THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 52, No. 6, July–August, 1973
Printed in U.S.A.

Coupled Mode Theory of Round Optical Fibers

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(Manuscript received January 12, 1973)

This paper presents a comprehensive theory of mode coupling in optical fibers with imperfections. The paper begins with the derivation of a general coupled wave theory based on the modes of the ideal fiber. The general theory is applied to a simplified description of guided and radiation modes of the fiber that is valid for small core-cladding index differences. The simplified theory results in expressions for the coupling coefficients that are nearly as simple as those of the slab waveguide. As an example, the theory is applied to the calculation of radiation losses caused by pure core diameter changes and by elliptical deformations of the fiber core.

I. INTRODUCTION

Dielectric optical waveguides support a finite number of guided modes and an infinite number of radiation modes.[1] Even if the number of guided modes that can be supported by the waveguide is reduced to one, the presence of the infinite number of radiation modes forces us to be concerned about mode conversion phenomena. Coupling among the guided modes of a multimode optical waveguide (multimode waveguide refers to the guided modes) is caused by imperfections in the refractive index distribution or the geometry of the optical waveguide. Its effects may be beneficial for reducing the delay distortion that results from uncoupled multimode operation.[2,3] Coupling between guided modes and the continuum of radiation modes is usually not desired unless the waveguide is intended to serve as an antenna. However, a certain amount of coupling is unavoidable and results in scattering losses.[1,4,5]

E. G. Rawson has calculated light scattering from fiber waveguides with irregular core surfaces by an approximate technique.[6] However, his method is not suitable to calculate coupling between guide modes.

A theory of radiation losses in round optical fibers has been presented in Ref. 7. This theory was based on a field expansion in terms of the exact modes of the fiber. Because of the complicated mode fields, a theory based on the exact modes of the guide is very tedious and results in equations whose numerical evaluation is difficult and costly. However, Snyder[8] and Gloge[9] have shown that the description of the modes of dielectric waveguides can be greatly simplified if it is assumed that the difference of the refractive indices of core and cladding is only very slight. This assumption allows an approximate treatment of the mode problem, resulting in very much simpler field expressions. The coupled mode theory based on approximate modes yields expressions for the coupling coefficients that are almost as simple as those for the slab waveguide. The entire coupled mode theory thus becomes simpler and its numerical evaluation becomes cheaper.

This paper starts out with a derivation of the coupled wave equations in terms of modes of the ideal guide. This mode description is somewhat different from the coupled mode theory in terms of local normal modes used by Snyder.[10,11] It results in simpler expressions for the coupling coefficients. The exact coupled mode theory is then applied to the problem of coupling between the simplified waveguide modes. We limit the discussion to coupling caused by changes or imperfections in the waveguide geometry. Coupling caused by refractive index inhomogeneities, which could be handled in a similar fashion with the use of the exact expressions for the coupling coefficients and the approximate mode description, is not discussed in this paper.

Finally, we apply our results to the problem of scattering losses of guided modes caused by diameter changes and elliptical deformations of the waveguide core. We also derive simplified expressions for the coupling coefficients between guided modes far from cutoff and discuss coupling between guided modes caused by deformations of the fiber core and by curvature of the waveguide axis.

II. EXACT COUPLED MODE THEORY

The dielectric optical waveguide with imperfections is defined by a certain refractive index distribution $n = n(x, y, z)$ that enters Maxwell's equations:

$$\nabla \times \mathbf{H} = i\omega\epsilon_o n^2 \mathbf{E} \tag{1}$$

$$\nabla \times \mathbf{E} = -i\omega\mu_o \mathbf{H}. \tag{2}$$

E and **H** are the electric and magnetic field vectors of a general field distribution in the waveguide. The fields are assumed to have the time dependence $e^{i\omega t}$, with radian frequency ω; ϵ_o and μ_o are the vacuum values of the electric permittivity and magnetic permeability. In addition to the refractive index distribution n of the real waveguide, we consider the index distribution n_o that defines an ideal guide from which the real guide deviates in some way.

