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The Impulse Response of an Optical Fiber With Parabolic Index Profile

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To the paraxial approximation there is no difference in the group delay of the modes of a parabolic index fiber. However, the wave optics treatment of the infinitely extended parabolic index medium predicts a slight difference in the group delay of the various modes. This result is used in this paper to predict the shape and width of the impulse response function of a parabolic index fiber with finite radius.

I. INTRODUCTION

The current interest in multimode optical waveguides is related to progress in the fabrication of luminescent diodes which have become cheap and dependable sources of incoherent light. Since incoherent light cannot be injected into a single-mode fiber with high efficiency, multimode waveguides must be used. A disadvantage of using multimode instead of single-mode waveguides is multimode pulse dispersion caused by the fact that the group velocity of the guided modes is not the same. Power injected at one end of the waveguide is shared by many or all of the possible guided modes. As each mode reaches the other end of the guide at a different time, the initial pulse is broadened.

It is the purpose of this paper to calculate the impulse response of a graded-index, multimode fiber.[1–3] The index distribution is assumed to be given by the expression

$$n = n_o\left(1 - \Delta\frac{r^2}{a^2}\right) \qquad 0 \leqq r \leqq a. \tag{1}$$

The parameter a represents the finite radius of the fiber, Δ determines the strength of the index gradient. The analysis is simplified by using the modes of the infinitely extended square-law medium (1) instead of the modes of the actual waveguide of radius a.

The impulse response of a graded-index fiber with parabolic index profile is much more favorable than that of the usual clad fiber with a rectangular index profile. In the paraxial approximation it is a delta function. The finite width of the impulse response function is attributable to rays that move on trajectories making relatively large angles with the waveguide axis.

II. THE MODES OF THE SQUARE-LAW MEDIUM

The electric or magnetic field components of the modes of the infinitely extended square-law medium are obtained from the reduced wave equation

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + (n^2 k^2 - \beta^2)\Psi = 0, \tag{2}$$

with

$$n^2 = n_o^2\left(1 - 2\Delta\frac{x^2 + y^2}{a^2}\right). \tag{3}$$

The coordinates x and y are oriented in transverse direction to the waveguide axis which points in z direction; $k = 2\pi/\lambda$ is the propagation constant of light in vacuum and β is the propagation constant of the guided modes. The distribution of the square of the refractive index is not simply the square of the index distribution (1). Equation (3) follows from (1) only if we neglect the square of the Δ term. However, we can turn the argument around and consider eq. (3) as correct and (1) as the approximation. The exact solution of (2) with the function (3) is given by[4]

$$\Psi = AH_p\left(\sqrt{2}\,\frac{x}{w}\right)H_q\left(\sqrt{2}\,\frac{y}{w}\right)\exp\left(-\frac{x^2 + y^2}{w^2}\right)e^{-i\beta z}. \tag{4}$$

H_p is the Hermite polynomial of order p. The beam half-width is defined as

$$w = \left(\frac{2}{\Delta}\right)^{\frac{1}{4}}\left(\frac{a}{n_o k}\right)^{\frac{1}{2}} \tag{5}$$

and the propagation constant is given by

$$\beta = n_o k\left[1 - 2\frac{\sqrt{2\Delta}}{n_o k a}(p + q + 1)\right]^{\frac{1}{2}}. \tag{6}$$

However, eqs. (4) through (6) are not an exact solution of Maxwell's equations since an additional term containing the gradient of n^2 has been neglected in (2). Since this additional term does not make a significant contribution in (6)—particularly at large mode numbers p and q—we use the solution in its present form for our discussion of the impulse response of the fiber with parabolic index profile.[5]

III. GROUP DELAY

We are interested in fibers satisfying the inequality

$$\frac{\sqrt{\Delta}}{n_o \mathrm{k} a} \ll 1. \tag{7}$$

We thus approximate (6) in the form

$$\beta = n_o \mathrm{k} - \frac{\sqrt{2\Delta}}{a}(p + q + 1) - \frac{\Delta}{n_o \mathrm{k} a^2}(p + q + 1)^2. \tag{8}$$

The group delay can now be expressed as

$$\tau = \frac{L}{v} = \frac{L}{c}\frac{d\beta}{d\mathrm{k}} = \frac{n_o L}{c}\left[1 + \frac{\Delta}{(n_o \mathrm{k} a)^2}(p + q + 1)^2\right]. \tag{9}$$

v is the group velocity and L is the length of the waveguide. The second term of β in (8) does not contribute to the group delay. In first approximation, if the third term in (8) is neglected, the group delay would be independent of the mode number. The difference in the group delay of the different modes is thus only slight in the square-law medium.

IV. CUTOFF CONDITION

In the infinite square-law medium there is an infinite number of modes; p and q can both assume values from 1 through infinity. The number of guided modes of a fiber with radius a must be finite. It seems reasonable to assume that those modes that interact strongly with the waveguide boundary at $r = a$ lose power at a high rate and become unimportant for the power transport. Low-order modes are concentrated near the waveguide axis while the modes spread out further away from the axis with increasing mode number. At a certain mode number the modes reach into the region of the fiber boundary and thus become very lossy.

It is known from the theory of the WKB approximation[6] that the mode field has an oscillatory behavior in the range

$$n(r)\mathrm{k} > \beta \tag{10}$$

and an exponentially decaying behavior in the range

$$n(r)\mathrm{k} < \beta. \tag{11}$$

It appears logical to let the cutoff point of the guided modes in the fiber with parabolic index profile and finite radius $r = a$ coincide with the condition

$$n(a)\mathrm{k} = \beta. \tag{12}$$

Using the square of (12) and eqs. (3) and (6) results in the cutoff condition

$$S = (p + q)_c = \sqrt{\frac{\Delta}{2}}\, n_o ka. \tag{13}$$

Using (5) we can also write the cutoff condition in the form

$$S = \frac{a^2}{w^2}. \tag{14}$$

The actual performance of the fiber with parabolic index distribution can now be approximated by assuming that all modes carry equal amounts of power up to the maximum mode number that is determined by (13) or (14).

