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A Theory of Traffic-Measurement Errors for Loss Systems With Renewal Input

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A theory of traffic-measurement errors for loss systems with renewal input is developed. The results provide an accurate approximation for the variance of any differentiable function of one or more of the following basic traffic measurements taken during a given time interval:

(*i*) *The total number of attempts (peg count)*
(*ii*) *The number of unsuccessful attempts (overflow count)*
(*iii*) *The usage based on discrete samples (TUR measurement) or on continuous scan.*

The approximation is given in terms of the individual variances and covariance functions of the three measurements. Asymptotic approximations for these moments are obtained using the concept of a generalized renewal process, and are shown to be sufficiently accurate for telephone traffic-engineering purposes.

As an application of the theory, we examine the variances of the standard estimates of the load and peakedness (variance-to-mean ratio) of an input traffic stream for a time interval of one hour. Other possible applications to Bell System trunking problems are discussed.

I. INTRODUCTION

In the Bell System, there are a number of traffic measurements which can be made on any given trunk group. For a standard time interval $(0, t]$ of one hour, the three most important measurements are:

(*i*) $A(t)$, the number of attempts (peg count);
(*ii*) $O(t)$, the number of unsuccessful attempts (overflow count); and
(*iii*) $L_d(t)$, an estimate of usage based on 36 discrete samples (TUR measurement).

When all three measurements are available, several statistics can be formed to estimate traffic parameters of interest. For instance, the ratio $O(t)/A(t)$ is an estimate of call congestion. Two other important parameters are the peakedness (variance-to-mean ratio) and the load of the input traffic. An estimate of the load is given by the function

$$\hat{\alpha} = \frac{L_d(t)/36}{1 - \dfrac{O(t)}{A(t)}},$$

while an estimate of peakedness is a complicated function of $A(t)$, $O(t)$, and $L_d(t)$ which is usually obtained by iteration using the Equivalent Random method.[1]

Since the trunk-engineering procedures are based on such estimates, it is important to know their statistical accuracy. For instance, it would be useful to know the error inherent in a prediction of the required size of a trunk group (to obtain a specified grade of service) based on the estimates of offered load and peakedness of the input traffic. Such a result could be used to determine the number of single-hour measurements necessary to ensure a desired accuracy in the prediction, to determine the optimum number of measurements from a cost-effectiveness point of view, or to evaluate the consequences for trunk provisioning of using a given number of measurements.

Many results concerning the accuracy of the individual traffic measurements (*i*) through (*iii*) have been obtained previously, but most of these have assumed the arrival process to be Poisson. For example, assuming Poisson arrivals, the variance of the usage measurement $L_d(t)$ was obtained by Beneš,[2] and the variance of the measured call-congestion $O(t)/A(t)$ was given by Descloux.[3] More recently, the variance of $O(t)/A(t)$ was obtained for arbitrary renewal input by Kuczura and Neal.[4] The variances of some nonstandard traffic counts were considered by Descloux,[5] and numerical results were obtained for the case of Poisson input.

Using the concept of a multidimensional renewal process, we develop a general theory of errors which provides an estimate of the variance of any differentiable function of the measurements (*i*) through (*iii*). Consequently, our results can be used to answer many questions similar to those mentioned above. The variances of the estimates of offered load and peakedness of the input traffic will be derived in Section IV as examples which illustrate our general theory.

Section II contains the derivation of an approximation for the variance of a function of the three traffic measurements. The approximation is given in terms of the individual variances of (*i*), (*ii*), and (*iii*) and the covariance functions between them.

Section III contains the mathematical model used to derive the variances and covariances. Section V contains a summary and an outline of other possible applications.

II. STANDARD ERRORS OF FUNCTIONS OF RANDOM VARIABLES

For completeness, we present those results from the theory of standard errors[6] which are required below. Let ξ_1, ξ_2, and ξ_3 be random variables and g a real-valued function. Assume that ξ_i has a mean θ_i, $g(\xi_1, \xi_2, \xi_3)$ has finite mean and variance, and g is differentiable at the point $(\theta_1, \theta_2, \theta_3)$. Using a Taylor series expansion we have, to first order,

$$g(\xi_1, \xi_2, \xi_3) - g(\theta_1, \theta_2, \theta_3) \approx \sum_{i=1}^{3} (\xi_i - \theta_i) \frac{\partial g}{\partial \theta_i} \cdot \tag{1}$$

We are assuming that the observation period will be sufficiently large so that the contribution of the higher-order terms can be neglected. This assumption will be justified in our model.

Taking the expectation of (1), we see that the mean value of g is approximately $g(\theta_1, \theta_2, \theta_3)$ since $E[\xi_i - \theta_i] = 0$. We also have

$$\begin{aligned} \operatorname{Var}[g(\xi_1, \xi_2, \xi_3)] & \\ \approx E\left[\sum_{i=1}^{3} (\xi_i - \theta_i) \frac{\partial g}{\partial \theta_i}\right]^2 & \\ = \sum_{i=1}^{3} \left(\frac{\partial g}{\partial \theta_i}\right)^2 \operatorname{Var}[\xi_i] + 2 \frac{\partial g}{\partial \theta_1} \frac{\partial g}{\partial \theta_2} \operatorname{Cov}[\xi_1, \xi_2] & \\ + 2 \frac{\partial g}{\partial \theta_1} \frac{\partial g}{\partial \theta_3} \operatorname{Cov}[\xi_1, \xi_3] + 2 \frac{\partial g}{\partial \theta_2} \frac{\partial g}{\partial \theta_3} \operatorname{Cov}[\xi_2, \xi_3] \cdot & \end{aligned} \tag{2}$$

After setting $\xi_1 = A(t)/t$, $\xi_2 = O(t)/t$, and $\xi_3 = L_d(t)/36$ the above relation becomes the starting point for our theory of traffic-measurement errors. It approximates the variance of any differentiable function of the measurements in terms of their variances and covariances.

In the next section we derive expressions for the required moments. These, together with the first partial derivatives of g, approximately determine the variance of g. If the function g is too complicated to be differentiated analytically, differencing may be used to approximate the partial derivatives. An example of this procedure is given in Section IV where we discuss the variance of an estimate of the peakedness of a stream of offered traffic.

III. MATHEMATICAL MODEL

Consider a system of c servers serving customers whose arrival epochs constitute a renewal process. We assume that the interarrival times are independent and identically distributed according to the distribution F having mean $1/\lambda$, and that the service times are independent and identically distributed according to a negative-exponential distribution with unit mean. If all servers are occupied when a customer arrives, he leaves and has no further effect on the system. If an idle server is available when a customer arrives, service begins immediately.

Let $(0, t]$ denote a time interval of length t which commences at a stationary point for the arrival process.* (Such a point is often said to be chosen at random on the time axis.) Let $A(t)$ be the number of arrivals and $O(t)$ the number of blocked attempts in $(0, t]$. Finally, let

$$L(t) = \int_0^t C(u)du \,, \tag{3}$$

where $C(u)$ is the number of busy servers at time u, be the total usage in $(0, t]$. Note that $L(t)/t$ is a continuous-scan estimate of the carried load.

