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Losses and Impulse Response of a Parabolic Index Fiber With Random Bends

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The coupling coefficients of the modes of a parabolic index fiber with randomly curved axis are derived and are used to compute its excess losses and impulse response. It is found that bends with a period comparable to the natural ray oscillation period in the parabolic index medium are catastrophic. The average radius of curvature R_c of a guide composed of circular sections with an average length of 1 cm must not decrease below approximately $R_c = 1$ m. Mode coupling by random bends has the tendency to reduce the width of the impulse response function. However, this improvement is accompanied by losses. Reducing the width of the impulse response for coupled mode operation to half its uncoupled width causes 0.7 dB additional loss, a ten-fold reduction of the pulse width costs 18 dB.

I. INTRODUCTION

Optical fibers with parabolic refractive index profile[1,2] have less pulse delay distortion than conventional fibers with piecewise constant, discontinuous index distribution.[3] The width of the impulse response increases in direct proportion to the length of the fiber. It is well known that an improvement of the impulse response results if the modes are coupled among each other.[4,5] In the presence of mode coupling the width of the impulse response increases only proportionally to the square root of the length of the fiber.

Pulse propagation in multimode parabolic index fibers is studied in this paper by means of converting the coupled power equations to a partial differential equation.[5–7] Random changes of the direction of the waveguide axis are considered as the coupling mechanism.

The problem is simplified by assuming that the modes of the parabolic index fiber are essentially the same as the modes of an infinitely extended square-law medium. The fiber boundary is included in the description by requiring that modes interacting with it suffer high losses so that an effective cutoff exists. Modes below the cutoff value propagate as if they were in an infinitely extended medium. At cutoff we demand that the modes do not carry power. The effect of the waveguide wall is thus taken into account as a boundary condition that has to be satisfied by the solutions of the partial differential equation.

This formalism provides information about the width of the impulse response function and the losses associated with the coupling mechanism. The achievable improvement in pulse width due to mode coupling can thus be expressed in terms of the associated loss penalty. We conclude that mode coupling is capable of improving the already favorable impulse response of the parabolic index fiber. However, this improvement of the width of the impulse response is accompanied by excess losses. The product of the square of the pulse width ratio (width of the impulse response of coupled modes to the uncoupled pulse width) times the loss penalty is independent of the waveguide parameters and the statistics of the axis deformation. There is thus no hope of reducing the loss penalty of delay distortion improvement by optimizing the waveguide parameters.

II. MODES AND COUPLING COEFFICIENTS

We use the modes of the infinite square-law medium. The refractive index distribution is assumed to be of the form

$$n^2 = n_0^2\left(1 - 2\bar{\Delta}\frac{r^2}{a^2}\right). \tag{1}$$

The fiber radius is at $r = a$. However, the modes are assumed to be unaffected by the fiber boundary if their mode number remains below a cutoff value. We use linearly polarized modes and obtain for the y component of the electric field[8]

$$E_{pq} = \frac{2\left(\sqrt{\frac{\mu_0}{\epsilon_0}}P\right)^{\frac{1}{2}} H_p\left(\sqrt{2}\,\frac{x}{w}\right) H_q\left(\sqrt{2}\,\frac{y}{w}\right) e^{-r^2/w^2}}{(n_0\pi 2^{p+q}p!\,q!)^{\frac{1}{2}}w}\, e^{-i\beta z}. \tag{2}$$

There is also an electric field component in axial direction. But, for small refractive index changes, it is negligible. The functions H_p and H_q are Hermite polynomials of order p and q, the radius r is defined by

$$r^2 = x^2 + y^2, \tag{3}$$

and the mode radius w is

$$w = \left(\frac{\sqrt{2}a}{n_0 k\sqrt{\bar{\Delta}}}\right)^{\frac{1}{2}} \tag{4}$$

with the free-space propagation constant

$$\mathrm{k} = \omega\sqrt{\epsilon_0\mu_0}. \tag{5}$$

The propagation constant of the mode is given by the expression[8]

$$\beta = n_0\mathrm{k}\left[1 - \frac{2\sqrt{2\bar{\Delta}}}{n_0\mathrm{k}a}(p + q + 1)\right]^{\frac{1}{2}}. \tag{6}$$

The orthogonality of the modes and their normalization follows from the following equation:

$$\frac{\beta}{2\mathrm{k}}\sqrt{\frac{\epsilon_0}{\mu_0}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{pq}E^*_{p'q'}dxdy = P\delta_{pp'}\delta_{qq'}. \tag{7}$$

The asterisk indicates complex conjugation.