We decompose the fields into their transverse and longitudinal parts. The electric field is thus represented by the equation

$$\mathbf{E} = \mathbf{E}_t + \mathbf{E}_z \tag{3}$$

and the magnetic field is given by

$$\mathbf{H} = \mathbf{H}_t + \mathbf{H}_z. \tag{4}$$

By using a similar decomposition of the ∇ operator ($\mathbf{e}_z$ is a unit vector in the z-direction),

$$\nabla = \nabla_t + \mathbf{e}_z \frac{\partial}{\partial z}, \tag{5}$$

Maxwell's equations can be written in the form

$$\nabla_t \times \mathbf{H}_z + \left(\mathbf{e}_z \times \frac{\partial \mathbf{H}_t}{\partial z}\right) = i\omega\epsilon_o n^2 \mathbf{E}_t \tag{6}$$

and

$$\nabla_t \times \mathbf{E}_z + \left(\mathbf{e}_z \times \frac{\partial \mathbf{E}_t}{\partial z}\right) = -i\omega\mu_o \mathbf{H}_t. \tag{7}$$

The longitudinal field components are expressed in terms of the transverse field components

$$\mathbf{E}_z = \frac{1}{i\omega\epsilon_o n^2} \nabla_t \times \mathbf{H}_t \tag{8}$$

and

$$\mathbf{H}_z = -\frac{1}{i\omega\mu_o} \nabla_t \times \mathbf{E}_t. \tag{9}$$

The modes of the ideal waveguide with index distribution $n_o(x, y)$ are defined as solutions of the equations

$$\nabla_t \times \mathcal{H}_{\nu z} - i\beta_\nu(\mathbf{e}_z \times \mathcal{H}_{\nu t}) = i\omega\epsilon_o n_o^2 \mathcal{E}_{\nu t} \tag{10}$$

and

$$\nabla_t \times \mathcal{E}_{\nu z} - i\beta_\nu(\mathbf{e}_z \times \mathcal{E}_{\nu t}) = -i\omega\mu_o \mathcal{H}_{\nu t}. \tag{11}$$

The index ν is a mode label and β_ν is the propagation constant of the νth mode. The longitudinal components of the mode fields are similarly expressed as

$$\mathcal{E}_{\nu z} = \frac{1}{i\omega\epsilon_o n_o^2} \nabla_t \times \mathcal{H}_{\nu t} \tag{12}$$

and

$$\mathcal{H}_{\nu z} = -\frac{1}{i\omega\mu_o} \nabla_t \times \mathcal{E}_{\nu t}. \tag{13}$$

The transverse field of the waveguide with the index distribution $n(x, y, z)$ is now expressed as a superposition of modes of the ideal waveguide.

$$\mathbf{E}_t = \sum_\nu a_\nu \mathcal{E}_{\nu t} \tag{14}$$

and

$$\mathcal{H}_t = \sum_\nu b_\nu \mathcal{H}_{\nu t}. \tag{15}$$

The summation symbol in (14) and (15) indicates summation over guided modes and integration over radiation modes. Indicating the index ν by the symbol ρ in case that it belongs to the continuum of radiation modes, we have to replace

$$\sum_\nu \rightarrow \sum \int_0^\infty d\rho. \tag{16}$$

The sum in front of the integral on the right-hand side of (16) indicates a summation over the various types of radiation modes.

For the derivation of coupled differential equations for the expansion coefficients a_ν and b_ν we need the orthogonality relations of the modes of the ideal waveguide.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{e}_z \cdot (\mathcal{E}_{\nu t} \times \mathcal{H}_{\mu t}^*) dx dy = 2 \frac{\beta_\mu^*}{|\beta_\mu|} P \delta_{\nu\mu}. \tag{17}$$

The asterisks indicate complex conjugation. The symbol $\delta_{\nu\mu}$ indicates Kronecker's delta for discrete values of ν and μ; it is zero if one of the indices labels a guided mode while the other labels a radiation mode, and it becomes Dirac's delta function if both indices label radiation modes.

The series expansions (14) and (15) are now substituted into the equations (6) through (9). Making use of the fact that the mode fields satisfy the equations (10) through (13), we obtain

$$\sum_\nu \left\{ \left(\frac{db_\nu}{dz} + i\beta_\nu a_\nu \right) (\mathbf{e}_z \times \mathcal{H}_{\nu t}) - i\omega\epsilon_o (n^2 - n_o^2) a_\nu \mathcal{E}_{\nu t} \right\} = 0 \tag{18}$$

and

$$\sum_\nu \left\{ \left(\frac{da_\nu}{dz} + i\beta_\nu b_\nu \right) (\mathbf{e}_z \times \mathcal{E}_{\nu t}) + \frac{1}{i\omega\epsilon_o} b_\nu \nabla_t \times \left[\left(\frac{1}{n^2} - \frac{1}{n_o^2} \right) (\nabla_t \times \mathcal{H}_{\nu t}) \right] \right\} = 0. \tag{19}$$