As was pointed out in Section II, the individual first two moments of $A(t)$, $O(t)$, and $L_d(t)$ and the corresponding three covariance functions are sufficient to obtain an estimate of the variance of any function of these measurements. By numerical experimentation, we found that the covariance functions $\mathrm{Cov}[A(t)/t,\ L_d(t)/36]$ and $\mathrm{Cov}[O(t)/t,\ L_d(t)/36]$ are, for our purposes, well approximated by $\mathrm{Cov}\,[A(t)/t,\ L(t)/t]$ and $\mathrm{Cov}\,[O(t)/t,\ L(t)/t]$, respectively. However, the variance of $L(t)$ can be significantly smaller than the variance of $L_d(t)$ so that, in general, we must use $\mathrm{Var}\,[L_d(t)]$ in our applications.

In the next two sections, we derive the individual and joint moments of $A(t)$, $O(t)$, and $L(t)$. In Section 3.3, we obtain the variance of $L_d(t)$.

3.1 *A Multidimensional Renewal Process*

Assume that the system described above is in statistical equilibrium†, let t_n, $n = 0, 1, 2, \cdots$, be the instant of time at which the nth overflow

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* That is, the time until the first arrival after $t = 0$ has the remaining life-time distribution $H(t) = \lambda \int_0^t [1 - F(x)]dx$.

† That is, the system has been in operation sufficiently long prior to $t = 0$ so that system-state probability distribution at $t = 0$ is the limiting (or stationary) distribution $P\{C(0) = k\} = p_k = \lim_{u\to\infty} P\{C(u) = k\}$. It follows that for any $t \geqq 0$, $P\{C(t) = k\} = p_k$, i.e., the process $\{C(t),\ t \geqq 0\}$ is stationary.

occurs, $t_0 < 0 < t_1 < t_2 < \cdots$, and define $X_n = t_n - t_{n-1}$. Now let K_n, $n = 1, 2, \cdots$, be the number of arrivals in $(t_{n-1}, t_n]$ and

$$I_n = \int_{t_{n-1}}^{t_n} C(u)du\,, \quad n = 1, 2, \cdots,$$

be the total usage in $(t_{n-1}, t_n]$.

Since holding times are exponential and arrival epochs constitute a renewal process, the sequence of times t_n, $n = 0, 1, 2, \cdots$, are regeneration or renewal points in our model. Hence X_n, K_n, and I_n, $n = 1, 2, \cdots$, are sequences of independent and identically distributed random variables.

If we now define the row vector

$$\chi_n = (1, K_n, I_n), \quad n = 1, 2, \cdots,$$

then it follows that $\{\chi_n, X_n\}$, $n = 1, 2, \cdots$, is a multidimensional renewal process.[7] Moreover, setting

$$\eta(t) = \Sigma \chi_n\,,$$

where the sum is taken over all n such that $0 < t_n \leqq t$, we see that for large t,

$$\eta(t) \approx (O(t), A(t), L(t)).$$

Since this formulation corresponds to the concept of a generalized renewal process as communicated by J. M. Hammersley in the discussion of W. L. Smith's paper,[7] his results apply directly to our model. In particular, we shall use his equations (25) and (26) to compute the moments of $\eta(t)$.

Let $\mu_n(c) = E(X_1^n)$ be the nth moment of the interoverflow times from a group of c servers and

$$\nu_n = \int_0^\infty \xi^n dF(\xi)\,\cdot$$

For brevity, we denote the arrival intensity ν_1^{-1} by λ. From Ref. 4, we have the first moments of $A(t)$ and $O(t)$ already computed:

$$E[A(t)] = \lambda t\,,$$
$$E[O(t)] = \frac{t}{\mu_1(c)}\,\cdot \qquad (4)$$

From eq. (25) of Ref. 7, we have

$$E[L(t)] = \frac{t\omega_1(c)}{\mu_1(c)}\,, \qquad (5)$$

where

$$\omega_n(c) = E[I_1^n].$$

Again, from Ref. 4 and Ref. 7, the variances and covariances of the three measurements for large t, omitting terms which behave as $o(t)$, are given by

$$\operatorname{Var}[A(t)] \sim \frac{t}{\nu_1^3}[\nu_2 - \nu_1^2],$$

$$\operatorname{Var}[O(t)] \sim \frac{t}{\mu_1^3(c)}[\mu_2(c) - \mu_1^2(c)],$$

$$\operatorname{Var}[L(t)] \sim \frac{t}{\mu_1^3(c)}\{\mu_1^2(c)\omega_2(c) + \mu_2(c)\omega_1^2(c) - 2\mu_1(c)\omega_1(c)E[X_1I_1]\}, \quad (6)$$

$$\operatorname{Cov}[A(t), O(t)] \sim \frac{t}{\mu_1^2(c)}\{\lambda\mu_2(c) - E[K_1X_1]\},$$

$$\operatorname{Cov}[O(t), L(t)] \sim \frac{t}{\mu_1^3(c)}\{\mu_2(c)\omega_1(c) - \mu_1(c)E[X_1I_1]\},$$

$$\operatorname{Cov}[A(t), L(t)] \sim \frac{t}{\mu_1^2(c)}\{\mu_1(c)E[K_1I_1] + \lambda\mu_2(c)\omega_1(c) - \lambda\mu_1(c)E[X_1I_1] - \omega_1(c)E[X_1K_1]\}.$$

We now need to compute the various moments and joint moments of X_1, K_1, and I_1 appearing on the right-hand side of (4), (5), and (6) in order to evaluate the approximate expressions for the moments and joint moments of $A(t)$, $O(t)$, and $L(t)$. Note that the mean and variance of $A(t)$ are known since $\lambda = \nu_1^{-1}$ and ν_2 are computed directly from F.

3.2 *The Joint Distribution of K_1, X_1, and I_1*

The development here parallels that of Section 2.2 in Ref. 4. Let $h_c(w, r, n)$ be the joint density function defined by

$$h_c(w, r, n) = \frac{\partial^2}{\partial w \partial r} P\{X_1 \leqq w, I_1 \leqq r, K_1 = n\}.$$

By considering the two mutually exclusive events {the cth trunk remains busy throughout $(0, w)$} and {its complement}, and using a renewal-type argument, we arrive at the following integral equation:

$$h_c(w, r, n) = e^{-w}h_{c-1}(w, r - w, n) + \sum_{k=1}^{n-1}\int_0^w\int_0^u\int_0^{r-v} e^{-v}h_{c-1}(u, s, k) \times h_c(w - u, r - s - v, n - k)\,ds\,dv\,du, \quad (7)$$

in which the time variables u and v run concurrently from an overflow epoch.