The cutoff condition for the guided mode has been derived in Ref. 9. The permitted maximum values of p and q are defined by the relation

$$(p + q)_c = \sqrt{\frac{\bar{\Delta}}{2}}\, n_0\mathrm{k}a = \frac{a^2}{w^2}. \tag{8}$$

Any deviation of the parabolic index fiber from its perfect geometry can be expressed by a change of its refractive index distribution.

Changes in the direction of the waveguide axis can be expressed by the following index distribution:

$$n^2 = n_0^2\left\{1 - 2\frac{\bar{\Delta}}{a^2}[(x - f(z))^2 + y^2]\right\}. \tag{9}$$

We consider waveguides bends in only one plane for simplicity. The results thus obtained can easily be extended to the general case. The appropriate coupling coefficient for this type of index change is [10,11]

$$K_{pq,p'q'} = \frac{\omega\epsilon_0}{4P(\beta_{pq} - \beta_{p'q'})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial n^2}{\partial z}\,\mathcal{E}^*_{pq}\mathcal{E}_{p'q'}dxdy. \tag{10}$$

Bends of the waveguide axis couple even modes to their immediate odd neighbors and odd modes to their immediate even neighbors. Only the following coupling coefficients are different from zero:

$$K_{pq,p\pm1,q} = \frac{n_0\mathrm{k}w\bar{\Delta}}{a^2}\sqrt{p}\,\frac{\dfrac{df}{dz}}{\beta_p - \beta_{p\pm1}} = i\,\frac{n_0\mathrm{k}w\bar{\Delta}}{a^2}\sqrt{p}f(z). \tag{11}$$

Because we restricted the waveguide curvature to the $x - z$ plane there is no coupling between the modes with different values of q.

All coupling coefficients with different q values vanish. The derivative of $f(z)$ was replaced by the function itself with the help of the relation

$$\frac{1}{\beta - \beta'} \frac{df}{dz} \rightarrow i f(z). \tag{12}$$

This replacement is permissible since it is the spatial frequency $\beta - \beta'$ of $f(z)$ that is responsible for the coupling process.

III. COUPLED POWER EQUATIONS

Pulse propagation in multimode optical fibers can be described by the following set of coupled equations for the average power P_ν carried by the modes:[5]

$$\frac{\partial P_\mu}{\partial z} + \frac{1}{v_\mu} \frac{\partial P_\mu}{\partial t} = -\alpha_\mu P_\mu + \sum_{\nu=1}^{N} h_{\mu\nu}(P_\nu - P_\mu). \tag{13}$$

The single label ν indicates both p and q. The power coupling coefficients $h_{\nu\mu}$ are defined by [5,6]

$$h_{\mu\nu} = h_{\nu\mu} = |\hat{K}_{\mu\nu}|^2 F(\beta_\mu - \beta_\nu). \tag{14}$$

The coupling coefficient (11) enters the power coupling coefficients via the definition

$$K_{\mu\nu} = \hat{K}_{\mu\nu} f(z). \tag{15}$$

The power spectrum $F(\beta_\mu - \beta_\nu)$ is defined by the equation

$$F(\theta) = \left\langle \left| \frac{1}{\sqrt{L}} \int_0^L f(z) e^{-i\theta z} dz \right|^2 \right\rangle. \tag{16}$$

The symbol $\langle \ \rangle$ indicates an ensemble average.

