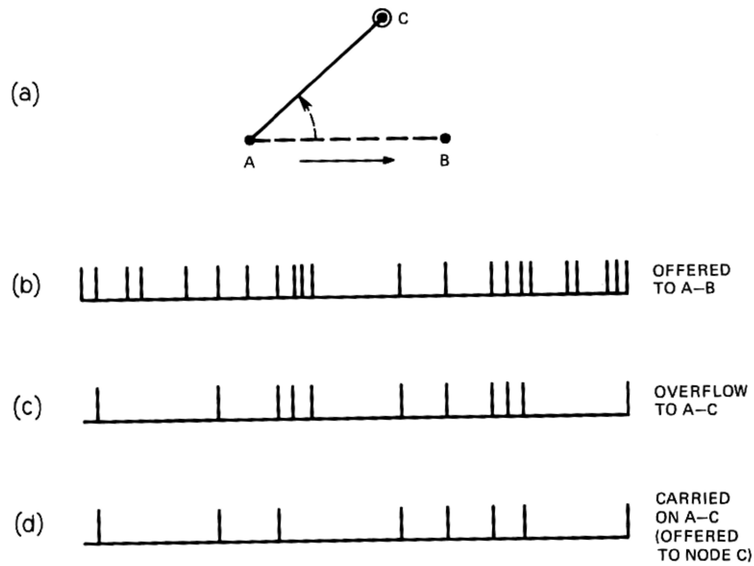

Fig. 1—Carried Arrival Process.

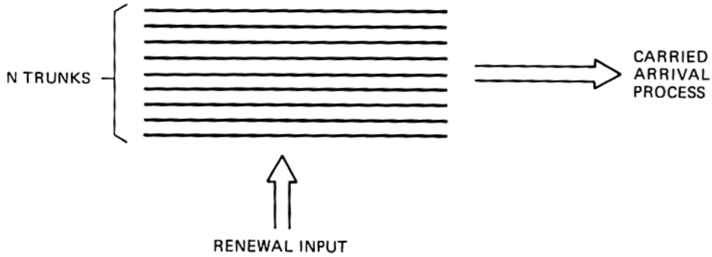

Fig. 2—Carried arrival model.

are independent, identically distributed, exponential random variables (with service rate μ) which are *independent* of the holding times on the incoming trunks.*

The peakedness of the CAP should be distinguished from the variance-to-mean ratio of busy trunks on the incoming group, a quantity which is discussed, for example, in Section 8.4 of Reference 2 for the case of Poisson input. This distinction can be made clear by considering an example of Poisson traffic of intensity λ(calls/second) offered to N trunks. As λ approaches zero, the two measures approach one. Clearly, as λ gets large, the variance of busy circuits in the trunk group goes to zero and the mean goes to N, giving a variance-to-mean ratio of zero. On the other hand, as λ gets large, the time differences between successive carried calls approach independent, exponential random variables with rate $N\mu$ (i.e., a Poisson stream) and the peakedness z_c approaches unity. This is illustrated graphically in Fig. 3 which plots z_c and $(v/m)_{B.S.}$ (variance-to-mean ratio of busy servers on the N trunks) as a function of offered load for $N = 10$. The example is a special case of the general results derived in this paper for arbitrary renewal input to the trunk group.

By modeling the CAP as a semi-Markov process (SMP), it becomes possible to calculate peakedness z_c as a function of the peakedness of the traffic offered to the incoming trunk group and of the congestion encountered on the group. The resulting z_c may then be used in the determination of machine performance.[1] Numerical results illustrate the reduction in peakedness, or smoothing, introduced by the trunk group congestion. In the course of determining the peakedness, the transform of the distribution of the time between carried calls is derived.

* That is, although carried arrivals are accepted simultaneously on the finite and infinite trunk groups, the departure times are different.

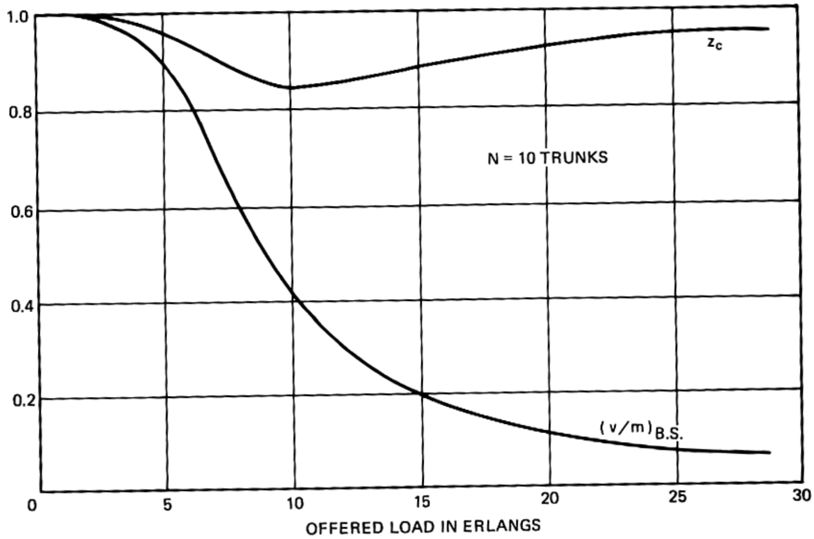

Fig. 3—Distinction between peakedness and variance-to-mean ratio of busy servers.

II. OUTLINE OF RESULTS

In this section, we give an informal overview of the results. In particular, we give the key equations that a user can employ to determine the peakedness, z_c, of a Carried Arrival Process. The equation numbers are the same as will be used in the derivation. Stationarity is assumed throughout. In all that follows, we assume unity holding time (or the time unit is the mean holding time).

First of all,

$$z_c = M^+ - \frac{1}{\beta}, \tag{11}$$

where M^+ is the mean number of calls up on the infinite trunk just after the time that a call is accepted onto the finite (and infinite) trunk group. $(1/\beta)$ is the mean number of carried calls (on both the finite and infinite trunk group).*

M^+ is determined from the following equation:

$$M^+ = \frac{1}{1 - \phi(1)} - M_N^+ P_N^+ \left[\frac{\phi(N+1)}{1 - \phi(N+1)} \right], \tag{32}$$

where $\phi(s)$ is the Laplace-Stieltjes transform of the interarrival time

* Note that, since we are assuming unity holding time, M^+ and $(1/\beta)$ are in erlangs.

distribution. M_j^+ is the mean number of calls up on the infinite trunk group immediately after arrival of a carried call, given j calls in progress on the finite trunk group immediately following the arrival of the carried call. P_j ($j = 0, 1, \cdots, N$) is the probability that an arrival finds j trunks busy on the finite access group, and

$$P_j^+ = \frac{P_{j-1}}{1 - P_N} \tag{24}$$ *

is the probability of j calls up on the finite trunk group immediately following a carried call arrival.

The only quantity left to be determined in (32) is M_N^+ which is calculated by solving the linear equations

$$[M_m^+ - 1]P_m^+ = \sum_{l=\max(1,m-1)}^{N} C_{lm} M_l^+, \qquad m = 1, 2, \cdots, N, \tag{33}$$

where

$$C_{lm} = P_l^+ \binom{l}{m-1} \sum_{\eta=0}^{l-m+1} \binom{l-m+1}{\eta} (-1)^\eta \phi(\eta + m)$$

$$m - 1 \leqq l \leqq N - 1, \quad 1 \leqq m \leqq N, \tag{34}$$ †

$$C_{Nm} = \frac{P_N^+ \binom{N}{m-1} \sum_{\eta=0}^{N-m+1} \binom{N-m+1}{\eta} (-1)^\eta \phi(\eta + m)}{1 - \phi(N+1)}. \tag{35}$$ †

A simple method of solving (33) is discussed at the end of Section VI. In the course of deriving the expressions which ultimately determine the peakedness of the CAP, the Laplace-Stieltjes transform of the distribution of time between carried call arrivals is obtained. This is given by

$$\phi_c(s) = \phi(s) - \frac{[1 - \phi(s)]\phi(s + N)}{1 - \phi(s + N)} P_N^+. \tag{23}$$

Note that the CAP is not completely characterized by (23), since it is not generally a renewal process.