The following boundary conditions hold:

$$h_c(w, r, n) = 0, \quad \text{for} \quad r > cw \quad \text{or} \quad r < 0,$$
$$h_c(w, cw, n) = e^{-cw} f(w) \delta_{1,n},$$

where $f = F'$ and $\delta_{1,n} = 1$ for $n = 1$ and is zero otherwise.*

If we define

$$\gamma_c(x, y, z) = \sum_{n=1}^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-xw-yr} h_c(w, r, n) z^n dr dw, \tag{8}$$

then it follows from (7) that

$$\gamma_c(x, y, z) = \frac{(y+1)\gamma_{c-1}(x+y+1, y, z)}{y+1-\gamma_{c-1}(x, y, z)+\gamma_{c-1}(x+y+1, y, z)}. \tag{9}$$

Motivated by the work of Riordan[8] and the success of the approach taken in Ref. 4, we set

$$\gamma_c(x, y, z) = \frac{(y+1)D_c(x, y, z)}{D_{c+1}(x, y, z)}, \tag{10}$$

where $D_0(x, y, z) = 1$, and, as can be seen by setting $c = 0$ in (8),

$$D_1(x, y, z) = \frac{y+1}{z\phi(x)}, \tag{11}$$

where

$$\phi(x) = \int_0^{\infty} e^{-xt} dF(t).$$

Furthermore, for $m \geqq 1$,

$$D_{m+1}(x, y, z) = D_m(x, y, z) + \left[\frac{y+1}{z\phi(x)} - 1\right] D_m(x+y+1, y, z). \tag{12}$$

If we now define

$$\lambda_j = \lambda_j(x, y, z) = 1 - \frac{y+1}{z\phi(x+jy+j)}, \tag{13}$$

then using (11) and (12) and mathematical induction one can show that

$$D_m(x, y, z) = 1 + \sum_{j=1}^{m} (-1)^j \binom{m}{j} \lambda_0 \lambda_1 \cdots \lambda_{j-1}. \tag{14}$$

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* In our model we assume that the interarrival-time probability distribution function is differentiable. However, with more formalism, the same results can be obtained for the more general case; e.g., the one-point distribution function for constant interarrival times.

Now, since

$$E[X_1^i I_1^j K_1^k] = (-1)^{i+j} \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k} \gamma_c(x, y, z) \Big|_{\substack{x=y=0 \\ z=1}}$$

for $k = 0$, 1 and i, $j \geqq 0$, we can compute the required moments directly by means of differentiation. Omitting all of the details of the operations indicated, we obtain

$$\begin{aligned} \mu_1(c) &= \nu_1 D_c \\ \omega_1(c) &= D_c - 1. \end{aligned} \tag{15}$$

where

$$D_c = D_c(1, 0, 1) = 1 + \sum_{j=1}^{c} (-1)^j \binom{c}{j} \Lambda_0 \Lambda_1 \cdots \Lambda_{j-1},$$

with

$$\Lambda_k = 1 - \frac{1}{\phi(k+1)}.$$

Note that D_c is the reciprocal of the generalized Erlang-B blocking probability B_c. Moreover, with the aid of the results obtained in Ref. 4, we have

$$\omega_2(c) = 2[\omega_1(c) + 1] \sum_{j=1}^{c} [\omega_1(j) + 1] - 2[D_c^{(100)} + D_c^{(010)}],$$

$$\mu_2(c) = \frac{\nu_2}{\nu_1} \mu_1(c) + 2\mu_1(c) \sum_{k=1}^{c} \mu_1(k) - 2\nu_1 D_c^{(100)}, \tag{16}$$

where

$$D_c^{(ijk)} = \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k} D_c(x, y, z) \Big|_{\substack{y=0 \\ x=z=1}}. \tag{17}$$

The required derivatives are given by

$$D_c^{(100)} = \sum_{j=1}^{c} (-1)^j \binom{c}{j} \Lambda_0 \Lambda_1 \cdots \Lambda_{j-1} \left[\frac{\Omega_0'}{\Lambda_0} + \frac{\Omega_1'}{\Lambda_1} + \cdots + \frac{\Omega_{j-1}'}{\Lambda_{j-1}} \right],$$

$$D_c^{(010)} = \sum_{j=1}^{c} (-1)^j \binom{c}{j} \Lambda_0 \Lambda_1 \cdots \Lambda_{j-1} \left[\frac{\Omega_0''}{\Lambda_0} + \frac{\Omega_1''}{\Lambda_1} + \cdots + \frac{\Omega_{j-1}''}{\Lambda_{j-1}} \right],$$

where Ω_k' is the derivative of $\lambda_k(x, 0, 1)$ evaluated at $x = 1$, i.e.,

$$\Omega_k' = \frac{\phi'(k+1)}{\phi^2(k+1)},$$

and Ω_k'' is the derivative of $\lambda_k(1, y, 1)$ evaluated at $y = 0$, i.e.,

$$\Omega_k'' = \frac{k\phi'(k+1)}{\phi^2(k+1)} - \frac{1}{\phi(k+1)}.$$

Similarly, we have the joint moments

$$
\begin{aligned}
E[X_1I_1] &= 2\nu_1[\omega_1(c) + 1] \sum_{j=1}^{c} [\omega_1(j) + 1] \\
&\qquad\qquad - (1 + \nu_1)D_c^{(100)} - \nu_1 D_c^{(010)}, \\
E[K_1X_1] &= \mu_1(c) + \frac{2}{\nu_1}\mu_1(c) + \sum_{k=1}^{c} \mu_1(k) + \nu_1 D_c^{(001)} - D_c^{(100)}, \qquad (18) \\
E[K_1I_1] &= 2[\omega_1(c) + 1] \sum_{j=1}^{c} [\omega_1(c) + 1] \\
&\qquad\qquad - D_c^{(100)} - D_c^{(010)} + D_c^{(001)},
\end{aligned}
$$

where

$$D_c^{(010)} = \sum_{j=1}^{c} (-1)^j \binom{c}{j} \Lambda_0\Lambda_1 \cdots \Lambda_{j-1}\left[\frac{1}{\Lambda_0} + \frac{1}{\Lambda_1} + \cdots + \frac{1}{\Lambda_{j-1}} - j\right].$$

3.3 *Variance of Discrete-Scan Estimate of Usage*

Our present mathematical model assumes that the measurement of usage, $L(t)$, is made by means of continuous scanning, as can be seen from the definition of $L(t)$ in eq. (3). In practice, however, usage is estimated by discrete scanning. The number of busy trunks is sampled at constant intervals of time, say τ, and the integral in (3) is replaced by the finite sum

$$L_d(t) = \sum_{k=1}^{n(t)} C(k\tau), \qquad (19)$$

where

$$n(t) = \max(k : k\tau \leqq t).$$

This procedure introduces a sampling error in the evaluation of the integral. As we shall see later, the difference between the variances of $L(t)$ and $L_d(t)$ can sometimes be large enough that the discrete-scan variance of usage must be used to estimate accurately the variance of g in eq. (2). In this section we indicate how the variance of $L_d(t)$ is computed.

Let $n = n(t)$ be the number of discrete samples in $(0, t]$. In trunking applications, t is usually taken to be one hour (about 20 mean holding times) and, since τ is normally set at 100 seconds, $n = 36$. Since the process $\{C(t), t \geqq 0\}$ is stationary, from (19) we have

$$\operatorname{Var}[L_d(t)] = nR(0) + 2\sum_{j=1}^{n} (n - j)R(j\tau), \qquad (20)$$

where R is the covariance function defined by

$$R(t) = E[C(0)C(t)] - E[C(0)]E[C(t)]. \qquad (21)$$

Since $P\{C(0) = k\} = P\{C(t) = k\} = p_k$,

$$R(t) = \sum_{k=0}^{c} kp_k \sum_{j=0}^{c} jP_{kj}(t) - m_1^2, \tag{22}$$

where

$$P_{kj}(t) = P\{C(t) = j | C(0) = k\}, \tag{23}$$

and

$$m_1 = \sum_{k=1}^{\infty} kp_k. \tag{24}$$

The problem of determining the transition function in (23) has been treated by Takács in Chapter 4 of Ref. 9. However, he uses a renewal point for $t = 0$, i.e., his origin is chosen at a point immediately after an arrival has occurred. His result, though not directly applicable, can be modified in a straightforward manner to take account of our different location of the origin. We state here the analogue of his Theorem 3 for the case of a stationary origin and give a proof in the appendix. We use $1/\mu$ to denote the mean service time throughout the statement and proof of the theorem.