Coupling between the modes of the parabolic index fiber is an ideal application for a diffusion theory of the power coupling process.[7] Since only nearest neighbors couple directly to each other, power redistributes itself by jumping from mode to mode in the same way as particles diffuse through real space. If the mode number is very large we can consider the set of discrete modes as a quasi-continuum and change the equation system (13) into a partial differential equation. To accomplish this transformation we consider the following expression using $h_{\mu\nu} = h_{\nu\mu}$:

$$\sum_{\nu=1}^{N} h_{\mu\nu}(P_\nu - P_\mu) = h_{\mu+1,\mu}(P_{\mu+1} - P_\mu) - h_{\mu,\mu-1}(P_\mu - P_{\mu-1}). \tag{17}$$

Considering μ as a continuous variable $\mu = \theta$ we use the approximation

$$h_{\mu+1,\mu}(P_{\mu+1} - P_\mu) - h_{\mu,\mu-1}(P_\mu - P_{\mu-1})$$
$$= \Delta\theta \left\{ h(\theta + \Delta\theta) \left(\frac{\partial P}{\partial \theta} \right)_{\theta+\Delta\theta} - h(\theta) \left(\frac{\partial P}{\partial \theta} \right)_\theta \right\}$$
$$= (\Delta\theta)^2 \frac{\partial}{\partial \theta} \left(h \frac{\partial P}{\partial \theta} \right). \qquad (18)$$

With $\Delta\theta = 1$ we can write (13) as a partial differential equation

$$\frac{\partial P}{\partial z} + \frac{1}{v} \frac{\partial P}{\partial t} = -\alpha P + \frac{\partial}{\partial \theta} \left(h \frac{\partial P}{\partial \theta} \right). \qquad (19)$$

The propagation constant (6) can be approximated as follows:

$$\beta = n_0 \mathrm{k} - \frac{\sqrt{2\bar{\Delta}}}{a} (\theta + q + 1) \qquad (20)$$

with $p = \theta$. The difference of the propagation constants of adjacent modes,

$$\Delta\beta = \beta(\theta + \Delta\theta) - \beta(\theta) = -\frac{\sqrt{2\bar{\Delta}}}{a} = -\Omega, \qquad (21)$$

is independent of θ. The power spectrum entering (14) contributes to the coupling process only at one fixed spatial frequency and is independent of the variable θ. The power coupling coefficient (14) can be expressed with the help of (4), (11), and (21) as follows ($p = \theta$):

$$h(\theta) = \frac{\sqrt{2} n_0 \mathrm{k} \bar{\Delta}^{\frac{3}{2}}}{a^3} F(\Omega)\theta. \qquad (22)$$

With (22) the partial differential equation (19) assumes the form

$$\frac{\partial P}{\partial z} + \frac{1}{v} \frac{\partial P}{\partial t} = -\alpha P + K \left[\theta \frac{\partial^2 P}{\partial \theta^2} + \frac{\partial P}{\partial \theta} \right] \qquad (23)$$

with

$$K = \frac{\sqrt{2} n_0 \mathrm{k} \bar{\Delta}^{\frac{3}{2}} F(\Omega).}{a^3} \qquad (24)$$

We assume that the attenuation coefficient α in (19) is constant and describe the high loss, that we must attribute to modes interacting with the waveguide boundary, by means of the boundary condition

$$P(z, t, \theta) = 0 \quad \text{for} \quad \theta = \theta_c. \qquad (25)$$

The cutoff value $\theta = \theta_c$ follows from (8):

$$\theta_c = \frac{a^2}{w^2} - q = \frac{n_0 ka\sqrt{\Delta}}{\sqrt{2}} - q. \tag{26}$$

The slope $\partial P/\partial\theta$ determines the rate of power diffusion. Since no power can be lost at $\theta = 0$ we must also require

$$\frac{\partial P}{\partial \theta} = 0 \qquad \text{at} \qquad \theta = 0. \tag{27}$$

IV. STEADY-STATE POWER LOSS

We begin the discussion of the solutions of (23) by neglecting the fact that each guided mode has a slightly different group velocity and consider $v(\theta) = \text{const}$. We construct a solution of (23) by introducing the trial solution

$$P(z, t, \theta) = e^{-(\sigma+\alpha)z}G(\theta). \tag{28}$$

Substitution of (28) into (23) yields the ordinary differential equation

$$\theta\frac{d^2G}{d\theta^2} + \frac{dG}{d\theta} + \frac{\sigma}{K}G = 0. \tag{29}$$

