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A Geometric Theory of Intersymbol Interference

Part II: Performance of the Maximum Likelihood Detector

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In a companion paper,[1] a geometric approach to the study of intersymbol interference was introduced. In the present paper this approach is applied to the performance analysis of the Viterbi algorithm maximum likelihood detector (MLD) of Forney.[2–4] It is shown that a canonical relationship exists between the minimum distance, which Forney has shown determines the performance of the MLD, and the performance and tap-gains of the decision-feedback equalizer (DFE). Upper and lower bounds on the minimum distance are derived, as is an iterative technique for computing it exactly.

The performances of the MLD, DFE, and zero-forcing equalizer (ZFE) are compared on the $\sqrt{f}$ channel representative of coaxial cables and some wire pairs. One important conclusion is that, previous statements notwithstanding,[2,4] even the MLD experiences a substantial penalty in S/N ratio relative to the isolated pulse bound on this channel of practical interest.

I. INTRODUCTION

Forney[2,3] has detailed the Viterbi algorithm version of the maximum likelihood detector (MLD) of digital sequences in the presence of intersymbol interference. He asserts that the probability of bit error of the MLD in additive white Gaussian noise can be bounded at high S/N ratios in the form

$$K_L Q\left(\frac{d_{\min}}{2\sigma}\right) \leqq P_e \leqq K_u Q\left(\frac{d_{\min}}{2\sigma}\right), \tag{1}$$

where K_L and K_u are constants, Q is the Gaussian distribution function,

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}\, dy\,, \tag{2}$$

$d_{\min}$ is the minimum distance between any two transmitted signals (it will be defined more fully in Section 2.2), and σ^2 is the noise variance. For comparison purposes, the probability of error for a matched filter receiver in the absence of intersymbol interference is

$$P_e = Q\left(\frac{\sqrt{R_0}}{2\sigma}\right), \tag{3}$$

where R_0 is the energy of an isolated pulse [(1) reduces to (3) in this case].

Forney also asserts that the lower bound of (1) is also a lower bound on the error probability of any receiver.[4] Thus, the MLD achieves, within the multiplicative constant K_u/K_L, the minimum probability of error attainable by any receiver at high S/N ratios, and, in a very fundamental sense, the quantity

$$d_{\min}^2/R_0$$

is a measure of the effective decrease in the S/N ratio (relative to the detection of an isolated pulse) resulting from intersymbol interference.

The determination of the quantity $d_{\min}^2$ (known as the "minimum distance problem") is therefore a very important one for, even if the implementation of the MLD is not contemplated on a particular channel, $d_{\min}^2$ is a measure of the potential performance which can be obtained using receivers of arbitrary complexity. Unfortunately, on channels with severe intersymbol interference, the exact analytical determination of $d_{\min}^2$ does not appear feasible because of the nonlinear nature of the problem.

The minimum distance can be determined numerically by the "brute force" technique of calculating a sequence of converging upper bounds. A shortcoming of this method is that it gives no assurance as to when convergence to the desired accuracy has occurred. In addition, it gives no insight into the nature of $d_{\min}^2$ and its relationship to the intersymbol interference or to the performances of other receivers.

In this paper, we attack the minimum distance problem using a geometric theory of intersymbol interference developed in companion papers.[1,5] A canonical relationship will be shown between $d_{\min}^2$ and the decision-feedback equalizer (DFE). This relationship will be exploited to derive simple lower and upper bounds on $d_{\min}^2$ in terms of the tap-gains of the DFE transversal filter and the S/N ratio performance of the DFE. In addition, an iterative procedure will be derived for the calculation of $d_{\min}^2$ to any desired accuracy using a sequence of converging upper *and lower* bounds on $d_{\min}^2$. The lower bounds give us a measure of the degree of convergence and enable us to terminate the calculation when the desired accuracy is assured.

After consideration of the minimum distance problem in Section II, the performance of the zero-forcing equalizer (ZFE), DFE, and MLD is compared on a channel of practical interest in Section III.

II. PERFORMANCE OF THE MLD

The minimum distance problem will now receive consideration. The first step is to briefly review the notation of a companion paper.[1]

2.1 *Notation*

The reception from a PAM communication channel takes the form

$$r(t) = \sum_k B_k h(t - kT) + n(t), \tag{4}$$

where each B_k assumes one of a finite number of predetermined values (the data being transmitted), $h(t)$ is square-integrable (element of L_2),* and $n(t)$ is white Gaussian noise.

When we denote $h(t - kT)$ as an element of L_2 by h_k, $M(h_k, k \in I)$ is the smallest closed linear subspace of L_2 containing all finite linear combinations of elements of the set $\{h_k, k \in I\}$. The projection of a vector x on $M(h_k, k \in I)$ is denoted by $P[x, M(h_k, k \in I)]$. The forward matched-filter transversal-filter combination of the DFE corresponds to the L_2 inner product of the reception $r(t)$ with the element

$$e_k^+ \triangleq h_k - P[h_k, M(h_m, m > k)] \tag{5}$$

and is orthogonal to the subspace $M(h_m, m > k)$. The quantity

$$\frac{\|e_0^+\|^2}{R_0},$$

where

$$R_k \triangleq \langle h_m, h_{m+k} \rangle \tag{6}$$

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* We denote by L_2 the space of square integrable waveforms with inner product

$$\langle x,y \rangle = \int_{-\infty}^{\infty} x(t) y(t) dt$$

and norm $\|x\|^2 = \langle x,x \rangle$.

is the effective decrease in S/N ratio relative to an isolated pulse for the DFE. Thus, $\|e_0^+\|^2$ plays the same role for the DFE as $d^2_{\min}$ plays for the MLD.