Theorem: Let $t = 0$ be a stationary point for the arrival process described above. Then the Laplace transform of (23) is given by

$$\pi_{kj}(s) = \int_0^\infty e^{-st} P_{kj}(t) dt = \sum_{i=j}^{c} (-1)^{i-j} \binom{i}{j} \beta_{ki}(s), \tag{25}$$

where

$$\beta_{ki}(s) = \frac{1 - \phi(s + i\mu)}{\phi(s + i\mu)} \frac{\psi_{ki}(s)}{(s + i\mu)} + \binom{k}{i} \frac{1}{(s + i\mu)} \left[1 - \frac{\bar{\phi}(s + i\mu)}{\phi(s + i\mu)}\right],$$

$$\psi_{ki}(s) = \left[C_i(s) \Big/ \sum_{j=0}^{c} \binom{c}{j} \frac{1}{C_j(s)}\right]$$

$$\times \left\{\left[\sum_{j=i}^{c} \binom{c}{j} \frac{1}{C_j(s)}\right]\left[\sum_{j=0}^{i} \binom{k}{j} \frac{1}{C_{j-1}(s)} \frac{\bar{\phi}(s + j\mu)}{\phi(s + j\mu)}\right] - \left[\sum_{j=0}^{i-1} \binom{c}{j} \frac{1}{C_j(s)}\right]\left[\sum_{j=i+1}^{c} \binom{k}{j} \frac{1}{C_{j-1}(s)} \frac{\bar{\phi}(s + j\mu)}{\phi(s + j\mu)}\right]\right\},$$

$$C_j(s) = \prod_{i=0}^{j} \frac{\phi(s + i\mu)}{1 - \phi(s + i\mu)}, \qquad j = 0, 1, 2, \cdots,$$

$$C_{-1}(s) = 1,$$

$$\phi(s) = \int_0^\infty e^{-st} dF(t),$$

and $\bar{\phi}(s)$ is the Laplace-Stieltjes transform of the distribution (24), that is,

$$\bar{\phi}(s) = \frac{\lambda}{s}[1 - \phi(s)].$$

Taking the Laplace transform of (22), substituting for $\pi_{kj}(s)$, and simplifying, we obtain the following expression for $\rho(s)$, the Laplace transform of the covariance function $R(t)$:

$$\begin{aligned}\rho(s) = {} & \frac{m_2}{s+1}\left\{1 - \frac{\lambda}{s+1}\left[\frac{1-\phi(s+1)}{\phi(s+1)}\right]\right\} - \frac{m_1^2}{s} \\ & + \left[1\Big/(s+1)\sum_{j=0}^{c}\binom{c}{j}\frac{1}{C_j(s)}\right]\left\{\frac{\lambda m_1}{s}\sum_{j=1}^{c}\binom{c}{j}\frac{1}{C_j(s)}\right. \\ & + \frac{\lambda m_2}{(s+1)}\frac{C_0(s)}{C_1(s)}\sum_{j=1}^{c}\binom{c}{j}\frac{1}{C_j(s)} \\ & - \lambda^2\sum_{j=2}^{c}\frac{1}{(s+j)C_j(s)}\left[\frac{1-\phi(j)}{\phi(j)}B_j\right. \\ & \left.\left. + \frac{1-\phi(j+1)}{\phi(j+1)}B_{j+1}\right]\right\}, \qquad (26)\end{aligned}$$

where

$$B_j = C_j\sum_{i=j}^{c}\binom{c}{i}\frac{1}{C_i}\Big/\sum_{i=0}^{c}\binom{c}{i}\frac{1}{C_i},$$

$$C_j = \prod_{i=1}^{j}\frac{\phi(i\mu)}{1-\phi(i\mu)}, \qquad j = 1, 2, \cdots,$$

$$C_0 = 1,$$

and m_1 and m_2, the first and second moments of the distribution $\{p_k\}$, are given by[9]

$$m_1 = \sum_{k=1}^{c} kp_k = \frac{\lambda}{\mu}(1 - B_c),$$

$$m_2 = \sum_{k=1}^{c} k^2 p_k = m_1 + \lambda[B_1 - cB_c].$$

Note that m_1 is the carried load and B_c is the generalized Erlang-B blocking

$$B_c = 1\Big/\sum_{j=0}^{c}\binom{c}{j}\frac{1}{C_j}.$$

Equation (26) has been inverted analytically for the case of Poisson input.[2] When $\phi(s)$ does not have the simple expression of this special case, analytical inversion appears to be complicated. However, for the purpose of computing the variance of $L_d(t)$ for our trunking application, it is unnecessary to obtain an explicit inverse of $\rho(s)$. We found that the numerical inversion scheme described by Eisenberg[10] is computationally efficient and gives satisfactory results.

To illustrate the difference between continuous-scan and discrete-scan measurements for the case when the input traffic is of the overflow type, we computed the estimates of the variances of $L(t)/t$, the continuous-scan estimate of the carried load, and $L_d(t)/36$, the discrete-scan estimate, for various trunk-group sizes and $t = 20$ mean holding times–i.e., about one hour. For these results, the interarrival-time distribution of the arriving traffic was obtained by using the Interrupted Poisson process with a three-moment match.[11]

The case for ten trunks is typical and is presented in Fig. 1 where $\sigma_{L_d} = \sqrt{\text{Var}\,[L_d(t)/36]}$ and $\sigma_L = \sqrt{\text{Var}\,[L(t)/t]}$ vs α are graphed for $z = 1$, 2, and 4.

Since the variance of $L_d(t)/36$ must be at least as large as the variance of $L(t)/t$, our results show that the asymptotic estimate of Var $[L(t)/t]$ has a small positive bias, especially for low loads. Our simulation results verify this observation and also indicate that the asymptotic approximation becomes more accurate as the input load increases. Notice that the variance of $L_d(t)/36$ is about equal to the variance of $L(t)/t$ at low loads. As the load increases, the relative error introduced by discrete scanning can increase substantially. Finally, we found that for fixed load and peakedness, the relative difference between Var $[L_d(t)/36]$ and Var $[L(t)/t]$ decreases as the trunk-group size increases (an effect not shown in the figure).

IV. TWO APPLICATIONS

We give two applications of our results, in which we obtain the accuracy of the estimates of two traffic parameters. In the first example, the parameter is the offered load as given in Section I. For the second example, we discuss an estimate of the peakedness of the offered traffic.