Examples are given in Section VII.

---

* One method of computing P_j is via the equations given on p. 179 of Reference 3. Alternate methods which may avoid some of the numerical difficulties inherent in this approach will be discussed in Appendix C.

† Alternate expressions for special cases, more suitable for computation, are given in Appendix C.

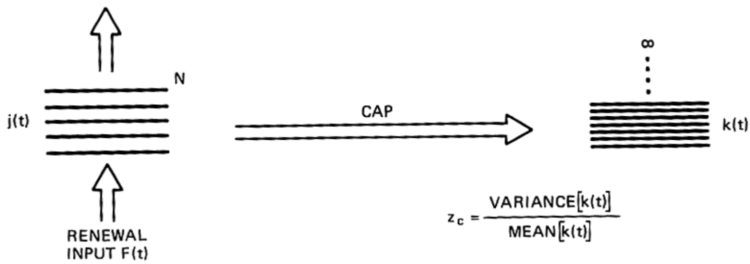

Fig. 4—Peakedness of the Carried Arrival Process.

III. CHARACTERIZATION OF THE CARRIED ARRIVAL PROCESS AS A SEMI-MARKOV PROCESS

Consider a renewal process, with a nonlattice interarrival time distribution $F(t)$, as an input to N trunks with mutually independent exponential holding times each with mean unity (or the time unit is the mean holding time). $F(0^+)$ is assumed to be zero. Blocked calls are cleared. Such a system is analyzed in Chapter 4 of Reference 3.

Let $\{\tau_i,\ i = 1, 2, \cdots\}$ denote the sequence of times at which calls are accepted by the N servers (this is, of course, a subset of the times at which calls are offered). Let $j(t)$ be the number of servers busy at time t. Then $j(\tau_n^+)$ is the number of servers busy just after the nth carry (nth carried call). Note that $P\{j(\tau_n^+) = 0\} = 0$. It is clear that the $j(t)$ process held fixed at $j(\tau_n^+)$ for $\tau_n < t < \tau_{n+1}$ is an SMP.* The transition probabilities for the embedded Markov chain are derived as follows.

Since there is a death process on the finite trunk group between carried calls, we have for $m = 1, 2, \cdots, N$ and for $l = m - 1, \cdots, N - 1$ (in which case, the next arrival is the next carry)

$$P\{j(\tau_n^+) = m \mid j(\tau_{n-1}^+) = l\}$$
$$= \int_0^\infty \binom{l}{m-1} e^{-(m-1)t}[1 - e^{-t}]^{l-m+1} dF(t)$$
$$= \binom{l}{m-1} \sum_{\eta=0}^{l-m+1} \binom{l-m+1}{\eta} (-1)^\eta \phi(\eta + m - 1), \quad (1)$$

where $\phi(s)$ is the Laplace-Stieltjes transform of $F(t)$,

$$\phi(s) = \int_0^\infty e^{-st} dF(t) \cdot \quad (2)$$

* For an introduction to semi-Markov processes, see Reference 4, Chapter 5.

When $j(\tau_{n-1}^+) = N$, note that the next arrival need not be the next carry. With s.c. denoting service completion and $F_i(t)$ the ith-fold convolution of $F(t)$, we have

$$P\{j(\tau_n^+) = m \mid j(\tau_{n-1}^+) = N\}$$

$$= \sum_{i=1}^{\infty} P\left\{\begin{matrix}\text{no s.c. before the first } (i-1) \\ \text{arrivals after } \tau_{n-1},\ (N-m+1) \text{ s.c.'s} \\ \text{before } i\text{th arrival after } \tau_{n-1}\end{matrix} \,\middle|\, j(\tau_{n-1}^+) = N\right\}$$

$$= \sum_{i=1}^{\infty} \int_0^{\infty} \int_0^t e^{-Ns} \binom{N}{m-1} e^{-(m-1)(t-s)}[1 - e^{-(t-s)}]^{N-m+1} \times dF_{i-1}(s)dF(t-s)$$

$$= \frac{\binom{N}{m-1} \sum_{\eta=0}^{N-m+1} \binom{N-m+1}{\eta} (-1)^{\eta}\phi(\eta + m - 1)}{1 - \phi(N)}. \tag{3}$$

Letting $F_{lm}(t)$ be the conditional probability that a transition will take place within a time t, given that the process has just entered l and will next enter m, we have

$$dF_{lm}(t) = \begin{cases} \dfrac{\binom{l}{m-1} e^{-(m-1)t}[1 - e^{-t}]^{l-m+1} dF(t)}{P\{j(\tau_n^+) = m \mid j(\tau_{n-1}^+) = l\}}, & l = m-1, \cdots, N-1, \\[2ex] \dfrac{\sum_{i=1}^{\infty} \int_0^t e^{-Ns} \binom{N}{m-1} e^{-(m-1)(t-s)} \times [1 - e^{-(t-s)}]^{N-m+1} dF_{i-1}(s) dF(t-s)}{P\{j(\tau_n^+) = m \mid j(\tau_{n-1}^+) = N\}}, & l = N. \end{cases} \tag{4}$$

Although the SMP characterization of the CAP is of general interest, it is particularly useful in determining the peakedness of the CAP as we shall see in Section IV.

IV. PEAKEDNESS OF THE CARRIED ARRIVAL PROCESS

Recall that the peakedness of a process is the variance-to-mean ratio of the number of calls up on an infinite trunk group when that process is offered to the infinite trunk group. To determine the peakedness of the CAP, consider the situation shown in Fig. 4. In this figure, each time a call is carried on the finite group, a call is put up on the infinite group with an exponentially distributed holding time with the same mean as on the N-trunk group but independent of the N-trunk group holding time.* As before, $j(t)$ is the number of busy servers on the N-trunk group at time t. Let $k(t)$ be the number of busy servers on the infinite trunk group at time t. We use the following familiar result (given for renewal input in Reference 3 and for semi-Markov input in Reference 5, which is the form applicable to our problem). The following limits

$$\bar{P}_i = \lim_{n\to\infty} P\{k(\tau_n^-) = i\} \tag{5}$$

and

$$\bar{P}_i^* = \lim_{t\to\infty} P\{k(t) = i\} \tag{6}$$

exist and satisfy

$$\bar{P}_i^* = \frac{\bar{P}_{i-1}}{i\beta}, \tag{7}$$

where β is the mean time between transitions of the SMP (i.e., mean time between carried calls). From (7) we obtain

$$\lim_{t\to\infty} E\{k^2(t)\} = \frac{1}{\beta}[\lim_{n\to\infty} M(\tau_n^-) + 1] = \frac{1}{\beta}\lim_{n\to\infty} M(\tau_n^+), \tag{8}$$

where we have defined

$$M(t) = E\{k(t)\}. \tag{9}$$

Defining

$$M^+ = \lim_{n\to\infty} M(\tau_n^+), \tag{10}$$

the peakedness of the CAP (denoted z_c) is given by

$$z_c = M^+ - \frac{1}{\beta}. \tag{11}^\dagger$$

Note that $1/\beta$ corresponds to the mean of the carried load (recall that we are assuming unity mean holding time).

We are thus left with the problem of determining M^+. This determination will be in terms of the distribution of time between carried calls, to which the next section is devoted.

V. DISTRIBUTION OF TIME BETWEEN CARRIED CALLS

Consider an arrival at τ_n which finds a free circuit [i.e., $j(\tau_n^-) < N$]. Let $F_c(t)$ be the distribution of time until the next carry (carried call);

---

* It is this independence which distinguishes the peakedness from the variance-to-mean ratio of busy trunks on the N-trunk group (as discussed in Section I).