The sequence of vectors $\{w_k \triangleq e_k^+/\|e_k^+\|\}$ is an orthonormal sequence in L_2, and h_n has the orthogonal expansion

$$h_n = \sum_{m=0}^{\infty} C_m w_{m+n}, \tag{7}$$

where the coefficients $\{C_m\}$ can be determined by the method of Ref. 1 for channels with either a rational or nonrational spectrum. In particular, we have

$$C_0 = \|e_0^+\|. \tag{8}$$

Of course, it is apparent that (7) is valid only as long as $\|e_0^+\| > 0$, which is true if and only if a DFE exists.

2.2 *Interpretation of the Minimum Distance*

The MLD described by Forney[2] consists of a combination of a matched filter followed by a causal or anticausal transversal filter, the combination of which he calls a "whitened matched filter," followed by a dynamic programming algorithm known as the Viterbi algorithm.[3] The whitened matched filter forms a sequence of sufficient statistics for the detection of the data digits and has independent noise samples at the output. As pointed out by Price,[6] the anticausal whitened matched filter is identical to the forward linear filter portion of the DFE.

The signal at the output of the whitened matched filter (or DFE forward filter) is[1]

$$r_k = C_0^2 B_k + \sum_{m=1}^{\infty} C_0 C_m B_{k-m} + n_k, \tag{9}$$

where n_k is a noise sample. The DFE forms the quantity

$$r'_k = r_k - \sum_{m=1}^{\infty} C_0 C_m \hat{B}_{k-m} \tag{10}$$

and applies it to a decision threshold to determine the estimated digit $\hat{B}_k$. The MLD detector, on the other hand, assumes that the sum in (9) is truncated to M terms and determines the sequence $\{\hat{B}_k\}$ so as to minimize

$$\sum_{k=1}^{N} \left\{ r_k - \sum_{m=0}^{M} C_0 C_m \hat{B}_{k-m} \right\}^2. \tag{11}$$

Thus, the two receivers perform similar functions on the same sufficient statistics r_n, the major differences being the greater complexity of the MLD and the susceptibility of the DFE to decision errors. We will now demonstrate the less obvious conclusion that the *performance* of the MLD is closely related to the DFE as well.

The minimum distance, $d^2_{\min}$, is defined as[2]

$$d^2_{\min} \triangleq \inf_{\epsilon_0 \neq 0} \left\| \sum_{n=0}^{N} \epsilon_n h_n \right\|^2, \tag{12}$$

where the infimum is over all error sequences $(\epsilon_0, \cdots, \epsilon_N)$ and all N.* Each ϵ_k assumes the value $+1$, -1, or zero (for simplicity, the binary case with $B_k = 1$ or 0 is considered). Thus, $d_{\min}$ is the minimum distance in L_2 between two signals in the signal set. It is apparent that

$$d^2_{\min} \leqq R_0, \tag{13}$$

since R_0 corresponds to $\epsilon_n = 0$, $n > 0$. Thus, $d^2_{\min}/R_0$, which is the S/N ratio penalty, is a number between zero and unity as it should be.

It is apparent in (12) that without loss of generality we can choose $\epsilon_0 = 1$ and write

$$d^2_{\min} = \inf \left\| h_0 + \sum_{n=1}^{N} \epsilon_n h_n \right\|^2. \tag{14}$$

The sum in (14) is an element of $M(h_k, k \geqq 1)$, and the minimization in (14) is an attempt to find the element of $M(h_k, k \geqq 1)$ with manifold coefficients $(+1, -1, 0)$ which is closest (in $\mathfrak{L}_2$ metric) to h_0. We know that the closest element without the restriction in coefficients is the projection of h_0 on $M(h_k, k \geqq 1)$, $P[h_0, M(h_k, k \geqq 1)]$. Thus, intuitively, $d^2_{\min}$ is determined by how closely the projection can be approximated by an element with restricted manifold coefficients. To formalize this intuition, add and subtract the projection from (14) and utilize (5),

$$\begin{aligned} d^2_{\min} &= \inf \left\| e_0^+ + P[h_0, M(h_k, k \geqq 1)] + \sum_{n=1}^{N} \epsilon_n h_n \right\|^2 \\ &= \|e_0^+\|^2 + \inf \left\| P[h_0, M(h_k, k \geqq 1)] + \sum_{n=1}^{N} \epsilon_n h_n \right\|^2, \end{aligned} \tag{15}$$

where the fact that e_0^+ is orthogonal to $M(h_k, k > 0)$ has been used to eliminate the cross-product in (15). The most immediate consequence of (15) is that

$$d^2_{\min} \geqq \|e_0^+\|^2. \tag{16}$$

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* In most cases of interest, the infimum will be achieved for finite N.

We have thus succeeded in proving formally what should be obvious from considerations of the relative complexity of the two receivers: The effective S/N ratio of the MLD always exceeds that of the DFE (and hence ZFE[1]).* The second consequence of (15) is the formalization of our intuition through the assertion that the amount by which the S/N ratio of the MLD exceeds that of the DFE is governed by the coarseness of the best approximation to the projection by the element with restricted coefficients: The poorer the approximation, the better the S/N ratio of the MLD.