4.1 *Accuracy of an Estimate of Offered Load*

Suppose we have observations $A(t)$, $O(t)$, and $L_d(t)$ recorded. Then for the measurement period $(0, t]$, $\hat{\alpha}$, an estimate of the offered load (in erlangs), is given by

$$\hat{\alpha} = g[A(t), O(t), L_d(t)] = \frac{L_d(t)/36}{1 - \dfrac{O(t)}{A(t)}}. \tag{27}$$

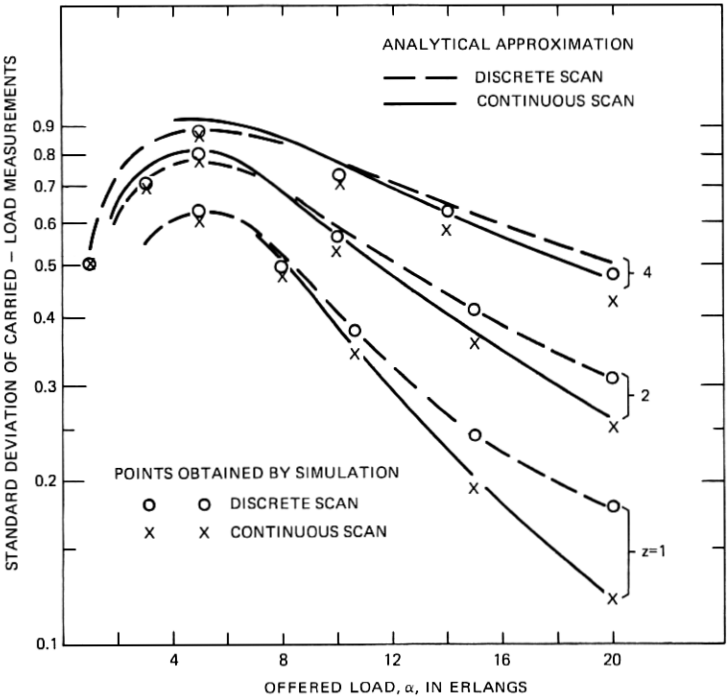

Fig. 1—Standard deviation of carried load measurements vs offered load using discrete-scan and continuous-scan measurements on a 10-trunk group for $t = 20$ mean holding times.

Obtaining the required derivatives of g as indicated in eq. (2), substituting into (2), and simplifying, we have the following expression for the variance of the offered-load estimate in (27):

$$\begin{aligned} \text{Var}\,[\hat{\alpha}] = \frac{1}{t^2(1-B_c)^2} \Big\{ & \text{Var}\,[O(t)] + \text{Var}\,[L_d(t)] \left(\frac{t}{36}\right)^2 \\ & + B_c^2 \,\text{Var}\,[A(t)] + 2\,\text{Cov}\,[L(t), O(t)] \\ & - 2B_c\,\text{Cov}\,[L(t), A(t)] - 2B_c\,\text{Cov}\,[O(t), A(t)] \Big\}. \end{aligned} \quad (28)$$

Now using eqs. (6), (15), (16), and (18) to substitute for the various quantities on the right-hand side of (28), we can compute Var $[\hat{\alpha}]$.

To test the approximation (28), we computed the variance of $\hat{\alpha}$, as outlined above, for trunk-group sizes of $c = 10$ and $c = 40$ trunks, for input traffic streams of the overflow type having different combinations of load and peakedness values. We also used a computer simulation to

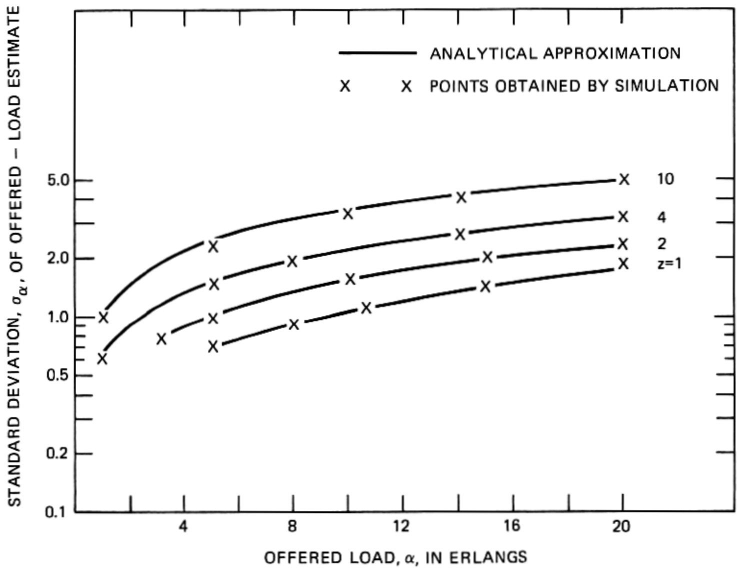

Fig. 2—Standard deviation of offered-load estimate vs offered load for $c = 10$ trunks. The measurement interval is 20 mean holding times.

estimate Var $[\hat{\alpha}]$ at several points. The numerical results are displayed in Fig. 2 for $c = 10$ trunks and Fig. 3 for $c = 40$ trunks where $\sigma_\alpha = \sqrt{\text{Var}\,[\hat{\alpha}]}$ vs α is displayed for $z = 1, 2, 4$, and 10 (again for $t = 20$ mean holding times).

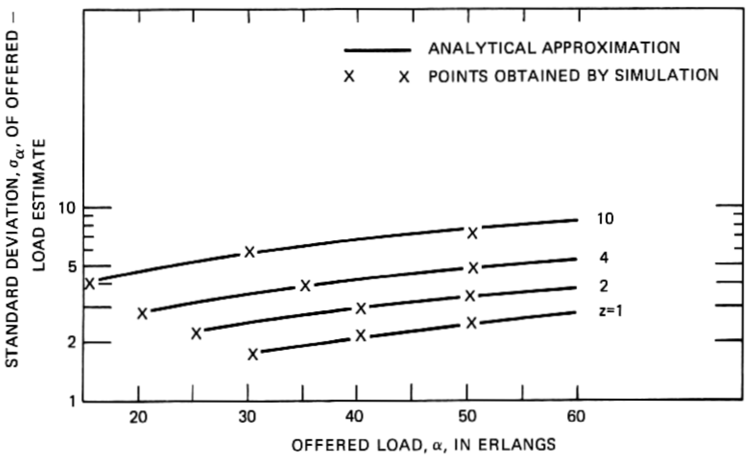

Fig. 3—Standard deviation of offered-load estimate vs offered load for $c = 40$ trunks. The measurement period is 20 mean holding times.

The simulation results indicate that for $c = 10$ and $c = 40$ the asymptotic approximation for Var $[\hat{\alpha}]$ is quite accurate for all ranges of load and peakedness of interest in trunking applications.

We obtained almost identical results for Var $[\hat{\alpha}]$ regardless of whether we used Var $[L_d(t)/36]$ or Var $[L(t)/t]$. Apparently, for low loads (low blocking probability) the accuracy of $\hat{\alpha}$ is dominated by the accuracy of the usage measurements while at high loads (blocking near 1) the accuracy of the call-congestion estimate is the dominant factor. Since the relative difference between Var $[L_d(t)/36]$ and Var $[L(t)/t]$ is small for low blocking probabilities, we see that the accuracy of $\hat{\alpha}$ is not significantly affected by the TUR sampling error.

4.2 *Accuracy of an Estimate of Traffic Peakedness*

When all three of the measurements $A(t)$, $O(t)$, and $L_d(t)$ are available for a final trunk group of c trunks in an alternate-route network, the peakedness z of the input traffic is estimated in the following manner: First an estimate of offered load $\hat{\alpha}$ is determined as described in the preceding section. Then an estimate $\hat{z}$ of z is obtained by iterative methods (using the Equivalent Random method[1]), such that a single overflow stream having load $\hat{\alpha}$ and peakedness $\hat{z}$ would experience the call congestion $O(t)/A(t)$ or, equivalently, the resulting carried load would be $L_d(t)/36$.