† This is given in Reference 6 for the case of renewal input and weaker assumptions on service times.

i.e.,

$$F_c(t) = P\{ict \leqq t \mid \text{carry at } \tau_n\}, \tag{12}$$

where *ict* denotes inter-carry time. Denoting

$$F_c[t \mid j(\tau_n^+) < N] = P\{ict \leqq t \mid j(\tau_n^+) < N\}$$

and

$$\bar{F}_c(t) = P\{ict \leqq t \mid j(\tau_n^+) = N\} \tag{13}$$

and recognizing that

$$F_c[t \mid j(\tau_n^+) < N] = F(t) \tag{14}$$

yields

$$F_c(t) = F(t)\left[1 - \frac{P_{N-1}}{1 - P_N}\right] + \bar{F}_c(t)\left[\frac{P_{N-1}}{1 - P_N}\right], \tag{15}$$

where

$$P_j = P\{j \text{ trunks busy on the finite group just before a call arrival}\} \tag{16}$$

is the stationary call congestion probability given on p. 179 in Chapter 4 of Reference 3. In particular, P_N is the blocking probability.

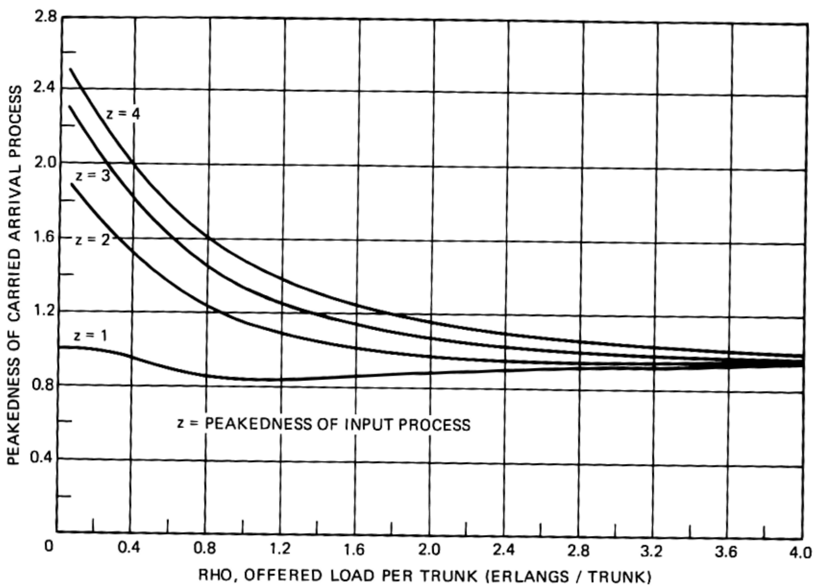

Fig. 5—Peakedness of CAP (5 trunks).

Let $\phi_c(s)$ and $\bar{\phi}_c(s)$ be the Laplace-Stieltjes transforms of $F_c(t)$ and $\bar{F}_c(t)$, respectively. Transforming (15) gives

$$\phi_c(s) = \phi(s)\left[1 - \frac{P_{N-1}}{1 - P_N}\right] + \bar{\phi}_c(s)\left[\frac{P_{N-1}}{1 - P_N}\right]. \tag{17}$$

The function $\bar{\phi}_c(s)$ can be obtained from the solution to the Type I counter problem given on p. 207 of Reference 3 with renewal input transform $\phi(s)$:

$$\bar{\phi}_c(s) = [1 - \phi(s)] \int_0^\infty e^{-sy} H(y) dm(y)\,, \tag{18}$$

where, for our problem,

$$H(t) = 1 - e^{-Nt} \tag{19}$$

and

$$m(t) = E\{\text{number of arrivals in } (0, t)\}. \tag{20}$$

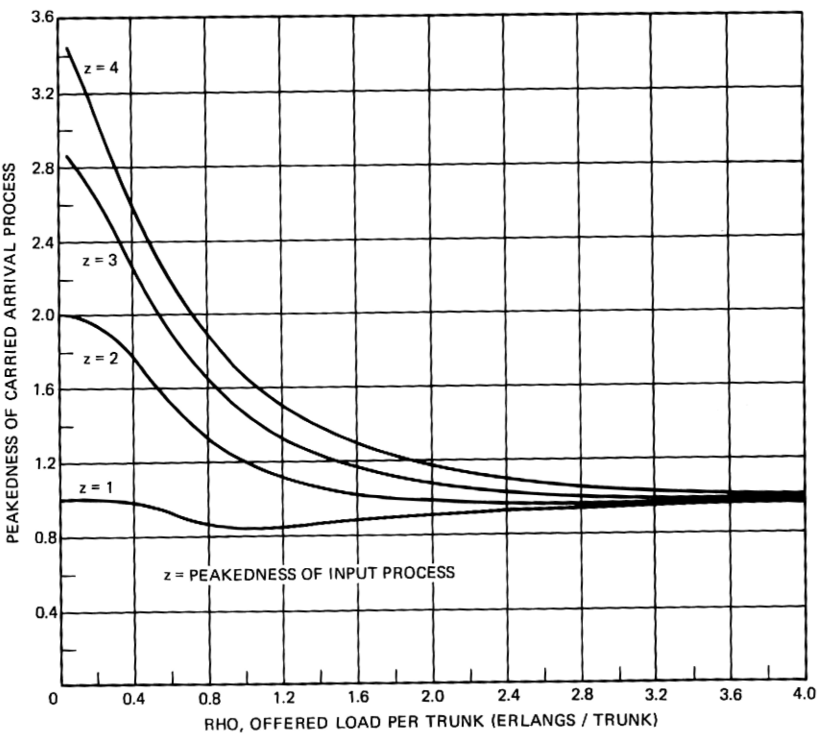

Fig. 6.—Peakedness of CAP (10 trunks).

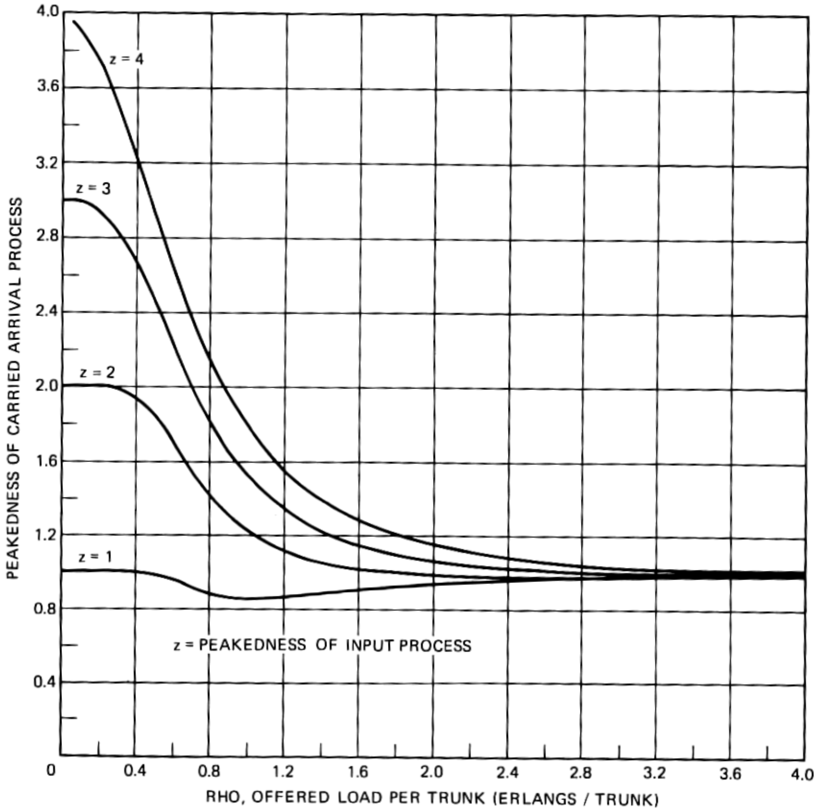

Fig. 7—Peakedness of CAP (20 trunks).

From (18) we have

$$\bar{\phi}_c(s) = [1 - \phi(s)] \left[\frac{\phi(s)}{1 - \phi(s)} - \frac{\phi(s+N)}{1 - \phi(s+N)} \right], \tag{21}$$

where we have used

$$\int_0^\infty e^{-st} dm(t) = \frac{\phi(s)}{1 - \phi(s)}. \tag{22}$$

Combining (17) and (21) gives the transform of the intercarry time distribution:

$$\phi_c(s) = \phi(s) - \frac{[1 - \phi(s)]\phi(s+N)}{1 - \phi(s+N)} \left[\frac{P_{N-1}}{1 - P_N} \right]. \tag{23}$$

We are now in position to determine M^+, defined by (10), and subsequently the peakedness of the CAP.