We take the scalar product of (18) with $\mathcal{E}_{\mu t}^*$ and of (19) with $\mathcal{H}_{\mu t}^*$. After integration over the infinite cross section, we obtain with the help of the orthogonality relation (17)

$$\frac{db_\mu}{dz} + i\beta_\mu a_\mu = 2\sum_\nu \bar{K}_{\mu\nu}\, a_\nu \tag{20}$$

and

$$\frac{da_\mu}{dz} + i\beta_\mu b_\mu = 2\sum_\nu \bar{k}_{\mu\nu} b_\nu \tag{21}$$

with the coupling coefficients

$$\bar{K}_{\mu\nu} = \frac{\omega\epsilon_o}{4iP}\frac{|\beta_\mu|}{\beta_\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(n^2 - n_o^2)\,\mathcal{E}_{\mu t}^*\cdot\mathcal{E}_{\nu t}dxdy \tag{22}$$

and

$$\bar{k}_{\mu\nu} = \frac{-1}{4i\omega\epsilon_o P}\frac{|\beta_\mu|}{\beta_\mu^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{H}_{\mu t}^*\cdot\nabla_t \times\left[\left(\frac{1}{n^2} - \frac{1}{n_o^2}\right)\nabla_t\times\mathcal{H}_{\nu t}\right]dxdy. \tag{23}$$

Equation (23) can be brought into a simpler form with the help of (12) and by performing a partial integration

$$\bar{k}_{\mu\nu} = \frac{\omega\epsilon_o}{4iP}\frac{|\beta_\mu|}{\beta_\mu^*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{n_o^2}{n^2}(n^2 - n_o^2)\,\mathcal{E}_{\mu z}^*\cdot\mathcal{E}_{\nu z}dxdy. \tag{24}$$

Finally, we introduce the amplitudes $c_\mu^{(+)}$ and $c_\mu^{(-)}$ of forward and backward traveling waves by means of the transformation

$$a_\mu = c_\mu^{(+)}e^{-i\beta_\mu z} + c_\mu^{(-)}e^{i\beta_\mu z} \tag{25}$$

and

$$b_\mu = c_\mu^{(+)}e^{-i\beta_\mu z} - c_\mu^{(-)}e^{i\beta_\mu z}. \tag{26}$$

Substitution into (20) and (21), addition and subtraction of the resulting equations, and regrouping of terms results in the desired coupled wave equations

$$\frac{dc_\mu^{(+)}}{dz} = \sum_\nu \{K_{\mu\nu}^{(+,+)}c_\nu^{(+)}e^{i(\beta_\mu-\beta_\nu)z} + K_{\mu\nu}^{(+,-)}c_\nu^{(-)}e^{i(\beta_\mu+\beta_\nu)z}\} \tag{27}$$

$$\frac{dc_\mu^{(-)}}{dz} = \sum_\nu \{K_{\mu\nu}^{(-,+)}c_\nu^{(+)}e^{-i(\beta_\mu+\beta_\nu)z} + K_{\mu\nu}^{(-,-)}c_\nu^{(-)}e^{-i(\beta_\mu-\beta_\nu)z}\}. \tag{28}$$

The coupling coefficients are defined as

$$K_{\mu\nu}^{(p,q)} = \frac{\omega\epsilon_o}{4iP}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(n^2 - n_o^2)\left\{p\frac{|\beta_\mu|}{\beta_\mu}\,\mathcal{E}_{\mu t}^*\cdot\mathcal{E}_{\nu t} + q\frac{|\beta_\mu|}{\beta_\mu^*}\frac{n_o^2}{n^2}\,\mathcal{E}_{\mu z}^*\cdot\mathcal{E}_{\nu z}\right\}dxdy. \tag{29}$$

The factors and superscripts p and q indicate the symbols $(+)$ or $(-)$ or the corresponding factors $+1$ and -1. The propagation constants β_μ are positive quantities. The coupled wave equations (27) and (28) provide an exact description of the imperfect waveguide in terms of normal modes of the perfect guide. The use of normal modes of the perfect guide results in the simple general form (29) for the coupling coefficients.