Writing the projection in the form

$$P[h_0, M(h_k, k > 0)] = -\sum_{m=1}^{\infty} a_m^+ h_m, \tag{17}$$

we note that the a_m^+ are the tap-gains of the DFE forward transversal filter, and rewrite (15) as†

$$d_{\min}^2 = \|e_0^+\|^2 + \inf \left\| \sum_{n=1}^{\infty} (\epsilon_n - a_n^+) h_n \right\|^2. \tag{18}$$

Equation (18) shows the fundamental relationship between the minimum distance, the effective S/N ratio of the DFE (in the form of $\|e_0^+\|^2$), and the tap-gains of the DFE transversal filter. In particular, we can assert that $d_{\min}^2 = \|e_0^+\|^2$ if and only if the tap-gains are all $+1$, -1, or zero.

2.3 *Bounds on the Minimum Distance*

Equation (18) can be used to derive bounds on $d_{\min}^2$ in terms of the DFE tap-gains. From the identity‡

$$\left\| \sum_{n=1}^{N} (\epsilon_n - a_n^+) h_n \right\|^2 = (\epsilon_k - a_k^+)^2 \left\| h_k + \sum_{\substack{n=1 \\ n \neq k}}^{N} \frac{\epsilon_n - a_n^+}{\epsilon_k - a_k^+} h_n \right\|^2, \tag{19}$$

we immediately get the bounds

$$\left\| \sum_{n=1}^{N} (\epsilon_n - a_n^+) h_n \right\|^2 \geqq \begin{cases} (\epsilon_1 - a_1^+)^2 \|e_0^+\|^2, & k = 1 \\ (\epsilon_k - a_k^+)^2 \|e_0\|^2, & k > 1, \end{cases} \tag{20}$$

* We are tempted to argue that (16) is implied by the assertion in Ref. 2 that the MLD achieves the lowest effective S/N ratio of any receiver. However, that is not the case, because of the effect of decision errors on the DFE. The effective S/N ratio of the DFE could be higher than that of the MLD, and yet the DFE could have at the same time a higher error probability because of error propagation.

† We have taken the liberty of writing a sum over infinite error sequences, where it is understood that the infimum is only over error sequences with a finite number of nonzero terms.

‡ In (19) it is assumed that $(\epsilon_k - a_k^+) \neq 0$. When $\epsilon_k - a_k^+ = 0$, (20) is trivially satisfied.

since

$$\sum_{\substack{n=1 \\ n \neq k}}^{N} \frac{(\epsilon_n - a_n^+)}{(\epsilon_k - a_k^+)} h_n$$

is an element of $M(h_m, m \neq k)$. In (20), e_0 is the ZFE filter defined in Ref. 1,

$$e_0 \triangleq h_0 - P[h_0, M(h_k, k \neq 0)]. \tag{21}$$

In addition, if we define $\lambda_{\min}(N)$ and $\lambda_{\max}(N)$ as the minimum and maximum eigenvalues of the correlation matrix

$$\mathbf{R}_N \triangleq [R_{m-n}] \qquad 1 \leqq m,n \leqq N,$$

then we can assert that

$$\lambda_{\min}(N) \sum_{n=1}^{N} (\epsilon_n - a_n^+)^2 \leqq \left\| \sum_{n=1}^{N} (\epsilon_n - a_n^+) h_n \right\|^2 \leqq \lambda_{\max}(N) \sum_{n=1}^{N} (\epsilon_n - a_n^+)^2. \tag{22}$$

A standard Toeplitz form result[7] asserts that*

$$\lim_{N \to \infty} \lambda_{\min}(N) = \frac{1}{T} \text{ ess inf } R(\omega)$$

$$\lim_{N \to \infty} \lambda_{\max}(N) = \frac{1}{T} \text{ ess sup } R(\omega).$$

Applying (18), (20), and (22), we get three lower and one upper bound on $d_{\min}^2$ in terms of the tap coefficients of the DFE,

$$d_{\min}^2 \geqq \|e_0^+\|^2 + \begin{cases} \|e_0^+\|^2 \min\limits_{\epsilon_1} (\epsilon_1 - a_1^+)^2 \\ \|e_0\|^2 \min\limits_{\epsilon_k} (\epsilon_k - a_k^+)^2, \qquad k > 1 \\ \dfrac{1}{T} \{\text{ess inf } R(\omega)\} \lim\limits_{N \to \infty} \min\limits_{\epsilon_1, \cdots, \epsilon_N} \sum\limits_{n=1}^{N} (\epsilon_n - a_n^+)^2 \end{cases}$$

$$d_{\min}^2 \leqq \|e_0^+\|^2 + \frac{1}{T} \{\text{ess sup } R(\omega)\} \lim_{N \to \infty} \min_{\epsilon_1, \cdots, \epsilon_N} \sum_{n=1}^{N} (\epsilon_n - a_n^+)^2. \tag{23}$$

In addition, an upper bound can be obtained by substituting any error sequence into (18); a reasonable choice is

$$\epsilon_k = \begin{cases} +1, & a_k^+ < -\frac{1}{2} \\ 0, & -\frac{1}{2} < a_k^+ < \frac{1}{2}. \\ -1, & a_k^+ > \frac{1}{2} \end{cases} \tag{24}$$

* For all practical purposes, "ess inf" and "ess sup" can be replaced by "min" and "max," respectively.