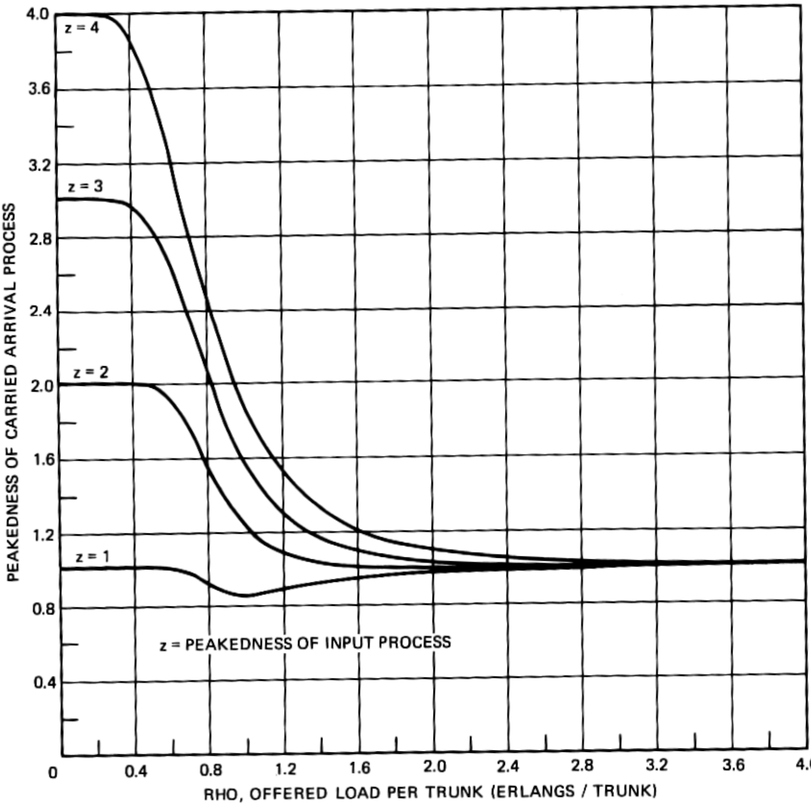

Fig. 8—Peakedness of CAP (50 trunks).

VI. DETERMINATION OF M^+

In order to determine the peakedness of the CAP [eq. (11)], we need to evaluate M^+ defined in (10). This will be done by characterizing the conditional mean of trunks up on the infinite trunk group, given j calls up on the finite trunk group.

Recall that we are considering an arrival at τ_n that finds a free circuit on the finite trunk group (i.e., $j(\tau_n^-) < N$). The state distribution on the finite trunk group at τ_n^+ is thus given by

$$P_j^+ = \Pr\{j(\tau_n^+) = j\} = \frac{P_{j-1}}{1 - P_N}, \qquad 1 \leqq j \leqq N, \tag{24}$$
$$P_0^+ = 0,$$

where the P_j's are the call congestion probabilities defined in (16).

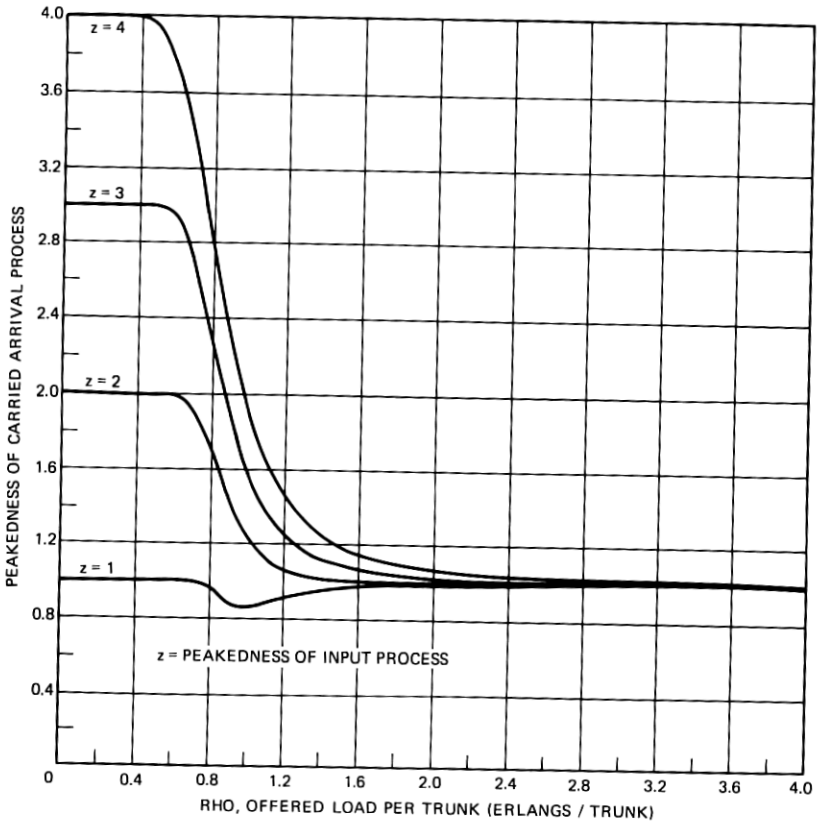

Fig. 9—Peakedness of CAP (100 trunks).

We define

$$P^+_{<N} = \Pr\{j(\tau_n^+) < N\}, \tag{25}$$

$$M^- = E\{k(\tau_n^-)\} = E\{k(\tau_n^+)\} - 1 = M^+ - 1, \tag{26}$$

$$M_l^+ = E\{k(\tau_n^+) \mid j(\tau_n^+) = l\}, \tag{27}$$

and

$$M^+_{<N} = E\{k(\tau_n^+) \mid j(\tau_n^+) < N\}, \tag{28}$$

where k corresponds to the infinite trunk group and j corresponds to the finite trunk group. In terms of these quantities, we have

$$M^+ = M^+_{<N}P^+_{<N} + M^+_N P^+_N. \tag{29}$$

Recall that, if $j(\tau_n^+) < N$, the distribution of time until the next

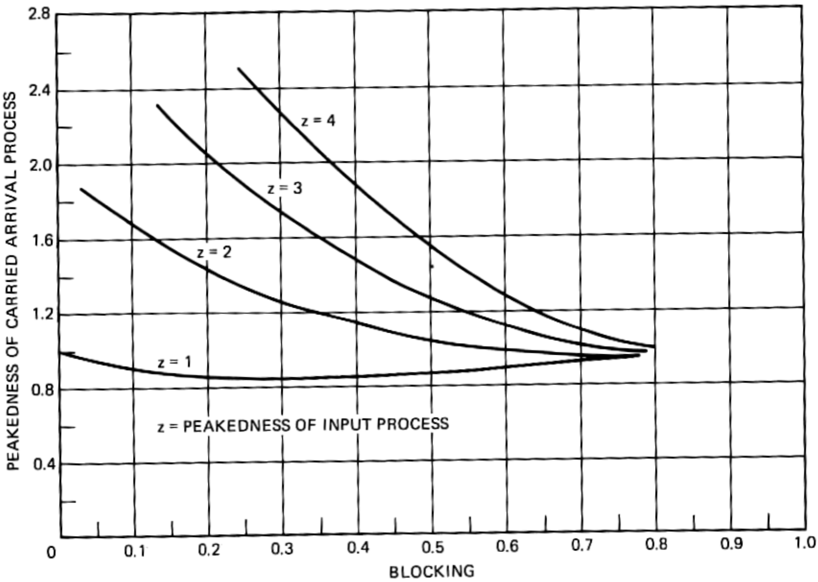

Fig. 10—Peakedness of CAP versus blocking (5 trunks).

carry is $F(t)$ and if $j(\tau_n^+) = N$ the distribution of time until the next carry is $\bar{F}_c(t)$. Using this together with some conditioning arguments, the following relationship is obtained (see Appendix A):

$$M^- = M^+ - 1 = P^+_{<N} M^+_{<N} \phi(1) + P^+_N M^+_N \bar{\phi}_c(1). \tag{30}$$

From (29) and (30) we obtain

$$M^+ = \frac{1}{1 - \phi(1)} + M^+_N P^+_N \left[\frac{\bar{\phi}_c(1) - \phi(1)}{1 - \phi(1)} \right]. \tag{31}$$

Use of (21) simplifies (31) to

$$M^+ = \frac{1}{1 - \phi(1)} - \frac{M^+_N P^+_N \phi(N+1)}{1 - \phi(N+1)}. \tag{32}$$

It should be noted that the first term in (32) corresponds to the value M^+ would assume if the renewal input process was offered directly to the infinite trunk group. The second term corresponds to the reduction in M^+ as a result of blocking on the finite trunk group. We are now left with the problem of determining M^+_N.