Our coupled mode theory can be applied to any type of waveguide problem such as waveguides with refractive index inhomogeneities, tapers, or bends. It may be that the expansion in terms of ideal modes of the waveguide does not provide the most convenient basis for some problems. For the description of tapers, for example, we face the following situation. Consider a waveguide which is perfectly straight and uniform up to a point where its cross section begins to increase. After some distance, the taper connects to a uniform waveguide of constant cross section. If we use the modes of the smaller guide for our mode expansion, we see immediately that the coupling coefficients have nonzero values not only on the taper itself but also throughout the entire waveguide of larger cross section. Even though our description is precise and yields the right answers, it is inconvenient for the problem at hand. It would be far better to use so-called local normal modes that do not themselves describe wave forms in any real waveguide but correspond at each point z along the nonuniform guide to the modes of a hypothetical uniform guide whose cross section coincides locally with that of the waveguide under study. Using local normal modes results in coupled wave equations of the form (27) and (28) but with different coupling coefficients. In case of the taper, these coupling coefficients would be nonzero only on the taper itself but would vanish on the uniform waveguide sections. Local normal modes are obviously better suited for the description of tapers. For the description of waves in bent waveguides it would be most convenient to use modes that locally correspond to a straight waveguide whose axis is tangential to that of the actual guide. In addition to these problems of convenience, there exist problems of convergence of the series expansions (14) and (15). A series expansion in terms of ideal modes may converge more slowly than an expansion in terms of local normal modes. However, we shall see that we can use the series expansion in terms of ideal modes to treat most problems of waveguides with only slight refractive index differences between the core and cladding materials. In addition, it is usually possible to guess the form of the coupling coefficients of a particular expansion from the coupling coefficients for the ideal mode expansion.

For the purposes of this paper, we restrict ourselves to the description of waveguides with piecewise constant refractive index distributions and allow only deformations of the cross section of the waveguide core. Figure 1 shows a typical waveguide imperfection. The refractive index distributions n and n_o coincide inside of the core and in the cladding region. They differ only near the core-cladding boundary. If the boundary has moved outward from its ideal position, the index difference $n^2 - n_o^2$ equals $n_1^2 - n_2^2$ in the region where the actual core overlaps the ideal cladding; it vanishes everywhere else. If the core boundary has moved inwards, we have $n^2 - n_o^2 = -(n_1^2 - n_2^2)$ in the region where the actual cladding overlaps the ideal core and zero values everywhere else. The field components that multiply the refractive index difference term in (29) are either continuous, if they are tangential to the boundary of the ideal waveguide, or they jump by a factor $(n_1/n_2)^2$ if they are normal to the ideal core boundary. If we restrict the discussion to core boundary displacements that are so slight that the fields can be considered constant over the region of the displacement and to weakly guiding fibers with $(n_1/n_2 - 1) \ll 1$, we obtain from (29)

$$K_{\mu\nu}^{(p,q)} = -\frac{i\omega\epsilon_o(n_1^2 - n_2^2)a}{4P}\int_0^{2\pi}[r(x, y, z) - a] \times [p\,\mathcal{E}_{\mu t}^*\cdot \mathcal{E}_{\nu t} + q\,\mathcal{E}_{\mu z}^*\cdot \mathcal{E}_{\nu z}]\}d\phi. \quad (30)$$

It is noteworthy that the approximate coupling coefficient (30) is identical to the coupling coefficient (13) of Ref. 12 for the local normal mode expansion. The only difference in the appearance of these two coupling coefficients consists in the fact that the derivative of the boundary function instead of the function itself appears in Ref. 12 and that the entire expression is divided by $\beta_\nu - \beta_\mu$. It has been explained

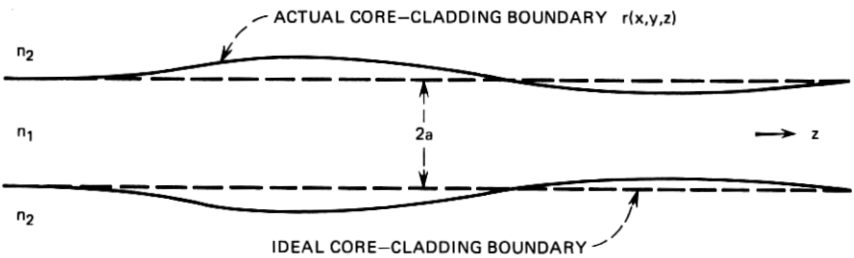

Fig. 1—Sketch of a fiber with distorted core-cladding interface.