These five bounds can be useful in estimating the penalty in S/N ratio for the MLD. They all require the existence of a DFE and require that the projection can be written as the convergent sum of (17).* The second and third bounds of (23) are an improvement on (16) only when the increasingly stringent requirements that a ZFE exist ($\|e_0\| > 0$) and $R(\omega)$ be uniformly bounded away from zero (almost everywhere) are imposed. The requirement of the upper bound of (23) that $R(\omega)$ be uniformly upper bounded (almost everywhere) will generally be satisfied in practice. All the bounds require a pointwise minimization over error sequences, a task much simpler than minimizing (12) directly.

As a simple application of these bounds, consider the exponential autocorrelation

$$R_k = A^{|k|}, \qquad 0 < A < 1. \tag{25}$$

Then we have[1,2]

$$\begin{aligned} d_{\min}^2 &= \begin{cases} 1, & 0 < A \leqq \frac{1}{2} \\ 2(1-A), & \frac{1}{2} < A < 1 \end{cases} \\ \|e_0\|^2 &= (1-A^2)/(1+A^2) \\ \|e_0^+\|^2 &= 1 - A^2 \\ a_1^+ &= -A, \qquad a_k^+ = 0, \qquad k > 1. \end{aligned} \tag{26}$$

The first and third bounds of (23) become

$$d_{\min}^2 \geqq \begin{cases} 1 - A^4, & 0 < A \leqq \frac{1}{2} \\ (1-A^2)(2 + A^2 - 2A), & \frac{1}{2} < A < 1 \end{cases} \tag{27}$$

$$d_{\min}^2 \geqq \begin{cases} 1 - 2A^3/(1+A), & 0 < A \leqq \frac{1}{2} \\ 2(1-A)(1+A^2)/(1+A), & \frac{1}{2} < A < 1 \end{cases} \tag{28}$$

and the upper bound of (23) becomes

$$d_{\min}^2 \leqq \begin{cases} 1 + \dfrac{2A^3}{1-A}, & 0 < A \leqq \frac{1}{2} \\ 2(1-A^2), & \frac{1}{2} < A < 1. \end{cases} \tag{29}$$

These bounds are plotted in Fig. 1. The upper bound of (24) is equal to $d_{\min}^2$ and is not plotted.

* If the projection of h_0 on $P(h_k, k \geqq 1)$ cannot be written in the form of (17), the bounds of (22) to (24) can be fixed up by considering the projection on $P(h_k, 1 \leqq k \leqq N)$ and taking limits as $N \to \infty$. The tap-gains will then be a function of N, and the process will be more difficult.

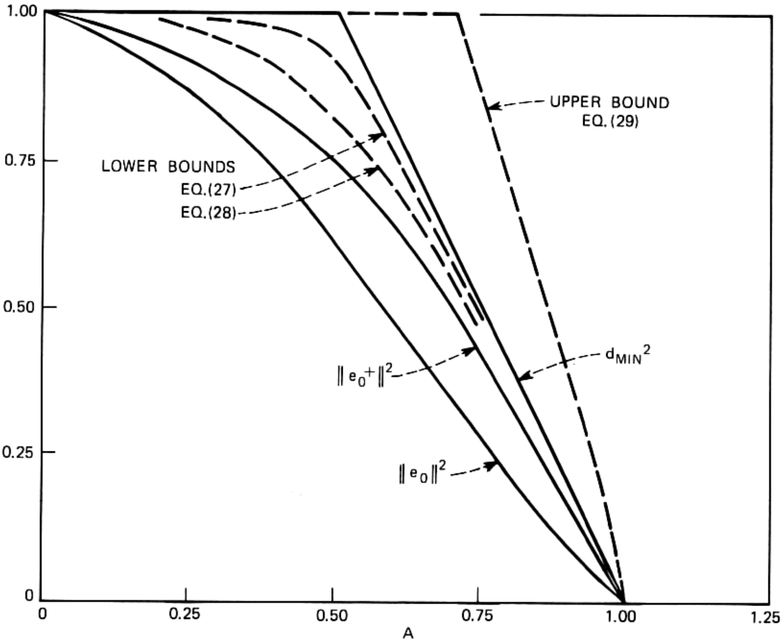

Fig. 1—Bounds on $d^2_{\min}$ for an exponential autocorrelation.

The bounds just determined have the disadvantages that (*i*) they require calculation of the DFE tap coefficients and (*ii*) they do not give precise results on $d^2_{\min}$. The exact value of $d^2_{\min}$ can be determined numerically by the direct minimization of (12); by letting $N \rightarrow \infty$ while exhaustively minimizing over error sequences, we get a sequence of upper bounds on $d^2_{\min}$ which approach $d^2_{\min}$ monotonically. The obvious difficulty with this method is that the number of error sequences which must be checked grows as 3^N, and the computational effort soon becomes unreasonable. What happens in practice is that the true minimum is achieved for a finite (and small) N. However, unless we have some method of determining when the true minimum is reached, there must always remain a degree of uncertainty as to whether the true minimum has been reached.