It is shown in Appendix B that M^+_m for $m = 1, 2, \cdots, N$ satisfies

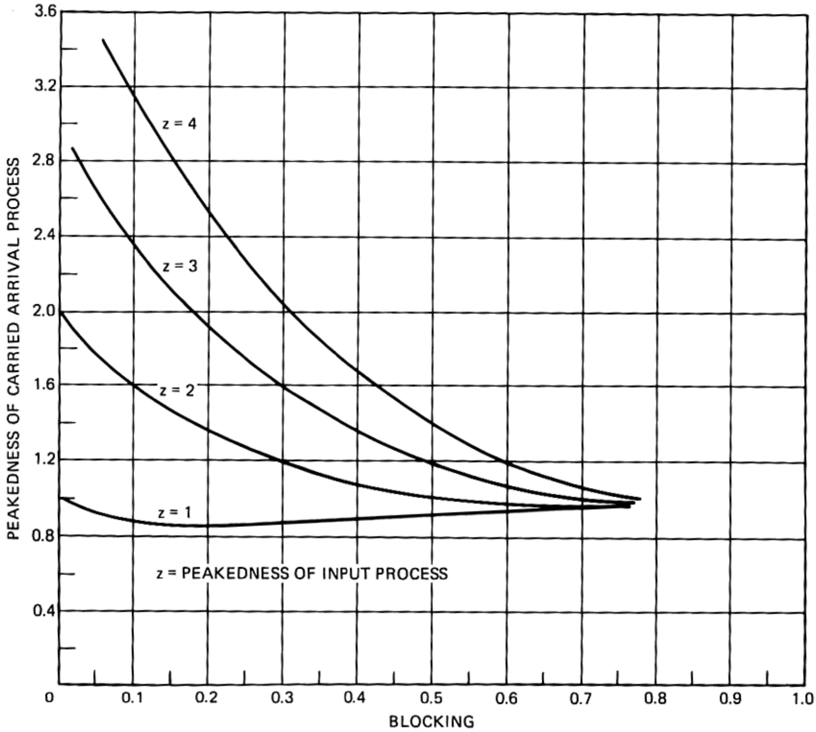

Fig. 11—Peakedness of CAP versus blocking (10 trunks).

the following set of equations:

$$[M_m^+ - 1]P_m^+ = \sum_{l=\max(1,m-1)}^{N} C_{lm}M_l^+, \qquad m = 1, 2, \cdots, N, \quad (33)$$

where for $m - 1 \leqq l \leqq N - 1$, $1 \leqq m \leqq N$,

$$C_{lm} = P_l^+ \binom{l}{m-1} \sum_{\eta=0}^{l-m+1} \binom{l-m+1}{\eta} (-1)^\eta \phi(\eta + m) \cdot \quad (34)$$

Further, for $l = N$ we have

$$C_{Nm} = \frac{P_N^+ \binom{N}{m-1} \sum_{\eta=0}^{N-m+1} \binom{N-m+1}{\eta} (-1)^\eta \phi(\eta + m)}{1 - \phi(N+1)} \cdot \quad (35)$$

It should be noted that the above set of equations is in a form which is amenable to solution for the desired quantity M_N^+. Written in matrix-vector form, the matrix in question is triangular with additional entries below the diagonal. Transforming the matrix associated with

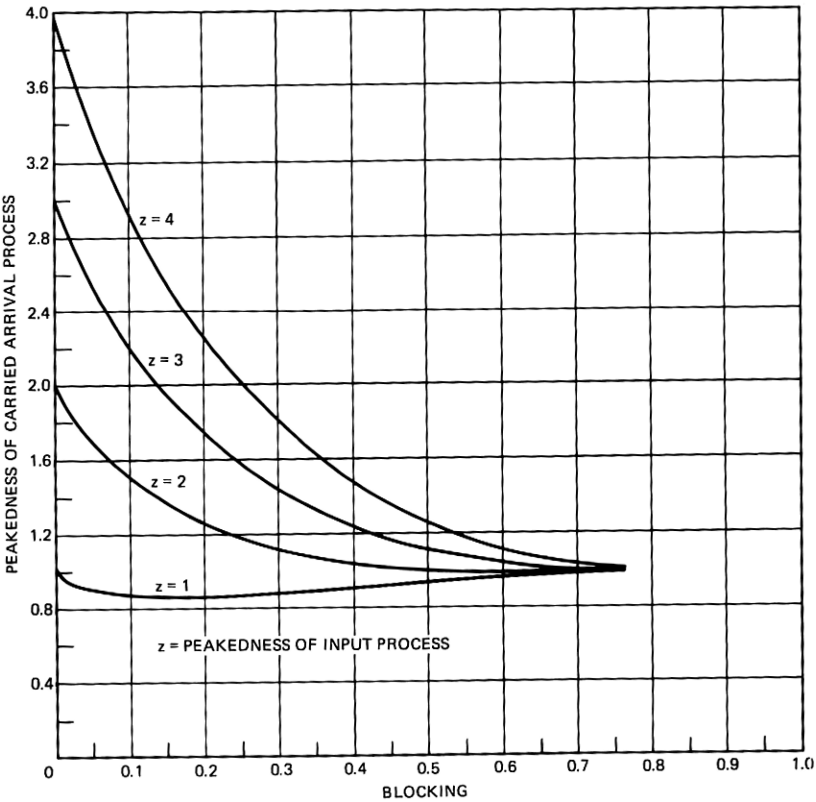

Fig. 12—Peakedness of CAP versus blocking (20 trunks).

(33) into triangular form* leads directly to the quantity of interest M_N^+ for use in (32), which is subsequently used to determine the peakedness of the CAP [eq. (11)].

VII. EXAMPLES

We ran some examples with a 2-moment match† interrupted Poisson process (Reference 7) as the renewal input to the finite trunk group (the computational aspects are discussed in Appendix C). Figures 5 to 9 show z_c, the peakedness of the CAP, as a function of ρ, the offered load per trunk for $N = 5$, 10, 20, 50, and 100 trunks, respectively.

* Details are in Reference 9.

† The blocking experienced by an overflow process is less than the blocking seen by a 2-moment match interrupted Poisson process and more than that seen by the 3-moment match process (all with the same mean and peakedness).