Thus, there is a well-defined procedure for determining a unique value for $\hat{z}$ corresponding to $A(t) \geqq O(t) > 0$ and $L_d(t) > 0$, i.e., we have the estimate

$$\hat{z} = g\left[\frac{A(t)}{t}, \frac{O(t)}{t}, \frac{L_d(t)}{36}\right]$$

in the required form. However, there is no explicit analytical expression for g which can be used to obtain the derivatives needed in (2) to obtain the variance of $\hat{z}$.

In such cases, it is natural to estimate the partial derivatives by first differences. For example,

$$\left.\frac{\partial g}{\partial x_1}\right|_{\theta_1,\theta_2,\theta_3} \approx \frac{g[\theta_1(1+\Delta), \theta_2, \theta_3] - g[\theta_1, \theta_2, \theta_3]}{\theta_1 \Delta}, \tag{29}$$

where Δ is a small positive number. Numerical experimentation indicated that $\Delta = 0.001$ gives sufficient accuracy for the present application. Using the first-difference approximations as illustrated in (29) for the derivatives in (2) we have an estimate for the variance of $\hat{z}$.

We computed the resulting approximation for $c = 10$ and $c = 40$ trunks for a range of offered loads α, several values of peakedness z,

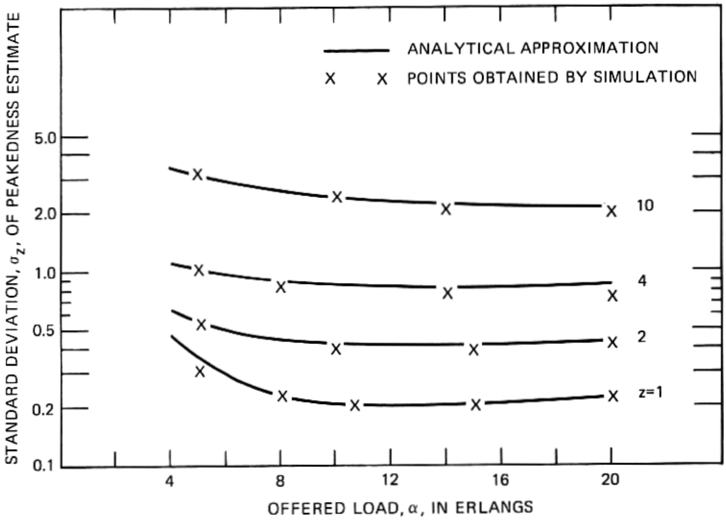

FIG. 4—Standard deviation of peakedness estimate vs offered load for $c = 10$ trunks. The measurement interval is 20 mean holding times.

and $t = 20$ mean holding times. We also compared our approximation with results obtained by simulation. The results are displayed in Figs. 4 (for $c = 10$) and 5 (for $c = 40$) where $\sigma_z = \sqrt{\text{Var}[\hat{z}]}$ vs α is given for $z = 1, 2, 4,$ and 10. Note that $\sigma_z/z \approx 0.2$ for large α, independent of c.

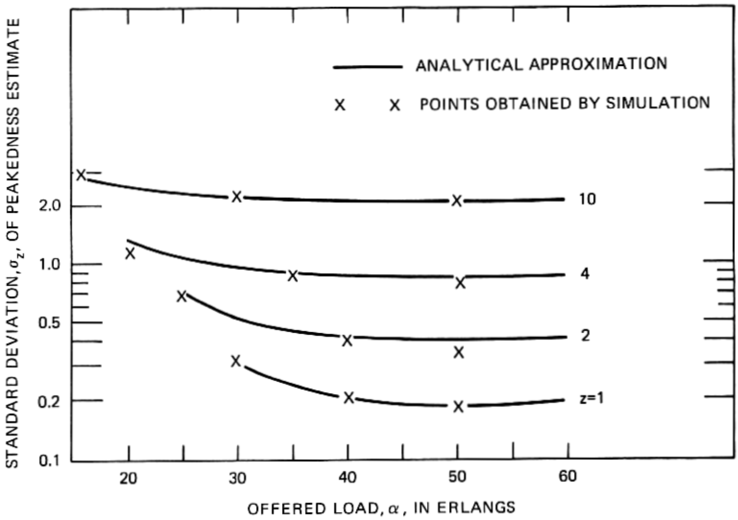

Fig. 5—Standard deviation of peakedness estimate vs offered load for $c = 40$ trunks. The measurement period is 20 mean holding times.

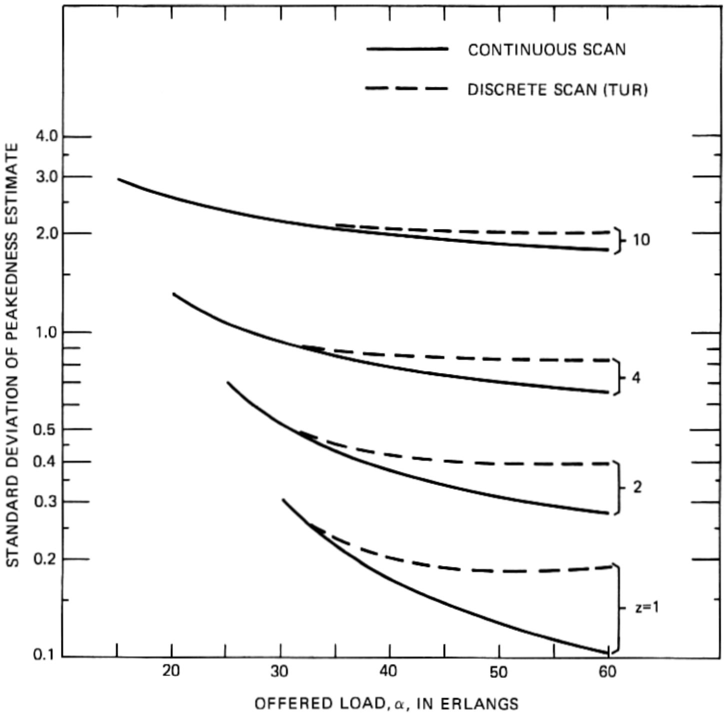

Fig. 6—Comparison of the standard deviation of peakedness estimates using discrete-scan and continuous-scan measurements of carried load on a 40-trunk group.

In general, the simulation results are in good agreement with the approximation. The curves are plotted either for $\alpha \geqq z$, or else for call-congestion exceeding 0.01, the range of interest for trunking applications. When α is smaller than z or the call-congestion is much smaller than 0.01 (not shown in the figures), the value obtained from the approximation for Var $[\hat{z}]$ occasionally tends to be larger than that obtained by simulation. Hence, the approximation may not be adequate for such applications.

In the preceding section, we noted that essentially the same results were obtained for Var $[\hat{\alpha}]$ regardless of whether we used Var $[L(t)/t]$ or Var $[L_d(t)/36]$ in the computations. In contrast, the variance of $\hat{z}$ is very sensitive to the variance of the usage measurements. That is, Var $[L_d(t)/36]$ is required to obtain an accurate approximation for Var $[\hat{z}]$. The errors which can result from using Var $[L(t)/t]$ instead of Var $[L_d(t)/36]$ are illustrated in Fig. 6 for the case of $c = 40$ trunks.