Our approach to this computational problem will be to derive a sequence of *lower* bounds on $d^2_{\min}$ which also approach $d^2_{\min}$ monotonically. We can then halt the process at a value of N where the upper and lower bounds are close enough to ensure knowledge of $d^2_{\min}$ within the desired accuracy. To this end, we will utilize the orthogonal expansion of (7). Substituting (7) into the sum of (12),

$$\sum_{n=0}^{\infty} \epsilon_n h_n = \sum_{n=0}^{\infty} \epsilon_n \sum_{m=0}^{\infty} C_m w_{n+m}$$

$$= \sum_{m=0}^{\infty} \beta_n w_n, \tag{30}$$

where

$$\beta_m = \sum_{k=0}^{m} \epsilon_k C_{m-k}. \tag{31}$$

Then, because the $\{w_n\}$ are orthonormal,

$$\left\| \sum_{n=0}^{\infty} \epsilon_n h_n \right\|^2 = \sum_{n=0}^{\infty} \beta_n^2. \tag{32}$$

It appears that we may have made life more difficult for ourselves, because even when we substitute a finite sum on the left of (32) we must still evaluate an infinite sum on the right. However, note that since the terms in the sum are positive,

$$\left\| \sum_{n=0}^{\infty} \epsilon_n h_n \right\|^2 \geqq \sum_{n=0}^{N} \beta_n^2, \tag{33}$$

where the sum on the right is always finite and is in terms of a finite length error sequence $(\epsilon_0, \cdots, \epsilon_N)$. Hence,

$$d_{\min}^2 \geqq \min_{\substack{\epsilon_1, \cdots, \epsilon_N \\ \epsilon_0 = 1}} \sum_{n=0}^{N} \beta_n^2 \tag{34}$$

and, furthermore, the right side of (34) approaches the left side monotonically as $N \to \infty$.

The minimization of (34) is no more or less difficult to perform than that of the direct minimization of (12). It does require the existence of a DFE and evaluation of the coefficients $\{C_m\}$. A reasonable procedure is, at each stage of N, to minimize the right side of (34) to obtain a lower bound on $d_{\min}^2$ and substitute the minimizing sequence into (12) to obtain the upper bound* on $d_{\min}^2$. When the lower and upper bounds are sufficiently close, the process can be terminated.

* Note that any sequence substituted into (12) yields an upper bound on $d_{\min}^2$, and the one which minimizes (34) is as good as any. On the other hand, only the sequence which minimizes (34) yields a valid lower bound, so it must be minimized.

The minimization of (34) can be assisted slightly by dynamic programming. Defining

$$f_{N-m}(\epsilon_1, \cdots, \epsilon_{m-1}) = \min_{\epsilon_m, \cdots, \epsilon_N} \sum_{n=m}^{N} \beta_n^2, \tag{35}$$

we note that

$$\min_{\epsilon_1, \cdots, \epsilon_N} \sum_{n=1}^{N} \beta_n^2 = \min_{\epsilon_1} [f_{N-2}(\epsilon_1) + \beta_1^2] \tag{36}$$

with a recursion relation for f_{N-m} $(\epsilon_1, \cdots, \epsilon_{m-1})$,

$$\begin{aligned} f_{N-m+1}(\epsilon_1, \cdots, \epsilon_{m-2}) &= \min_{\epsilon_{m-1}, \cdots, \epsilon_N} \sum_{n=m-1}^{N} \beta_n^2 \\ &= \min_{\epsilon_{m-1}} \left[\min_{\epsilon m \cdots \epsilon N} \sum_{n=m}^{N} \beta_n^2 + \beta_{m-1}^2 \right] \\ &= \min_{\epsilon_{m-1}} [f_{N-m}(\epsilon_1 \cdots \epsilon_{m-1}) + \beta_{m-1}^2]. \end{aligned} \tag{37}$$

Because there is no possibility of using forward dynamic programming in this case, the savings in computation for this method is not too spectacular. Each β_n must still be evaluated for 3^N error sequences; the savings is in eliminating the need for summing β_n^2 for most of the combinations of 3^N error sequences.

We note in passing that using the FFT algorithm to reduce the computational effort in the convolutional sum of (31) is a possibility. However, the 3^N sequences for which it must be evaluated becomes a limiting factor long before the savings of that method becomes substantial.

In the foregoing discussion, the existence of a DFE has been required [that is, $\|e_0^+\| > 0$ or equivalently $\log R(\omega)$ is integrable, where $R(\omega)$ is the equivalent power spectrum of the channel[1]]. When $\log R(\omega)$ is not integrable (as when it vanishes on an interval), there does not appear to exist an expansion of the type (31) to (32). What can be done is to use the Gram-Schmidt expansion of the form

$$h_m = \sum_{k=0}^{m} \langle h_m, w_k \rangle w_k, \tag{38}$$

where w_k is the orthonormal sequence obtained from $\{h_k\}$ by the usual Gram-Schmidt orthonogalization procedure. This expansion merely requires that $\{h_k\}$ be linearly independent, which is guaranteed by the existence of an interval on which $R(\omega)$ does not vanish.[1] From (38), it follows that

$$\sum_{m=0}^{\infty} \epsilon_m h_m = \sum_{m=0}^{\infty} \epsilon_m \sum_{k=0}^{m} \langle h_m, w_k \rangle w_k$$

$$= \sum_{k=0}^{\infty} \beta_k w_k \tag{39}$$

$$\beta_k = \sum_{m=k}^{\infty} \epsilon_m \langle h_m, w_k \rangle. \tag{40}$$

The key point is that the summation in (40) is infinite, so that evaluation of the lower bound of (34) is now necessarily over infinite error sequences. The finite sum in (31) results from the form of the expansion (7) in which h_n is expanded in terms of all future w_k's, and this expansion is in turn dependent on h_n not being an element of $M(h_k, k > n)$. Thus, when a DFE does not exist there appears to be no alternative to evaluating a sequence of upper bounds to $d_{\min}^2$ obtained by a finite sum approximation without the benefit of lower bounds to measure the degree of convergence.