The normalized solutions of this equation that satisfy the boundary conditions (25) and (27) are

$$G_\nu(\theta) = \frac{1}{\sqrt{\theta_c}}\frac{J_0\left(u_\nu\sqrt{\frac{\theta}{\theta_c}}\right)}{J_1(u_\nu)} \tag{30}$$

with

$$u_\nu = 2\sqrt{\frac{\sigma_\nu}{K}\theta_c}. \tag{31}$$

The parameters u_ν are determined as the roots of the equation

$$J_0(u_\nu) = 0. \tag{32}$$

The functions $G_\nu(\theta)$ are mutually orthogonal.

$$\int_0^{\theta_c} G_\nu(\theta)G_\mu(\theta)d\theta = \delta_{\nu\mu}. \tag{33}$$

The general solution of the power equation (23) is obtained as the superposition of the trial solutions

$$P(z, t, \theta) = e^{-\alpha z}\sum_{\nu=1}^{\infty} c_\nu G_\nu(\theta)e^{-\sigma_\nu z}. \tag{34}$$

The expansion coefficient c_ν can be determined from the power distribution at $z = 0$ with the help of the orthogonality condition (33),

$$c_\nu = \int_0^{\theta_c} G_\nu(\theta) P(0, t, \theta) d\theta. \tag{35}$$

The eigenvalues σ_ν are obtained from (31) and (32),

$$\sigma_\nu = \frac{K u_\nu^2}{4\theta_c}. \tag{36}$$

The eigenvalues increase with the increasing values of the roots u_ν. It is thus apparent that only the first term in the series (34) needs to be considered for large values of z. The steady-state power distribution is thus described by the equation

$$P(z, t, \theta) = c_1 e^{-(\alpha+\sigma_1)z} G_1(\theta) \quad \text{for} \quad z \to \infty. \tag{37}$$

After an initial transient has decayed, the power distribution (versus mode number θ) assumes the steady state (37). The power loss in the steady state is the sum of the constant loss α, that was assumed to be the same for every mode, plus the loss value σ_1 that stems from mode coupling due to waveguide curvature. With

$$u_1 = 2.405 \tag{38}$$

we obtain the steady-state curvature loss from (24), (26), and (36),

$$\sigma_1 = \frac{2.045 \cdot n_0 \mathrm{k} \bar{\Delta}^{3/2}}{\left(\dfrac{n_0 \mathrm{k} a \sqrt{\bar{\Delta}}}{\sqrt{2}} - q\right) a^3} F(\Omega). \tag{39}$$

Because of our assumption that the waveguide is curved only in one plane the steady state losses depend on the mode number q. For small values of q we find low curvature loss

$$\sigma_1 = \frac{2.89 \bar{\Delta}}{a^4} F(\Omega) \quad \text{for} \quad q = 0. \tag{40}$$

With increasing values of q the losses increase until they reach infinitely high values.

In any actual cases it is unrealistic to assume that the waveguide would be bent in only one plane. Bends in the perpendicular plane couple modes with different q values. The total steady-state loss is thus a weighted average of the losses (39). The weight factor is the number of modes for each value of q. According to (8) we have

$$p = N(q) = \sqrt{\frac{\bar{\Delta}}{2}}\, n_0 \mathrm{k} a - q \tag{41}$$

different modes for each value of q. The average loss that results from coupling all the modes by random bends in both planes is thus (see appendix)

$$\bar{\sigma}_1 = \frac{4}{(n_0ka)^2\bar{\Delta}} \int_0^{n_0ka\sqrt{\bar{\Delta}/2}} \sigma_1 N(q)dq = \frac{5.8\bar{\Delta}}{a^4} F(\Omega). \tag{42}$$

Comparison with the loss coefficient (40) for the mode group with $q = 0$ shows that the total fiber loss (42) is just twice as large. The loss coefficient (40) is representative of a slab waveguide. The actual loss of the round fiber can thus be deduced from the slab waveguide model. That the loss coefficient of the fiber is twice as large as the slab waveguide loss might be expected, since $F(\Omega)$ stands for the amplitude of the power spectrum for bends in only one plane. The fiber is assumed to be bent in both planes with equal power spectra. The effect of both bends add, doubling the loss coefficient.