V. SUMMARY AND OTHER APPLICATIONS

5.1 *Summary*

We derived an approximation for the variance of any differentiable function of the three basic traffic measurements–namely, peg count, overflow count, and usage (TUR). The approximation is expressed in terms of the first partial derivatives or first differences of the function, and the individual variances and covariances of the measurements. Except for the variance of the TUR measurements, asymptotic approximations for the required moments were obtained by an application of Hammersley's generalized renewal theory.

The variance of the TUR was given as a sum involving the covariance function for the number of busy servers at equally spaced scan intervals. The Laplace transform of the covariance function was derived and inverted numerically using an inversion technique described by Eisenberg.

The results were then applied to obtain approximations for the variances of estimates of offered load and peakedness (variance-to-mean ratio) of a stream of traffic of the overflow type submitted to a loss system. The approximations were in good agreement with results obtained by simulation.

5.2 *Other Applications*

5.2.1 *Offered Load Estimates Based on Usage Measurements*

At present, $A(t)$ and $O(t)$ are not always measured on primary high-usage trunk groups. Estimates of the single-hour offered load and call-congestion for such groups are obtained from the TUR measurement $L_d(t)$ with an iterative procedure based on the Erlang-B theory. Using the techniques presented above, one could compare the accuracy of these estimates with that which would be obtained by using all three measurements. It should then be possible to evaluate the difference in statistical accuracy that results from using (or not using) the additional measurements.

5.2.2 *The Optimum Number of Single-Hour Measurements*

Normally, 20 single-hour measurements are used to obtain estimates $\hat{s}$ of the correct number of trunks s required to obtain a specified grade of service. For example, on final trunk groups, the 20 single-hour estimates of call-congestion $O(t)/A(t)$ and usage $L_d(t)$ are first averaged and then used to estimate an average load and peakedness of the input traffic. These average values are used to obtain $\hat{s}$.

It has been proposed that the number of measurements be reduced in order to lower the data handling costs. However, reducing the number of measurements would increase the variance of $\hat{s}$, i.e., decrease the accuracy of the trunk estimates. It appears that an optimum number of measurements could be determined by minimizing a function of the form

$$\mathcal{C}(N) = \mathcal{K} \operatorname{Var}[\hat{s}(N)] + C_N, \tag{30}$$

where C_N is the cost of taking and processing N measurements, $\hat{s}(N)$ is the estimate of s based on N measurements, and $\mathcal{K}$ is a cost associated with inaccurate trunk estimates. The precise form of the function might require modification. However, the basic idea is to trade off an increase in the accuracy of the provisioning process due to more accurate trunk estimates (as N increases) against a corresponding increase in cost.

It appears that one can obtain an approximation for $\operatorname{Var}[\hat{s}(N)]$ using an extension of the ideas presented in Sections II and III in order to account for the effects of day-to-day variation in the offered loads. However, a realistic model to justify (30) or to obtain $\mathcal{K}$ and C_N will require further study.

APPENDIX

We prove here the theorem stated in Section 3.3. We shall need the following lemma.

Lemma: For the model described in the text, let $Y(t)$ be the number of busy servers at time t, $t = 0$ be a stationary point, and $Y(0) = i$. Now let Y_n be the number of busy servers found by the nth arrival and t_n be the time of the nth arrival. For $n = 1, 2, \cdots$, and $r = 0, 1, \cdots, c$, define

$$A_{ir}^{(n)}(s) = E\left\{e^{-st_n}\binom{Y_n}{r}\middle| Y(0) = i\right\}, \tag{31}$$

and

$$\psi_{ir}(s) = \sum_{n=1}^{\infty} A_{ir}^{(n)}. \tag{32}$$

Then we have

$$\begin{aligned}\psi_{ir}(s) = \left[C_r(s) \Big/ \sum_{j=0}^{c}\binom{c}{j}\frac{1}{C_j(s)}\right] &\cdot\left\{\left[\sum_{j=r}^{c}\binom{c}{j}\frac{1}{C_j(s)}\right]\left[\sum_{j=0}^{r}\binom{i}{j}\frac{1}{C_{j-1}(s)}\frac{\bar{\phi}(s+j\mu)}{\phi(s+j\mu)}\right]\right.\\ &\left.-\left[\sum_{j=0}^{r-1}\binom{c}{j}\frac{1}{C_j(s)}\right]\left[\sum_{j=r+1}^{c}\binom{i}{j}\frac{1}{C_{j-1}(s)}\frac{\bar{\phi}(s+j\mu)}{\phi(s+j\mu)}\right]\right\},\end{aligned} \tag{33}$$

where

$$C_j(s) = \prod_{i=0}^{j} \frac{\phi(s+i\mu)}{1-\phi(s+i\mu)}, \qquad j = 0, 1, 2, \cdots,$$

$$C_{-1}(s) \equiv 1,$$

$$\phi(s) = \int_0^\infty e^{-st} dF(t),$$

and

$$\bar{\phi}(s) = \int_0^\infty e^{-st} d\bar{F}(t) = \lambda \int_0^\infty e^{-st}[1 - F(t)]dt$$

$$= \frac{\lambda}{s}[1 - \phi(s)].$$

The proof below is essentially the same as the proof of Lemma 1 of Ref. 9, modified to account for the stationary origin. For $n = 1, 2, \cdots$, we have

$$E\left\{e^{-st_{n+1}} \binom{Y_{n+1}}{r} \middle| Y_n = j,\, t_{n+1} - t_n = x\right\}$$

$$= \begin{cases} e^{-(s+r\mu)x} \dbinom{j+1}{r} E\{e^{-st_n} | Y_n = j\}, & \text{for} \quad j < c, \\ e^{-(s+r\mu)x} \dbinom{c}{r} E\{e^{-st_n} | Y_n = c\} & \text{for} \quad j = c, \end{cases}$$

because under the given conditions Y_n has a binomial distribution with parameters $j + 1$ (for $j = 0, 1, \cdots, c - 1$) or c (for $j = c$) and $e^{-\mu x}$. If we remove the condition $t_{n+1} - t_n = x$, that is, multiply by $dP\{t_{n+1} - t_n \leqq x\}$ and integrate over all x, we obtain

$$E\left\{e^{-st_{n+1}} \binom{Y_{n+1}}{r} \middle| Y_n = j\right\} = \phi(s + r\mu) \binom{j+1}{r} E\{e^{-st_n} | Y_n = j\},$$

for $j = 0, 1, \cdots, c - 1$ and

$$E\left\{e^{-st_{n+1}} \binom{Y_{n+1}}{r} \middle| Y_n = c\right\} = \phi(s + r\mu) \binom{c}{r} E\{e^{-st_n} | Y_n = c\}.$$