III. THE PERFORMANCE OF THREE RECEIVERS ON THE $\sqrt{f}$ CHANNEL

Results of a calculation of the performance of the MLD, DFE, and ZFE will now be reported for the $\sqrt{f}$ channel, for which the attenuation in decibels increases as the square root of frequency. The $\sqrt{f}$ channel is a good approximation to coaxial cable, as well as to some cables consisting of wire pairs, and for this reason it is of great practical interest.

Many present high-speed digital transmission systems use some form of linear equalization, and their performance will be reasonably well approximated by that of the ZFE. Thus, the comparison between the ZFE and the MLD gives us an indication of the size of the gap in performance between common transmission systems in use today and what could theoretically be achieved by much more complex receiver designs.* The comparison with the DFE is much less interesting, because the susceptibility of the DFE to decision errors is not included in the present analysis and, as will be shown shortly, is of such a magnitude on the $\sqrt{f}$ channel as to essentially invalidate the performance estimate we calculate.

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* This comparison is, of course, very idealized. The only impairment we consider is additive Gaussian noise.

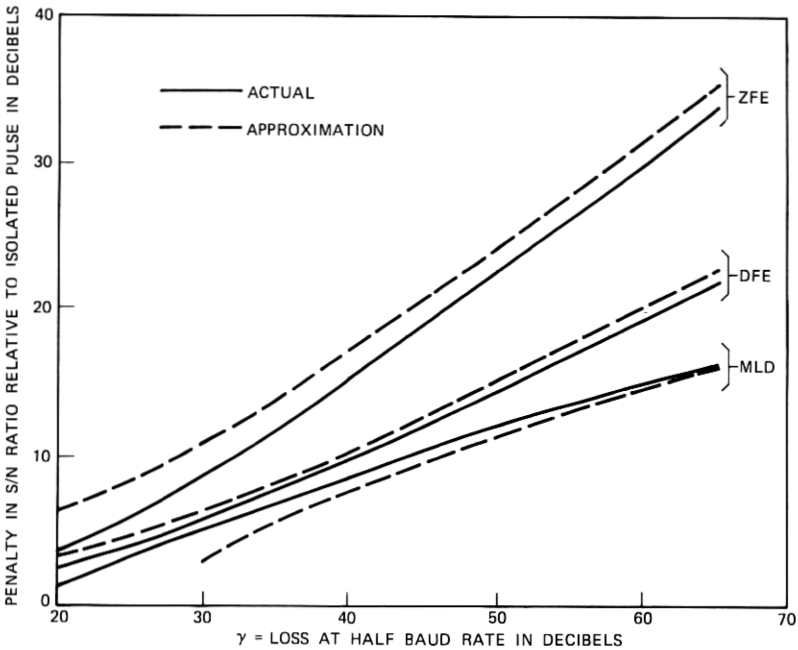

Fig. 2—Performance of three receivers on the $\sqrt{f}$ channel.

The power spectrum of the $\sqrt{f}$ channel is given by

$$|H(\omega)|^2 = 2\pi K^2 R_0 e^{-2K\sqrt{\omega}}, \tag{41}$$

where $H(\omega)$ is the frequency response of the channel and K is a parameter proportional to the line length. The usual convention is to designate the loss at the half-baud rate ($\omega = \pi/T$),

$$\gamma = -10 \log \frac{\left|H\left(\frac{\pi}{T}\right)\right|^2}{|H(0)|^2} \text{ (dB)}, \tag{42}$$

in which case

$$K = \sqrt{\frac{T}{\pi}} \frac{\gamma}{20 \log e}. \tag{43}$$

The effective penalties in S/N ratio relative to the isolated pulse bound can be calculated for the ZFE and DFE using the methods of Ref. 1, and for the MLD using the methods developed in Section II. The result is shown in Fig. 2 for the range of γ of practical interest. Most high-speed transmission systems in use today have a γ less than about 65 dB because of limitations in the maximum gain which can be incorporated into a repeater without excessive coupling of the output back into the input.

One interesting feature of Fig. 2 is that even the MLD has a substantial S/N ratio penalty (15 dB) on the $\sqrt{f}$ channel. Thus, Forney's statement[3] that on most channels intersymbol interference does not have to lead to a significant degradation in performance does not apply to channels with very severe intersymbol interference, such as are commonly used in high-speed transmission systems.

The value of $d^2_{\min}$, valid for Fig. 2, as well as many other examples considered by this author and Forney,[4] is

$$d^2_{\min} = 2(R_0 - R_1), \tag{44}$$

where R_k is the autocorrelation of the received pulse.* An approximation to (44) valid for large γ is derived in Appendix A and plotted in Fig. 2 as a dotted line. Approximations to the S/N ratio penalty of the ZFE and DFE are also derived in Appendix A and plotted in Fig. 2. An intuitive interpretation of eq. (44) is given in Appendix B.

As an illustration of the speed of convergence of (34), the sequence of upper and lower bounds is illustrated in Fig. 3 for a $\sqrt{f}$ channel with $\gamma = 60$ dB. These bounds are within 1 dB for $N = 1$ and 0.5 dB for $N = 3$. Thus, convergence is very rapid, even for severe intersymbol interference.