In a fiber cable the fibers may (or may not) be twisted around each other. Such twists could introduce an almost sinusoidal deformation of the fiber axis. The length Λ of the period of sinusoidal deformations that should be avoided follows from (21), $\Lambda = 2\pi/\Omega$. For numerical estimates we are using a fiber with radius $a = 4.85 \times 10^{-3}$ cm and $\bar{\Delta} = 0.014$. The critical period for this fiber is $\Lambda = 0.18$ cm. If such a period should be built into the fiber by the method of cable construction we can estimate the losses that would be caused by a given amplitude. $F(\Omega)$ has the dimension of cm^3. It can be interpreted as the ratio of the square of the amplitude of the sinusoidal deformation and the spatial bandwidth that may be caused by random phase changes. With $\Omega = 34.5$ cm^{-1} let us assume a spatial bandwidth of $\Delta\Omega = 3$ cm^{-1}. An excess loss of $\bar{\sigma}_1 = 10$ dB/km $= 2.3 \times 10^{-5}$ cm^{-1} would require $F(\Omega) = 1.57 \times 10^{-13}$ cm^3. The square of the amplitude A is given by the product of $F(\Omega)$ with the spatial bandwidth. We thus obtain the amplitude of the sinusoidal deformation of the fiber axis that causes an excess loss of 10 dB/km: $A = [F(\Omega)\Delta\Omega]^{\frac{1}{2}} = 6.9 \times 10^{-7}$ cm $= 69$ Å. Sinusoidal axis deformations at the critical wavelength are thus seen to be extremely dangerous.

V. LOSSES FOR A STATISTICAL MODEL

Even though it is only one spatial frequency of the power spectrum $F(\beta_\mu - \beta_\nu)$ that determines the steady state loss (42), it is hard to guess the amplitude $F(\Omega)$ that might be expected at this spatial frequency Ω. In order to gain insight into the expected steady-state curvature losses it is necessary to consider statistical models. We consider a model consisting of waveguide sections with constant curvature whose magnitude and sign varies randomly.

The power spectrum can be written as follows:

$$F(\Omega) = \frac{1}{L}\left\langle\left|\int_0^L f(z)e^{-i\Omega z}dz\right|^2\right\rangle$$
$$= \frac{1}{\Omega^4 L}\left\langle\left|\int_0^L \frac{d^2 f}{dz^2}\, e^{-i\Omega z}\right|^2\right\rangle. \qquad (43)$$

The step from the function $f(z)$ to its second derivative involved two partial integrations. The end points of the integration range do not contribute if we assume that the randomly disturbed guide is connected to two perfectly straight waveguide sections so that $f(z)$ and its first two derivatives vanish at $z = 0$ and at $z = L$.

For waveguides that are only slightly bent we can consider the second derivative of $f(z)$ as the curvature function $1/R_c(z)$. We denote with $C(u)$ the autocorrelation function of the curvature function,

$$C(u) = \left\langle \frac{1}{R_c(z)R_c(z+u)} \right\rangle. \qquad (44)$$

It is well known that the power spectrum of a function is equal to the Fourier transform of its autocorrelation function.[12] We may thus write

$$F(\Omega) = \frac{1}{\Omega^4}\int_{-\infty}^{\infty} C(u)e^{-i\Omega u}du. \qquad (45)$$

The autocorrelation function of waveguide sections with piecewise constant curvature and fixed length D is

$$C(u) = \begin{cases} \dfrac{D - |u|}{D}\,\kappa^2 & |u| \leqq D \\ 0 & |u| > D. \end{cases} \qquad (46)$$