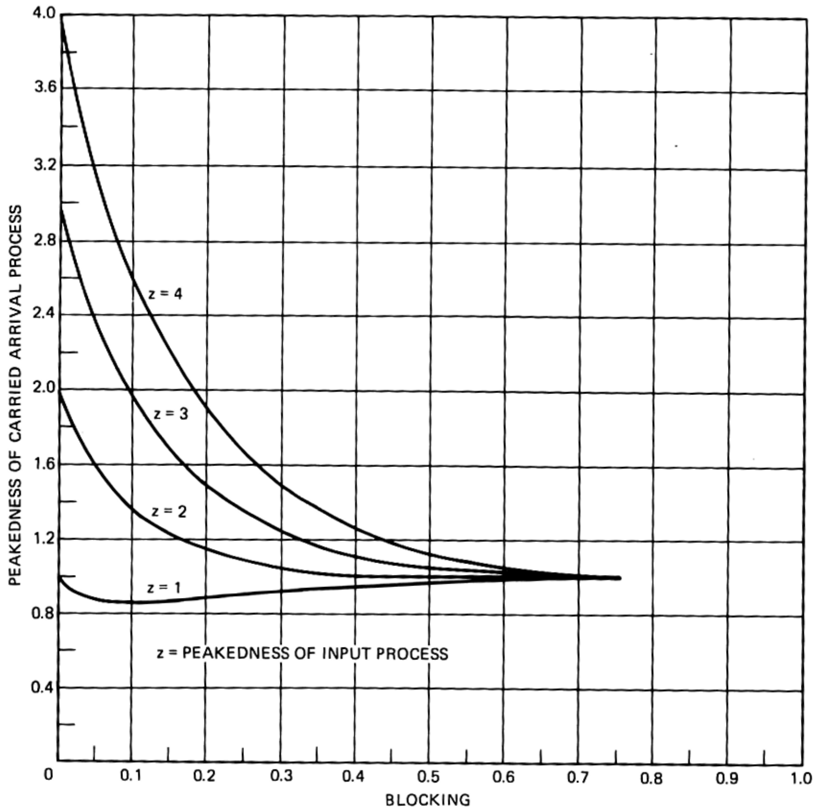

Fig. 13—Peakedness of CAP versus blocking (50 trunks).

On each figure are plots for offered z's of 1, 2, 3, and 4. It is seen that, for fixed N, as z increases, the smoothing effect (reduction of peakedness) becomes appreciable at lower ρ's. This is because of blocking remaining negligible for larger ρ's as z is decreased. If we fix z, we see that the smoothing effect becomes appreciable at lower ρ's as N is decreased. This is again explainable from the point of view of blocking, i.e., blocking is larger on the less efficient small trunk groups.

Since blocking is an important parameter, Figs. 10 to 14 show the peakedness of the CAP versus the blocking for the same cases as shown in Figs. 5 to 9. Note that, for final trunk groups which are normally operated with blockings of 0.01, the smoothing effect is very small, while for high-usage trunk groups which may reach blockings of a few tenths, the smoothing is substantial. Also, note that z_c, in all the cases, approaches unity as the load (and blocking) increases which

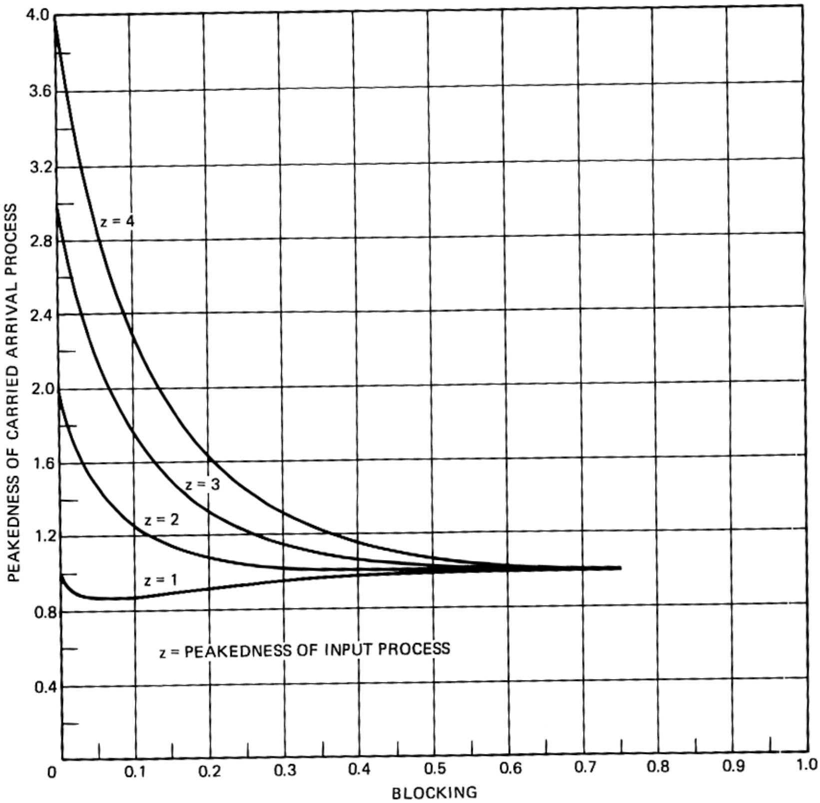

Fig. 14—Peakedness of CAP versus blocking (100 trunks).

is consistent with the explanation in Section I as to how peakedness differs from the variance-to-mean ratio of busy servers (which approaches zero).

It was observed in Reference 10 that, when $z > 1$, the blocking probability is bounded away from zero no matter how small the input mean is. This is evident in Fig. 10.

VIII. CONCLUSION

We have shown how to determine the peakedness of a CAP. The use of mean and peakedness to characterize a CAP is attractive from the point of view of simplicity and is consistent with the use of the equivalent random method (Reference 2) in trunking analyses. To approximately calculate the delays at a switching machine, we could replace the CAP (or, more usually, a superposition of CAP's) with an interrupted Poisson process with the same mean and variance and proceed as in Reference 1. This is being investigated.

We mention, in passing, that we also tried out a renewal approximation for z_c. That is, although the CAP is a semi-Markov process and not generally a renewal process, we tried to approximate z_c with

$$\frac{1}{1-\phi_c(1)} - \frac{1}{\beta},$$

which is the peakedness that the CAP would have if it were a renewal process. This approximation did not compare well enough with the true z_c to recommend its use. That is, although one might use a renewal approximation to a superposition of CAP's *after* the peakedness is determined (see the last paragraph), we do not recommend using a renewal assumption to determine the peakedness.

In the course of determining the CAP peakedness, we have more fully characterized the CAP as a semi-Markov process. Any queuing results available for semi-Markov inputs could be used with the CAP semi-Markov characterization given in Section III.

IX. ACKNOWLEDGMENT

The excellent programming of Mary Zeitler is gratefully acknowledged.

APPENDIX A

Derivation of Equation (30)

Consider the system in equilibrium with times of carried calls $\{\tau_n\}$. From (26) we have

$$M^- = E\{k(\tau_n^-)\}, \tag{36}$$

which can be expanded as

$$M^- = E\{k(\tau_n^-)\,|\,j(\tau_{n-1}^+) = N\}P\{j(\tau_{n-1}^+) = N\} + E\{k(\tau_n^-)\,|\,j(\tau_{n-1}^+) < N\}P\{j(\tau_{n-1}^+) < N\}. \tag{37}$$

This can be written as

$$\begin{aligned} M^- = P_N^+ \sum_{n=0}^{\infty} \sum_{i=1}^{n} iP\{k(\tau_n^-) = i\,|\,k(\tau_{n-1}^+) = n,\ j(\tau_{n-1}^+) = N\} \\ \times P\{k(\tau_{n-1}^+) = n\,|\,j(\tau_{n-1}^+) = N\} \\ + P_{<N}^+ \sum_{n=0}^{\infty} \sum_{i=1}^{n} iP\{k(\tau_n^-) = i\,|\,k(\tau_{n-1}^+) = n,\ j(\tau_{n-1}^+) < N\} \\ \times P\{k(\tau_{n-1}^+) = n\,|\,j(\tau_{n-1}^+) < N\}. \end{aligned} \tag{38}$$