If we multiply the corresponding equations by $P\{Y_n = j\}$ and add them for $j = 0, 1, \cdots, c$, then we get

$$A_{ir}^{(n+1)}(s) = \phi(s + r\mu) \left[A_{ir}^{(n)}(s) + A_{i,r-1}^{(n)}(s) - \binom{c}{r-1} A_{ic}^{(n)}(s) \right], \quad (34)$$

for $r = 1, 2, \cdots, c$ and

$$A_{i0}^{(n)}(s) = \bar{\phi}(s)[\phi(s)]^{n-1}.$$

Since $Y(0) = i$ and $t = 0$ is a stationary point, we have

$$A_{ir}^{(1)}(s) = \binom{i}{r} \bar{\phi}(s + r\mu).$$

Forming the sum (32) we get

$$\psi_{ir}(s) - \binom{i}{r} \bar{\phi}(s + r\mu) = \phi(s + r\mu) \left[\psi_{ir}(s) + \psi_{i,r-1}(s) - \binom{c}{r-1} \psi_{ic}(s) \right],$$

or

$$\psi_{ir}(s) = \frac{\phi(s + r\mu)}{[1 - \phi(s + r\mu)]} \left[\binom{i}{r} \frac{\bar{\phi}(s + r\mu)}{\phi(s + r\mu)} + \psi_{i,r-1}(s) - \binom{c}{r-1} \psi_{ic}(s) \right]. \quad (35)$$

Dividing both sides of this equation by $C_r(s)$ we get

$$\frac{\psi_{ir}(s)}{C_r(s)} = \frac{\psi_{i,r-1}(s)}{C_{r-1}(s)} + \frac{\binom{i}{r} \frac{\bar{\phi}(s + r\mu)}{\phi(s + r\mu)} - \binom{c}{r-1} \psi_{ic}(s)}{C_{r-1}(s)}.$$

Adding these equations over $r, r - 1, r - 2, \cdots, 1$ we obtain

$$\frac{\psi_{ir}(s)}{C_r(s)} = \sum_{j=0}^{r} \binom{i}{j} \frac{\bar{\phi}(s + j\mu)}{\phi(s + j\mu)} \frac{1}{C_{j-1}(s)} - \psi_{ic}(s) \sum_{j=1}^{r} \binom{c}{j-1} \frac{1}{C_{j-1}(s)}, \quad i = 1, 2, \cdots, c. \quad (36)$$

Setting $r = c$ in (36) we get

$$\psi_{ic}(s) = \sum_{j=0}^{c} \binom{i}{j} \frac{\bar{\phi}(s + j\mu)}{\phi(s + j\mu)} \cdot \frac{1}{C_{j-1}(s)} \Big/ \sum_{j=0}^{c} \binom{c}{j} \frac{1}{C_j(s)}. \quad (37)$$

Substituting (37) into (36) we obtain $\psi_{ir}(s)$ for $r = 1, 2, \cdots, c$. If $r = 0$, then

$$\psi_{i0}(s) = \sum_{n=1}^{\infty} \bar{\phi}(s)[\phi(s)]^{n-1} = \frac{\bar{\phi}(s)}{1 - \phi(s)}.$$

This completes the proof of the lemma.

We now prove the theorem stated in Section 3.3. Again, the proof is a slight modification of the proof of Theorem 3 in Chapter 4 of Ref. 6.

Let us define the binomial moments

$$B_{ir}(t) = \sum_{k=r}^{c} \binom{k}{r} P_{ik}(t), \qquad i, r = 0, 1, \cdots, c.$$

From the definition it follows that

$$P_{ik}(t) = \sum_{r=k}^{c} (-1)^{r-k} \binom{r}{k} B_{ir}(t), \tag{38}$$

so that setting

$$\beta_{ir}(s) = \int_0^\infty e^{-st} B_{ir}(t) dt$$

and forming the Laplace transform of (38), we get eq. (25) of the theorem. It remains to determine $\beta_{ir}(s)$.

Let $Y(t)$ be the number of busy servers at time t and let $t = 0$ be a stationary point. The times between those successive arrivals which find j servers busy are independent and identically distributed random variables. Hence, the sequence of epochs immediately preceding those arrivals which find j servers busy constitutes a renewal process. If $Y(0) = i$, we will denote the renewal function of such an imbedded renewal process by $M_{ij}(t)$.

It may be helpful to recall that $M_{ij}(t)$ is the expected number of those calls which arrive in the time interval $(0, t]$ and find exactly j servers busy, given that initially there are i servers busy. Hence, we can write

$$M_{ij}(t) = \sum_{n=1}^{\infty} P\{t_n \leqq t, Y_n = j \mid Y(0) = i\}, \tag{39}$$

where t_n and Y_n have the same meaning as in the statement of the lemma.

If no calls arrive in $(0, t]$, then $Y(t)$ has the binomial distribution with parameters i and $e^{-\mu t}$, and $B_{ir}(t)$ is given by

$$\binom{i}{r} e^{-r\mu t}[1 - \bar{F}(t)],$$

where

$$\bar{F}(t) = \lambda \int_0^t [1 - F(x)] dx.$$

If one or more calls arrive in $(0, t]$, let the last call's arrival epoch be u and at that instant let the number of busy servers be j. Now $Y(t)$ has the binomial distribution with parameters $j + 1$ (if $j = 0, 1, \cdots, c - 1$) or c (if $j = c$) and $e^{-\mu(t-u)}$. Thus, together we have

$$B_{ir}(t) = \binom{i}{r} e^{-r\mu t}[1 - \bar{F}(t)] + \sum_{j=r-1}^{c-1} \binom{j+1}{r} \int_0^t e^{-r\mu(t-u)}[1 - F(t-u)]dM_{ij}(u) + \binom{c}{r} \int_0^t e^{-r\mu(t-u)}[1 - F(t-u)]dM_{ic}(u). \quad (40)$$

If we introduce the Laplace-Stieltjes transform

$$\mu_{ij}(s) = \int_0^\infty e^{-st} dM_{ij}(t),$$

then from (40) we have

$$\beta_{ir}(s) = \frac{1 - \phi(s + r\mu)}{(s + r\mu)} \left[\binom{i}{r} \frac{1 - \bar{\phi}(s + r\mu)}{1 - \phi(s + r\mu)} + \sum_{j=r-1}^{c-1} \binom{j+1}{r} \mu_{ij}(s) + \binom{c}{r} \mu_{ic}(s) \right]. \quad (41)$$

From (39) we have

$$\mu_{ij}(s) = \sum_{n=1}^{\infty} P\{Y_n = j\} E\{e^{-st_n} \mid Y_n = j, Y(0) = i\}$$

and hence by (31) and (32) we get

$$\sum_{j=r}^{c} \binom{j}{r} \mu_{ij}(s) = \sum_{n=1}^{\infty} E\left\{ e^{-st_n} \binom{Y_n}{r} \,\middle|\, Y(0) = i \right\} = \psi_{ir}(s).$$

Thus, $\beta_{ir}(s)$ can be written in the following form

$$\beta_{ir}(s) = \frac{1 - \phi(s + r\mu)}{s + r\mu} \left[\binom{i}{r} \frac{1 - \bar{\phi}(s + r\mu)}{1 - \phi(s + r\mu)} + \psi_{ir}(s) + \psi_{i,r-1}(s) - \binom{c}{r-1} \psi_{ic}(s) \right].$$

If we take relation (35) into consideration, then this formula can be simplified to

$$\beta_{ir}(s) = \frac{1 - \phi(s + r\mu)}{\phi(s + r\mu)} \cdot \frac{\psi_{ir}(s)}{(s + r\mu)} + \binom{i}{r} \frac{1}{(s + r\mu)} \left[1 - \frac{\bar{\phi}(s + r\mu)}{\phi(s + r\mu)} \right].$$

This completes the proof of the theorem.

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