The parameter κ^2 is the variance (square of the rms value) of the curvature $1/R_c$. Substitution of (46) into (45) results in

$$F = \frac{2\kappa^2}{D\Omega^6}(1 - \cos \Omega D). \qquad (47)$$

An average over this expression, that allows us to consider D as an averaged quantity, leaves us with

$$F(\Omega) = \frac{2\kappa^2}{D\Omega^6}. \qquad (48)$$

From (21), (42), and (48) we obtain the following expression for the steady-state loss of our statistical waveguide model:

$$\sigma^{(1)} = \frac{1.4\kappa^2 a^2}{D\bar{\Delta}^2}. \tag{49}$$

As a numerical example, we consider a parabolic index waveguide with the following parameters:

$$\left.\begin{aligned} \lambda &= 1\ \mu\text{m, free-space wavelength } (\text{k} = 6.28 \times 10^4\ \text{cm}^{-1}) \\ a &= 4.85 \times 10^{-3}\ \text{cm, waveguide radius} \\ n_o &= 1.56 \\ \bar{\Delta} &= 0.014 \\ w &= 7.69 \times 10^{-4}\ \text{cm, mode radius} \\ \frac{a}{w} &= 6.3 \end{aligned}\right\}. \tag{50}$$

With these data we have (κ in cm^{-1}, D in cm)

$$\sigma^{(1)} = 0.17\,\frac{\kappa^2}{D}\ \text{cm}^{-1}. \tag{51}$$

We may now ask for the rms value of the curvature that is required to cause a steady-state loss of $2.3 \times 10^{-5}\ \text{cm}^{-1} = 10$ dB/km with an average length of the waveguide sections of $D = 1$ cm. We find from (51) $\kappa = 0.0116\ \text{cm}^{-1}$ or $1/\kappa = 86$ cm. Our result tells us that a waveguide composed of individual sections of constant curvature of average length $D = 1$ cm with $R_c \approx 1$ m radius of curvature has 10 dB/km additional loss. For our derivation we assumed that the high-order modes suffer very high losses since their fields reach into the vicinity of the waveguide wall. Whether the interaction with the outer waveguide boundary causes high losses depends on the construction of the waveguide. If the outer surface is rough or coated with an absorbing material to reduce crosstalk, the losses are high and our estimate applies.

VI. PULSE DELAY DISTORTION

It was shown in Ref. 5 that the width of the impulse response of a multimode fiber with coupled modes is given by the equation

$$\Delta t = 4\sqrt{\rho L}. \tag{52}$$

L is the length of the waveguide and ρ is the second-order perturbation of the eigenvalue σ_1 defined by (36). For the discrete case we write

$G_\nu(\theta) = G_p^{(\nu)}$ and obtain ρ in the form[5]

$$\rho = \sum_{\nu=2}^{N} \frac{\left[\sum_{p=1}^{N} \left(\frac{1}{v_p} - \frac{1}{v_o} \right) G_p^{(1)} G_p^{(\nu)} \right]^2}{\sigma_\nu - \sigma_1}. \tag{53}$$

The average group velocity v_o actually does not contribute to (53) on account of the orthogonality of the vectors $G^{(\nu)}$. With the assumption of a continuum of modes we obtain instead of (53)

$$\rho = \sum_{\nu=2}^{\infty} \left\{ \frac{1}{\sigma_\nu - \sigma_1} \left[\int_0^{\theta_c} \left(\frac{1}{v(\theta)} - \frac{1}{v_o} \right) G_1(\theta) G_\nu(\theta) d\theta \right]^2 \right\}. \tag{54}$$

The inverse group velocity is obtained by approximating (6),

$$\beta \approx n_o \mathrm{k} - \frac{\sqrt{2\bar{\Delta}}}{a} (p + q) - \frac{\bar{\Delta}}{n_o \mathrm{k} a^2} (p + q)^2, \tag{55}$$

and taking the derivative. With $v_o = c/n_o$ we obtain (with $p = \theta$ and c = light velocity in vacuum)

$$\frac{1}{v(\theta)} - \frac{1}{v_o} = \frac{1}{c} \frac{d\beta}{d\mathrm{k}} - \frac{n_o}{c} = \frac{\bar{\Delta}}{c n_o \mathrm{k}^2 a^2} (\theta^2 + 2\theta q + q^2). \tag{56}$$