Writing

$$P\{k(\tau_n^-) = i \mid k(\tau_{n-1}^+) = n,\ j(\tau_{n-1}^+) = N\}$$
$$= \int_0^\infty P\{k(\tau_n^-) = i \mid k(\tau_{n-1}^+) = n,\ j(\tau_{n-1}^+) = N,\ \tau_n - \tau_{n-1} = t\} d\bar{F}_c(t)$$
$$= \int_0^\infty \binom{n}{i} e^{-it}(1 - e^{-t})^{n-i} d\bar{F}_c(t) \tag{39}$$

and

$$P\{k(\tau_n^-) = i \mid k(\tau_{n-1}^+) = n,\ j(\tau_{n-1}^+) < N\}$$
$$= \int_0^\infty \binom{n}{i} e^{-it}(1 - e^{-t})^{n-i} dF(t) \tag{40}$$

and observing that

$$\sum_{i=1}^{n} i \binom{n}{i} e^{-it}(1 - e^{-t})^{n-i} = ne^{-t} \tag{41}$$

leads to the desired result

$$M^- = P^+_{<N} M^+_{<N} \phi(1) + P^+_N M^+_N \bar{\phi}_c(1). \tag{42}$$

APPENDIX B

Derivation of Equations (33) through (35) for M_m^+

Consider the events

$$A_{lm} = \{j(\tau_{n-1}^+) = l,\ (l - m + 1) \text{ s.c.'s before next arrival after } \tau_{n-1}\} \tag{43}$$

$$B_{km} = \{j(\tau_{n-1}^+) = N, \text{ no s.c. before } (k - 1)\text{st arrival after } \tau_{n-1},\ (N - m + 1) \text{ s.c.'s before } k\text{th arrival after } \tau_{n-1}\}, \tag{44}$$

where s.c. denotes service completion on the finite trunk group. From these definitions we obtain

$$P\{j(\tau_n^+) = m\} = \sum_{l=m-1}^{N-1} P\{A_{lm}\} + \sum_{k=1}^{\infty} P\{B_{km}\}. \tag{45}$$

The conditional mean of interest is thus given by

$$M_m^- = E\{k(\tau_n^-) \mid j(\tau_n^+) = m\}$$
$$= \frac{\sum_{l=m-1}^{N-1} E\{k(\tau_n^-) \mid A_{lm}\} P\{A_{lm}\} + \sum_{k=1}^{\infty} E\{k(\tau_n^-) \mid B_{km}\} P\{B_{km}\}}{P\{j(\tau_n^+) = m\}} \tag{46}^*$$

Defining the events

$$A_l = \{j(\tau_{n-1}^+) = l\} \tag{47}$$

* When $m = 1$, $P\{A_{0m}\} = 0$, because $P\{j(\tau_{n-1}^+) = 0\} = 0$. In that case, sum from $l = 1$ to $l = N - 1$.

and

$$A_{2lm} = \{(l - m + 1) \text{ s.c.,'s before next arrival after } \tau_{n-1}\} \tag{48}$$

gives

$$E\{k(\tau_n^-)|A_{lm}\}P\{A_{lm}\}$$
$$= \sum_{i=1}^{\infty} iP\{k(\tau_n^-) = i, A_{2lm}|A_l\}P_l^+ \tag{49}$$
$$= P_l^+ \sum_{r=0}^{\infty} P\{k(\tau_{n-1}^+) = r|A_l\} \sum_{i=1}^{r} iP\{k(\tau_n^-) = i, A_{2lm}|A_l, k(\tau_{n-1}^+) = r\}. \tag{50}$$

Letting $l < N$ and using

$$P\{k(\tau_n^-) = i, A_{2lm}|A_l, k(\tau_{n-1}^+) = r\} = \int_0^\infty \binom{r}{i} e^{-it}(1 - e^{-t})^{r-i} \times \binom{l}{m-1} e^{-(m-1)t}(1 - e^{-t})^{l-m+1}dF(t) \tag{51}$$

and

$$\sum_{i=1}^{r} i\binom{r}{i} e^{-it}(1 - e^{-t})^{r-i} = re^{-t} \tag{52}$$

yields

$$E\{k(\tau_n^-)|A_{lm}\}P\{A_{lm}\} = P_l^+M_l^+ \int_0^\infty \binom{l}{m-1} \times e^{-mt}[1 - e^{-t}]^{l-m+1}dF(t). \tag{53}$$

From (1) we have

$$E\{k(\tau_n^-)|A_{lm}\}P\{A_{lm}\} = P_l^+M_l^+ \binom{l}{m-1} \times \sum_{\eta=0}^{l-m+1} \binom{l-m+1}{\eta} (-1)^\eta \phi(\eta + m) \tag{54}$$
$$= C_{lm}M_l^+, \tag{55}$$

where we are using (54) to define C_{lm} in (55). This yields (34).

It now remains to show the desired relationship for C_{Nm}. We consider the second summation of (46)

$$E\{k(\tau_n^-)|B_{km}\}P\{B_{km}\} = \sum_{i=1}^{\infty} iP\{k(\tau_n^-) = i, B_1, B_{2km}\}, \tag{56}$$

where

$$B_1 = \{j(\tau_{n-1}^+) = N\}$$
$$B_{2km} = \{\text{no s.c. before } (k-1)\text{st arrival after } \tau_{n-1}, (N - m + 1) \text{ s.c.'s before } k\text{th arrival after } \tau_{n-1}\}. \tag{57}$$

Now

$$E\{k(\tau_n^-)|B_{km}\}P\{B_{km}\}$$

$$= \sum_{n=0}^{\infty}\sum_{i=1}^{n} iP\{k(\tau_n^-) = i,\, B_{2km}|B_1,\, k(\tau_{n-1}^+) = n\} \times P\{k(\tau_{n-1}^+) = n|B_1\}P_N^+ \quad (58)$$

$$= P_N^+ \sum_{n=0}^{\infty} nP\{k(\tau_{n-1}^+) = n|B_1\} \int_0^{\infty}\int_0^t \sum_{i=1}^{n} \binom{n-1}{i-1} e^{-it} \times (1 - e^{-t})^{(n-i)} e^{-Ns} \binom{N}{m-1} e^{-(m-1)(t-s)} \times (1 - e^{-(t-s)})^{N-m+1} dF_{k-1}(s)dF(t-s). \quad (59)$$

Upon performing the summation over i, (59) simplifies to

$$E\{k(\tau_n^-)|B_{km}\}P\{B_{km}\}$$

$$= P_N^+ M_N^+ \binom{N}{m-1} \int_0^{\infty}\int_0^t e^{-(N+1)s} e^{-m(t-s)} \times \sum_{\eta=0}^{N-m+1} \binom{N-m+1}{\eta} (-1)^{\eta} e^{-\eta(t-s)} dF_{k-1}(s)dF(t-s)$$

$$= P_N^+ M_N^+ \binom{N}{m-1} \sum_{\eta=0}^{N-m+1} (-1)^{\eta} \binom{N-m+1}{\eta} \times \phi^{k-1}(N+1)\phi(\eta+m). \quad (60)$$

Summing over k gives

$$\sum_{k=1}^{\infty} E\{k(\tau_n^-)|B_{km}\}P\{B_{km}\} = M_N^+ \frac{P_N^+ \binom{N}{m-1} \sum_{\eta=0}^{N-m+1} (-1)^{\eta} \binom{N-m+1}{\eta} \phi(\eta+m)}{1 - \phi(N+1)}, \quad (61)$$

which is the desired result for (35).

APPENDIX C

Computational Considerations

In this appendix we briefly* discuss some computational problems experienced in the numerical solution of the carried process problem and point out some possible approaches to circumvent them.† In

* A more detailed description of our computational experience is in Reference 9.
† We first discuss the approaches and then the numerical experience we have had.

particular, we are concerned with the computation of the state probabilities, P_j, $j = 0, \cdots, N$, defined in (16) and the computation of the quantities C_{lm} defined in (34) and (35).

We also specialize and subsequently simplify the above results for the case where the interarrival time distribution is a sum of exponentials. In particular, we consider the interrupted Poisson process[7] as the renewal input. This specialization appears to have eliminated the numerical problems associated with the required calculations for large trunk groups.