A word of caution is in order with respect to the curve for the DFE in Fig. 2. This curve does not take into account the effect of decision errors on the performance of the receiver. The DFE subtracts, prior to the decision threshold on data digit B_k, the quantity

$$\sum_{m=1}^{\infty} b_m \hat{B}_{k-m}, \tag{45}$$

where $\hat{B}_{k-m}$ is the receiver's previous decision on $\hat{B}_{k-m}$ and b_m is the tap-gain of the DFE feedback filter. The resulting quantity which is applied to the threshold is[1]

$$b_0 B_k + \sum_{m=1}^{\infty} b_m (B_{k-m} - \hat{B}_{k-m}) + n_k, \tag{46}$$

where n_k is a noise sample. Whenever the b_m's are large with respect to b_0, a single decision error will likely cause many more errors. The

* This corresponds to the error sequence $(1, -1, 0, 0, \cdots)$ or, in the notation of Forney, $(1 - D)$.

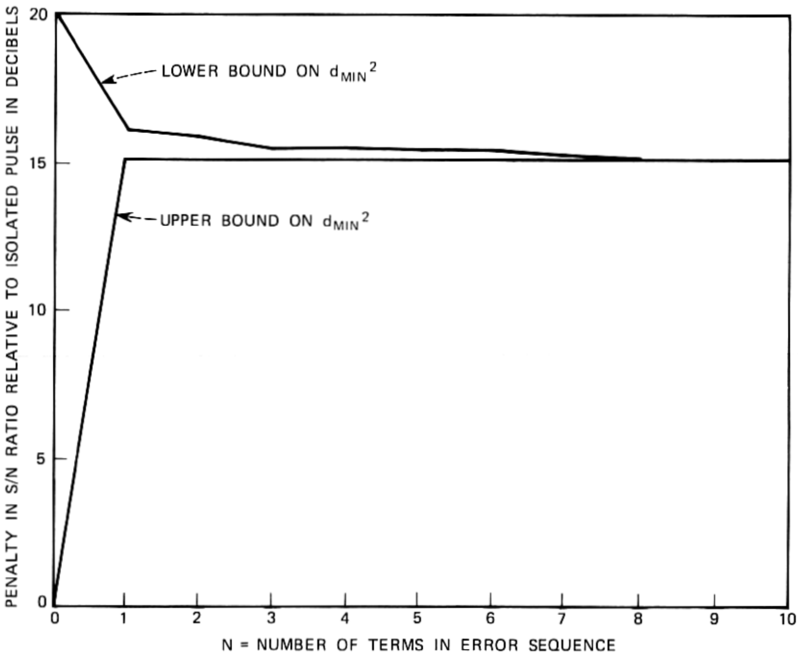

Fig. 3—Convergence of lower and upper bounds on $d_{\min}^2$ ($\sqrt{f}$ channel with $\gamma = 60$ dB).

coefficients of (46), given by (9), are tabulated in Table I for several values of γ.

Needless to say, the situation is hopeless for the large γ; the effect of a single decision error will be major and will last for a long time. Even for $\gamma = 20$, the reduction in noise margin resulting from a pre-

Table I—Coefficients of the DFE Feedback Filter (b_m)

m	b_m		
	$\gamma = 20$	$\gamma = 40$	$\gamma = 60$
0	1	1	1
1	0.61	1.4	2.2
2	0.36	1.3	2.8
3	0.25	1.1	2.9
4	0.18	0.94	2.9
5	0.14	0.80	2.8
10	0.06	0.42	1.9
47	0.006	0.06	0.38
174	0.001	0.009	0.06

vious decision error will be significant for five or ten subsequent decisions. We must conclude, then, that Fig. 2 will not be representative of the true performance of the DFE, and further that the DFE may not be a suitable receiver for the $\sqrt{f}$ channel*.

In terms of repeater spacing and baud rate, Fig. 1 can be interpreted in two ways. If the ZFE is replaced by an MLD, the same level of performance can be maintained while either increasing the repeater spacing with a constant baud rate or increasing the baud rate with the same repeater spacing. To illustrate this, consider the example of a ZFE operating at a given level of performance on a $\sqrt{f}$ channel with $\gamma = 40$ dB. Then γ can be increased to 60 dB at the same effective S/N ratio. This corresponds to a 50-percent increase in repeater spacing at a constant baud rate (since γ goes up linearly with the repeater spacing). However, since the repeater spacing has increased, the transmitted power must also be increased by 3.5 dB to maintain a constant isolated pulse energy at the receiver.†

If the repeater spacing is held constant, an increase in baud rate by a factor of $(1.5)^2$, or 125 percent, will also result in a 50-percent increase in γ. Here too, the average (but not peak) transmitted power is increased by 3.5 dB.

The conclusion of these results is that there is a fairly large gap between the performance of linear equalizers and the theoretical limit on the $\sqrt{f}$ channel. It is probably fair to say, however, that practical constraints on repeater complexity, speed of operation, and gain makes the attainment of a substantial portion of this potential improvement on high-speed transmission systems very difficult, at least for the present. Such is not the case for low-speed applications, such as voiceband data, where the implementation of the MLD can be contemplated on the basis of existing technology.

IV. CONCLUSIONS

In this paper, the minimum distance measure has been interpreted geometrically, related to equalization (the decision-feedback equalizer in particular), and bounded in several ways. A practical numerical technique has been developed for calculating the minimum distance without considering unnecessarily long error sequences.