The term with q^2 does not contribute to the following integral because of the orthogonality relation (33):

$$\int_0^{\theta_c} \left(\frac{1}{v(\theta)} - \frac{1}{v_o} \right) G_1(\theta) G_\nu(\theta) d\theta$$

$$= \frac{16\bar{\Delta}\theta_c}{c n_o \mathrm{k}^2 a^2} \frac{u_\nu u_1}{(u_\nu^2 - u_1^2)^2} \left\{ \theta_c \left[1 - \frac{12(u_\nu^2 + u_1^2)}{(u_\nu^2 - u_1^2)^2} \right] + q \right\}. \tag{57}$$

Each mode group with a given value of q has a different spread of the group velocities of its uncoupled modes. However, we have seen that the waveguide losses could be obtained from the simpler slab waveguide model. This simplification is expected to apply also to the pulse distortion problem. The slab model is obtained by setting $q = 0$. The spread of inverse group velocities is largest for $q = 0$ since the allowed θ range is largest in this case. However, even though the spread of the group velocities is reduced for increasing values of q, the mode groups with different q values arrive at different times. This delay distortion is reduced by coupling of the different mode groups by means of waveguide bends in the perpendicular plane (perpendicular to the plane coupling the modes with different p values). The mode group with $q = 0$ can be excited by shining light into the fiber that is collimated in one plane but spreads in the plane of the bends in such a way that all modes with $q = 0$ and p values in the range $0 < p < n_o ka(\bar{\Delta}/2)^{\frac{1}{2}}$ are excited. The bends in one plane do not cause coupling to modes with different q values but reduce the delay distortion of the modes with different p values. From this physical picture we see that the delay distortion problem is reduced to studying delay distortion in a slab waveguide. Bends of the fiber in the perpendicular plane couple the modes with different q but fixed p values. Their velocity spread is the same as that considered in the first problem. We thus expect to obtain the correct result by considering the delay distortion reduction for the mode group with $q = 0$.

With $q = 0$ we obtain from (26) and (57)

$$\int_0^{\theta_c} \left(\frac{1}{v(\theta)} - \frac{1}{v_o} \right) G_1(\theta) G_\nu(\theta) d\theta = \frac{8 n_o \bar{\Delta}^2}{c} \frac{u_\nu u_1}{(u_\nu^2 - u_1^2)^2} \left[1 - \frac{12(u_\nu^2 + u_1^2)}{(u_\nu^2 - u_1^2)^2} \right]. \tag{58}$$

Using (24), (36), (54), and (58) we have

$$\rho = \frac{128 n_o^2 a^4 \bar{\Delta}^3}{c^2 F(\Omega)} \left\{ \sum_{\nu=2}^{\infty} \frac{u_\nu^2 u_1^2}{(u_\nu^2 - u_1^2)^5} \left[1 - \frac{12(u_\nu^2 + u_1^2)}{(u_\nu^2 - u_1^2)^2} \right]^2 \right\} = 2.26 \times 10^{-4} \frac{n_o^2 a^4 \bar{\Delta}^3}{c^2 F(\Omega)}. \tag{59}$$

The width of the impulse response follows from (52),

$$\Delta t = 0.06 \frac{n_o a^2}{c} \bar{\Delta}^{\frac{3}{2}} \left(\frac{L}{F(\Omega)} \right)^{\frac{1}{2}}. \tag{60}$$

From Ref. 9 we obtain the width of the impulse response for uncoupled modes

$$\Delta\tau = \frac{n_o L}{2c} \bar{\Delta}^2. \tag{61}$$

The improvement that is caused by mode coupling is the ratio of the widths of these two impulse responses

$$R = \frac{\Delta t}{\Delta\tau} = \frac{0.12 a^2}{[F(\Omega) L \bar{\Delta}]^{\frac{1}{2}}}. \tag{62}$$

For the statistical model of a sequence of circularly bent waveguide sections we obtain with (48) and (21)

$$R = \frac{0.24 \bar{\Delta}}{\kappa a} \left(\frac{D}{L} \right)^{\frac{1}{2}}. \tag{63}$$

We may ask for the average radius of curvature that is required to cause a ten-fold improvement of the width of the impulse response due to mode coupling. Using $L = 1$ km and an average length of the bent sections of $D = 1$ cm and the numbers in (50) we obtain for $R = 0.1$, $\kappa = 0.022$ cm^{-1} or an average radius of curvature $R = 1/\kappa = 45$ cm. This relatively small radius of curvature may cause very substantial excess loss according to the loss example in Section V.