The problem of determining the call congestion state probabilities for a renewal input to a BCC system with N independent exponential servers is considered on p. 179 of Reference 3. The results are as follows: Let

$$C_{j+1} = \left(\frac{\phi(j+1)}{1 - \phi(j+1)} \right) C_j , \qquad j = 0, \cdots, N-1 , \tag{62}$$

with $C_0 = 1$. Then B_r, the rth binomial moment of the P_j's, is given by

$$B_r = C_r \frac{\sum_{j=r}^{N} \binom{N}{j} \frac{1}{C_j}}{\sum_{j=0}^{N} \binom{N}{j} \frac{1}{C_j}} . \tag{63}$$

The B_r satisfies the backward recursion

$$B_r = \left(\frac{1 - \phi(r+1)}{\phi(r+1)} \right) B_{r+1} + \binom{N}{r} B_N , \tag{64}$$

with

$$\frac{1}{B_N} = \frac{1}{P_N} = \sum_{j=0}^{N} \binom{N}{j} \frac{1}{C_j} \tag{65}$$

(Reference 11, p. 93). The P_j's are given by

$$P_j = \sum_{r=j}^{N} (-1)^{r-j} \binom{r}{j} B_r . \tag{66}$$

The computation of the C_j coefficients and the binomial moments B_r is fairly straightforward and does not pose much of a numerical problem. It is the computation of the P_j's, using (66), that is sensitive to numerical errors. The alternating sign in (66), together with the facts that $\binom{r}{j} B_r$ can be quite large and the summation in (66) is between zero and unity, lead to a numerical problem.*

* Actually, for an $N = 10$ case seven significant decimal digits were lost in one subtraction and the resultant probability was computed to be zero. This plays havoc with the solution to (33). Details are in Reference 9.

An alternate approach* to the computation of the state probabilities is by way of the equations

$$P_k = \sum_{j=k-1}^{N} p_{jk} P_j, \tag{67}$$

where the transition probabilities p_{jk} represents the probability of going from state j prior to one arrival to state k prior to the next arrival. The p_{jk}'s are given by

$$p_{jk} = \binom{j+1}{k} \int_0^\infty e^{-kt}(1-e^{-t})^{j+1-k} dF(t), \qquad j < N, \tag{68}$$

and

$$p_{N,k} = p_{N-1,k}. \tag{69}$$

From (67) and (68), we obtain the backward relation

$$P_{k-1} = \frac{P_k}{\phi(k)} - \frac{1}{\phi(k)} \sum_{j=k}^{N} p_{jk} P_j. \tag{70}$$

Expanding (68) gives

$$p_{jk} = \binom{j+1}{k} \sum_{\eta=0}^{j+1-k} \binom{j+1-k}{\eta} (-1)^\eta \phi(\eta + k). \tag{71}$$

The computational procedure is outlined as follows: Compute P_N from (65)† and use (70) to compute P_j for $j < N$. It should be noted that, although the terms in (71) alternate in sign, ϕ never exceeds unity and is monotonically decreasing. Also note the relation between p_{jk} given by (71) and C_{lm} given by (34) and (35). At this point, the accuracy in computing the P_j's should be comparable to the accuracy in computing the C_{lm}'s.

For the case where $F(t)$ is the sum of exponentials (e.g., interrupted Poisson process) we can further simplify (and more accurately compute) the P_j's and C_{lm}. We go to the integrals from which the sums (with alternating signs) appearing in (34), (35), and (71) were derived. Note that we have

$$\binom{l}{m-1} \sum_{\eta=0}^{l-m+1} \binom{l-m+1}{\eta} (-1)^\eta \phi(\eta + m) = \int_0^\infty \binom{l}{m-1} e^{-mt}(1-e^{-t})^{l-m+1} dF(t) \tag{72}$$

---

* Motivation for investigating this approach stems from remarks made by P. J. Burke. (In recent unpublished work, Burke showed a more accurate approach for the case where the renewal interarrival time distribution is a sum of exponentials.)

† Note that each term in the sum of (65) is positive.

[see (53) and (54)]. Let

$$F(t) = \sum_{i=1}^{S} k_i(1 - e^{-r_i t}); \tag{73}$$

then the integral in (72) can be identified as a beta function.* Repeated integration by parts in (72) gives

$$\int_0^\infty \binom{l}{m-1} e^{-mt}(1 - e^{-t})^{l-m+1} dF(t) = \sum_{i=1}^{S} f_{lm}(k_i, r_i), \tag{74}$$

where

$$f_{lm}(k_i, r_i) = \left(\frac{k_i r_i}{l + r_i + 1}\right)\left(\frac{l}{l + r_i}\right)\left(\frac{l-1}{l + r_i - 1}\right) \cdots \left(\frac{l - (l - m)}{l + r_i - (l - m)}\right) \quad \text{for} \quad l > m - 1. \tag{75}$$

For $l = m - 1$, we obtain from (72)

$$f_{m-1,m}(k_i, r_i) = \frac{k_i r_i}{m + r_i}. \tag{76}$$

Note that f_{lm} can be computed recursively from

$$f_{l+1,m}(k_i, r_i) = \left(\frac{l+1}{l + r_i + 2}\right) f_{l,m}(k_i, r_i) \tag{77}$$

with initialization from (76).

The direct calculation of the integral has thus led to a computationally tractable method of computing the C_{lm}'s and the P_j's. C_{lm} is calculated from

$$C_{lm} = P_l^+ \left(\sum_{i=1}^{S} f_{lm}(k_i, r_i)\right), \qquad m - 1 \leqq l \leqq N - 1, \tag{78}^\dagger$$

[see (53)] and

$$C_{Nm} = \frac{P_N^+}{1 - \phi(N + 1)} \left(\sum_{i=1}^{S} f_{N,m}(k_i, r_i)\right) \tag{79}^\dagger$$

[see (35) and (72)].

The P_j's are calculated from (70) where (68) is simplified as above (using integration by parts) and then used to compute the transition probabilities. Note that for an interrupted Poisson process,[7] $S = 2$, and k_1, k_2, r_1, and r_2 are given in Reference 7 in terms of the switch

* Identification made by D. L. Jagerman.

† Note that this procedure does not involve the calculation of either binomial moments or binomial coefficients.

parameters. An alternate method of computing the P_j's for the interrupted Poisson process is via the use of birth and death equations and conditioning the results on the switch being closed. A computer program for doing this was available (Reference 8).

At this point, it is of interest to discuss our computational experience using some of the aforementioned procedures. The first method considered was to calculate the P_j's by first obtaining the binomial moments [eqs. (62) to (66)] and to compute the C_{lm}'s from (34) and (35). Using single precision arithmetic the procedure worked for $N = 2$, but failed for $N = 10$. The problem was traced to inaccurate calculation of the probabilities from binomial moments. Double precision arithmetic extended the range of N (worked for $N = 10$). The method failed at $N = 20$. The failure was traced to the same cause as above. At this point we used the birth and death equation approach to calculate the P_j's,[8] which assumes an interrupted Poisson input. This extended the range to $N = 20$. For $N = 30$ we ran into problems computing the C_{lm}'s from (34) and (35).* Modification of the C_{lm} computation using (78) and (79) significantly extended the useful range on N. The results presented in Section VII were computed using this method of calculation.

REFERENCES

1. Heffes, H., "Analysis of First-Come First-Served Queuing Systems With Peaked Inputs," B.S.T.J., *52*, No. 7 (September 1973), pp. 1215–1228.
2. Wilkinson, R. I., "Theories for Toll Traffic Engineering in the U.S.A.," B.S.T.J., *35*, No. 2 (March 1956), pp. 421–514.
3. Takács, L., *Introduction to the Theory of Queues*, New York: Oxford University Press, 1962.
4. Ross, S. M., *Applied Probability Models with Optimization Applications*, San Francisco: Holden-Day, 1970.
5. Franken, P., "Erlang Formulas for Semi-Markovian Input," Elektron Informationsverarbeit Kybernetik, *4* (1968), pp. 197–204.
6. Descloux, A., "On Markovian Servers with Recurrent Input," Proc. Sixth International Teletraffic Congress, 1970, pp. 331/1–331/6.
7. Kuczura, A., "The Interrupted Poisson Process as an Overflow Process," B.S.T.J., *52*, No. 3 (March 1973), pp. 437–448.
8. Marzec, R. P., unpublished work.
9. Zeitler, M. J., unpublished work.
10. Holtzman, J. M., "The Accuracy of the Equivalent Random Method With Renewal Inputs," B.S.T.J., this issue, pp. 1673–1679.
11. Riordan, J., *Stochastic Service Systems*, New York: Wiley, 1962.

* This implicitly indicates the range of accuracy for the computation of the P_j's from (70) and (71).