V. THE IMPULSE RESPONSES

The impulse response of the fiber with parabolic index profile is obtained by counting the number of modes that arrive at the fiber output simultaneously. The power carried by these modes is proportional to their number. The waveguide losses do not influence the shape of the impulse response function if we assume that all modes suffer equal amounts of loss.

Equation (9) shows that modes with constant values of

$$u = p + q \tag{15}$$

arrive simultaneously at the end of the waveguide. The number of modes with equal group delay is obtained by inspection of Fig. 1. The modes of the waveguide occupy the area of the triangle indicated in the figure. Modes with equal transit time lie on the straight line labeled u = const. The area in p, q space is equal to the number of modes contained in it. The number of modes in the interval dh is thus given by the length of the line u = const times dh. The length of the lines u = const is $2h$. We thus have for the number of modes in the interval dh

$$M(h)dh = 2hdh. \tag{16}$$

The total number of modes is

$$N = \int_0^{(1/\sqrt{2})S} M(h)dh = \tfrac{1}{2}S^2. \tag{17}$$

The ratio of $M(h)dh/N$ is equal to the ratio of the power ΔP, that corresponds to the interval dh, divided by the total power P arriving at the end of the waveguide. We thus have

$$\frac{1}{P}\frac{dP}{dh} = \frac{4h}{S^2}. \tag{18}$$

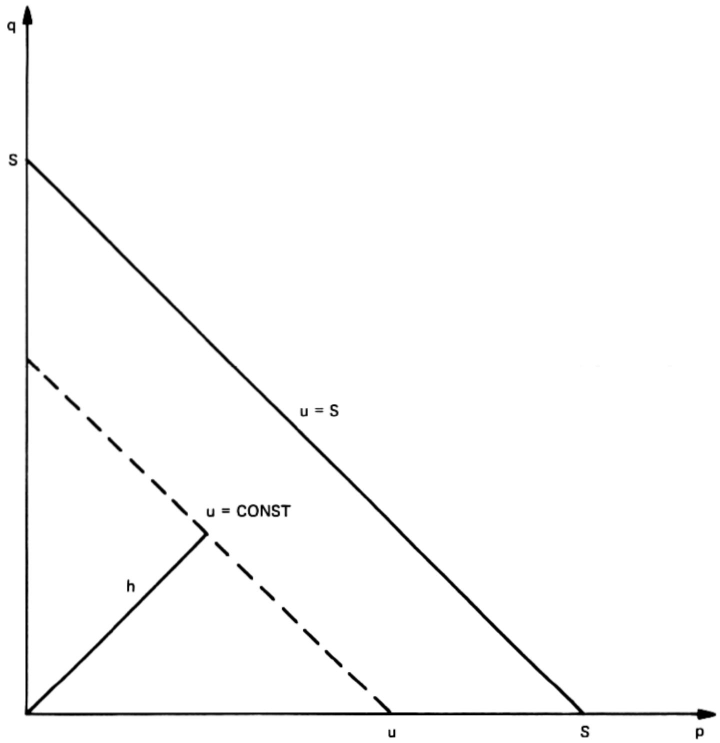

Fig. 1—Mode distribution in p-q space. The dotted line labeled u = const corresponds to modes with equal group delay.

We define the impulse response function $F(\tau)$ as the relative amount of power arriving per unit delay time τ,

$$F(\tau) = \frac{1}{P}\frac{dP}{d\tau} = \frac{1}{P}\frac{dP}{dh}\frac{dh}{d\tau}. \tag{19}$$

All that is left to do is the determination of the function $h(\tau)$. We see from Fig. 1 that

$$h = \frac{1}{\sqrt{2}}u = \frac{1}{\sqrt{2}}(p + q). \tag{20}$$

Neglecting the 1 compared to $p + q$ in (9) we have

$$h = \frac{n_o ka}{\sqrt{2\Delta}}\left(\frac{c\tau}{n_o L} - 1\right)^{\frac{1}{2}}. \tag{21}$$

The impulse response function is now obtained by combining eqs. (13)

and (18) through (21).

$$F(\tau) = \begin{cases} 0 & \dfrac{c\tau}{n_o L} - 1 < 0 \\ \dfrac{2c}{n_o L \Delta^2} & 0 < \dfrac{c\tau}{n_o L} - 1 < \tfrac{1}{2}\Delta^2. \\ 0 & \dfrac{c\tau}{n_o^2 L} - 1 > \tfrac{1}{2}\Delta^2 \end{cases} \tag{22}$$

VI. DISCUSSION

Equation (22) shows that an impulse, shared equally by all the modes at the beginning of the parabolic index fiber, reaches the end as a rectangularly shaped pulse whose width is

$$d\tau = \frac{n_o L}{2c} \Delta^2. \tag{23}$$

The pulse width is thus $d\tau = 0.25$ ns/km for $n_o = 1.5$ and $\Delta = 0.01$. The pulse width increases rapidly with increasing values of Δ. For $\Delta = 0.015$ we have a pulse width of $d\tau = 0.55$ ns/km. However, the impulse response of the parabolic index fiber is much more favorable than the corresponding impulse response of the conventional fiber with discontinuous index distribution whose impulse response width is directly proportional to[7] $n_1/n_2 - 1$ (n_1 = core index, n_2 = cladding index).

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