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* Tomlinson[8] has invented a method of avoiding the error propagation problem by subtracting out interference from past data digits in the transmitter.

† The received pulse energy is proportional to γ^{-2}, so that the peak and average transmitted power must be increased by $20 \log(60/40) = 3.5$ dB.

Numerical results for the $\sqrt{f}$ channel reveal that the penalty in S/N ratio relative to the isolated pulse bound for the MLD can be substantial for this channel, and that the gap in performance between the MLD and linear equalization is also substantial. The latter suggests that further attempts at finding receivers without the complexity of the Viterbi algorithm MLD but which nevertheless improve on the performance of linear equalization might well be fruitful. The decision-feedback equalizer does not appear to fit this bill because of its serious error propagation problem when confronted with intersymbol interference as severe as that found on the $\sqrt{f}$ channel.

APPENDIX A

Autocorrelation of the $\sqrt{f}$ Channel

From (41), the autocorrelation is

$$R_k = \frac{1}{\pi}\int_0^\infty |H(\omega)|^2 \cos(\omega kT)d\omega$$

$$= \frac{4K^2R_0}{T}\int_0^\infty x\exp\left(-\frac{2K}{\sqrt{T}}x\right)\cos(kx^2)dx. \tag{47}$$

Integrating by parts with $u = \exp\left(-\frac{2K}{\sqrt{T}}x\right)$ and $dv = x\cos(kx^2)dx$, we get

$$R_k = \frac{4K^3R_0}{(kT)^{3/2}}\int_0^\infty \exp\left(-\frac{2K}{\sqrt{kT}}x\right)\sin x^2dx\,,$$

which is given in terms of the Fresnel Integral,[9]

$$R_k = \sqrt{\frac{\pi}{2}}\frac{4K^3R_0}{(kT)^{3/2}}\left\{\left[\frac{1}{2} - C\left(\frac{K}{\sqrt{kT}}\sqrt{\frac{2}{\pi}}\right)\right]\cos\left(\frac{K^2}{kT}\right) + \left[\frac{1}{2} - S\left(\frac{K}{kT}\sqrt{\frac{2}{\pi}}\right)\right]\sin\left(\frac{K^2}{kT}\right)\right\}, \tag{48}$$

where

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}y^2\right)dy$$

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}y^2\right)dy\,.$$

An accurate approximation to R_1 valid for large γ is easily obtained from (47) by substituting the first two terms of a Taylor series for

$\cos x^2$,

$$R_1 \cong \beta^2 R_0 \int_0^\infty x \left(1 - \frac{x^4}{2}\right) e^{-\beta x} dx$$

$$= R_0 \left(1 - \frac{60}{\beta^4}\right), \tag{49}$$

where

$$\beta = \frac{2K}{\sqrt{T}}.$$

Hence

$$2(R_0 - R_1) \cong \frac{120}{\beta^4}$$

and

$$-10 \log \frac{2(R_0 - R_1)}{R_0} \cong 40 \log \gamma - 56.2. \tag{50}$$

Approximations to $\|e_0\|^2$ and $\|e_0^+\|^2$ can also be derived by assuming that $H(\omega) = 0$, $|\omega| > \pi/T$, or equivalently that $|H(\omega)|^2 = R(\omega)$. The resulting S/N ratio penalties are

$$-10 \log \|e_0\|^2/R_0 \cong \gamma + 25.15 - 30 \log \gamma \tag{51}$$

$$-10 \log \|e_0^+\|^2/R_0 \cong \tfrac{2}{3}\gamma + 15.76 - 20 \log \gamma\ . \tag{52}$$

Equations (50) to (52) are plotted in Fig. 2 as dotted lines.

APPENDIX B

Interpretation of Equation (44)

It is straightforward to show that whenever

$$\frac{R_1}{R_0} \geqq 0.5 \tag{53}$$

we have

$$d_{\min}^2 \leqq 2(R_0 - R_1) \leqq R_0. \tag{54}$$

Noting that

$$R_1 = \langle h_0, h_1 \rangle = \|h_0\| \, \|h_1\| \cos \theta$$
$$= R_0 \cos \theta,$$

where θ is the angle between h_0 and h_1, eq. (53) becomes

$$\theta \leqq 60°. \tag{55}$$

The geometric interpretation of (55) is shown in Fig. 4, where it is seen that (54) is satisfied until $\theta = 60°$, when the triangles become

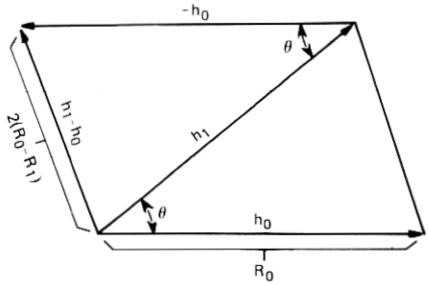

Fig. 4—Geometric interpretation of eq. (44).

equilateral. As long as (55) is satisfied, $h_0 - h_1$ is a shorter vector than h_0.

In the case of the $\sqrt{f}$ channel, R_1/R_0 is very close to unity. Thus, $h_0 - h_1$ is a very short vector. Although it will certainly not always be the case, a plausible explanation for the fact that longer error sequences do not yet yield a shorter vector is that the addition of other translates of h_k (such as $\pm h_2$) adds further components in other dimensions. Presuming that it does not reduce the component in the $h_0 - h_1$ plane, it can then only increase the length of the total vector.

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