In order to relate the excess loss to the delay distortion improvement we consider the loss penalty that must be paid for a given amount of improvement in the width of the impulse response. Both the loss formula (40) and the improvement factor R, (62), contain the power spectrum of the distortion function $f(z)$. By taking the square of R and multiplying it with the loss per length L, $\sigma_1 L$, we obtain†

$$R^2\sigma_1 L = 0.042 = 0.18 \text{ dB}. \tag{64}$$

This important formula is independent of any of the waveguide parameters and of the statistics of the axis deformation. This means that the loss penalty, $\sigma_1 L$, for parabolic index fibers depends only on the delay distortion improvement that one wants to achieve. For $R = 1$ we have a loss of $\sigma_1 L = 0.18$ dB. Clearly, the range of applicability of (64) is exceeded in this case since $R = 1$ means that there is no improvement at all. For $R = 0.5$ we pay a loss penalty of 0.7 dB, $R = 0.1$ increases the loss to 18 dB. The already favorable delay distortion of the parabolic index fiber can be improved by intentional curvature of the waveguide axis.

VII. DISCUSSION

We have studied the performance of the parabolic index fiber with randomly curved axis. The curvature of the waveguide axis has the tendency to force a light beam inside of the fiber towards the fiber boundary. In terms of wave optics this means that the wave field begins to interact with the boundary of the fiber. If this boundary is perfectly smooth no particular harm may be done except that the impulse response of the fiber is likely to deteriorate. However, the interfaces between two dielectric regions tend to be rough. Surface roughness leads to scattering losses. We have thus assumed that the interaction of the mode fields with the fiber boundary causes significant losses to high-order modes. On this basis we were able to calculate the fiber loss caused by random bends of the waveguide axis. For bends that approach a sinusoidal shape, with a period comparable to

† We use σ_1 of (40) instead of σ_1 of (42) since R was computed for $q = 0$.

the ray oscillation period in the parabolic index medium, the excess losses are extremely high. Bending of the waveguide axis with a period equal to the ray oscillation period must be avoided. For a statistical model, based on the assumption that the waveguide is composed of a sequence of circularly bent sections with random length and random radius of curvature, the waveguide losses have been predicted. We conclude that average radii of curvature of approximately 1 m can be allowed if an excess loss of 10 dB/km can be tolerated. The waveguide sections were assumed to have an average length of 1 cm.

It is possible to reduce the width of the impulse response of a parabolic index fiber by coupling its modes by random bends of the fiber axis. The impulse response of parabolic index fibers is already quite favorable compared to the impulse response of the conventional optical fiber with a discontinuous but piecewise constant index distribution. Our analysis shows that additional reduction of pulse delay distortion is accompanied by losses. A reduction of the pulse width to half its uncoupled width increases the loss by 0.7 dB, a ten-fold pulse width reduction increases the fiber loss by 18 dB.

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APPENDIX

The averaging process used to obtain (42) can be justified as follows. Each mode group characterized by the mode number q comprising all modes with

$$0 < p < \left(\sqrt{\frac{\Delta}{2}}\, n_o ka - q\right)$$

has the loss coefficient $\sigma_1(q)$. By definition this can be written

$$\sigma_1(q) = \frac{\Delta P(q)}{P(q)}.$$

$\Delta P(q)$ is the power lost from the mode group per unit length and $P(q)$ is the power carried by these modes. If we assume, for simplicity, that each mode carries the power P we can write $P(q) = N(q)P$ so that we have

$$\sigma_1(q) = \frac{\Delta P(q)}{N(q)P},$$

with $N(q)$ indicating the number of modes in the group. The total loss is

$$\bar{\sigma}_1 = \frac{\sum_q \Delta P(q)}{\sum_q P(q)} = \frac{\sum_q \sigma_1(q) N(q)}{\sum_q N(q)}.$$

Replacing the sum by an integral yields formula (42).

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