in a different place[13] that it is the Fourier component of the boundary function at the spatial frequency $\beta_\nu - \beta_\mu$ that determines the coupling behavior of the modes. This fact makes it clear that (we write f instead of $r - a$ for simplicity)

$$\frac{1}{i(\beta_\nu - \beta_\mu)} \frac{df}{dz} \tag{31}$$

is fully equivalent to the function f itself as far as its effect on mode coupling is concerned. We can carry the argument one step further and replace f with

$$-\frac{1}{(\beta_\nu - \beta_\mu)^2} \frac{d^2 f}{dz^2}. \tag{32}$$

If we replace $f = r - a$ in (30) with (32), we obtain a coupling coefficient that vanishes in straight, uniform waveguide sections. It is not hard to guess that a coupling coefficient of this type belongs to an expansion in terms of local normal modes of a hypothetical waveguide that is tangential to the curved axis of the actual guide. The modification of (30) that is indicated by (32) is thus particularly suitable for the description of mode coupling caused by bends of the waveguide axis. A description in terms of ideal modes or even in terms of local normal modes of a hypothetical guide with straight axis is unsuitable for a description of waveguide bends since it leads to coupling coefficients that do not vanish on the straight waveguide section behind the bend. This brief discussion shows that it is not hard to modify the coupling coefficients of the ideal mode coupling theory to extend it to the case of local normal mode expansions of different types.

III. SIMPLIFIED DESCRIPTION OF GUIDED MODES OF THE FIBER

A. W. Snyder[8] realized that the modes of round fibers and their eigenvalue equations simplify considerably if use is made of the fact that $(n_1/n_2 - 1) \ll 1$ applies to most fibers of practical interest. D. Gloge[9] went one step further and showed that the mode fields become simple in appearance if they are expressed in Cartesian instead of the more conventional description in cylindrical coordinates. Gloge's technique is useful for even more complicated waveguide structures such as tubes.[14]

We write down the field expressions for the guided modes of the round fiber without derivation.[9] The mode fields can be polarized in two mutually orthogonal directions. We have for one polarization in the core region for $r < a$

$$\mathcal{E}_z = \frac{iA\kappa}{2\beta_\nu}\left[J_{\nu+1}(\kappa r)\begin{Bmatrix}\sin(\nu+1)\phi\\ -\cos(\nu+1)\phi\end{Bmatrix} + J_{\nu-1}(\kappa r)\begin{Bmatrix}\sin(\nu-1)\phi\\ -\cos(\nu-1)\phi\end{Bmatrix}\right] \quad (33)$$

$$\mathcal{E}_y = AJ_\nu(\kappa r)\begin{Bmatrix}\cos\nu\phi\\ \sin\nu\phi\end{Bmatrix} \quad (34)$$

$$\mathcal{H}_z = -\frac{iA\kappa}{2k}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[J_{\nu+1}(\kappa r)\begin{Bmatrix}\cos(\nu+1)\phi\\ \sin(\nu+1)\phi\end{Bmatrix} - J_{\nu-1}(\kappa r)\begin{Bmatrix}\cos(\nu-1)\phi\\ \sin(\nu-1)\phi\end{Bmatrix}\right] \quad (35)$$

$$\mathcal{H}_x = -nA\frac{\beta_\nu}{|\beta_\nu|}\sqrt{\frac{\epsilon_o}{\mu_o}}J_\nu(\kappa r)\begin{Bmatrix}\cos\nu\phi\\ \sin\nu\phi\end{Bmatrix}. \quad (36)$$

The field in the (infinite) cladding region for $r > a$ is obtained from the field expressions (33) through (36) by replacing the amplitude constant A with $[J_\nu(\kappa a)/H_\nu^{(1)}(i\gamma a)]A$. In addition, we replace κ with $i\gamma$ and the Bessel function $J_\nu(\kappa r)$ with the Hankel function of the first kind $H_\nu^{(1)}(i\gamma r)$. The parameters κ and γ are defined as ($k^2 = \omega^2\epsilon_o\mu_o$)

$$\kappa = (n_1^2k^2 - \beta_\nu^2)^{\frac{1}{2}} \quad (37)$$

and

$$\gamma = (\beta_\nu^2 - n_2^2k^2)^{\frac{1}{2}}. \quad (38)$$

The remaining field components vanish, $\mathcal{E}_x = 0$ and $\mathcal{H}_y = 0$. The two sets of circular functions that are shown in the field equations are necessary to obtain a complete set of orthogonal modes. The functions in the upper as well as those in the lower position belong together. We have used n to indicate $n \approx n_1 \approx n_2$.

The set of guided modes is still not complete unless we also include the orthogonal polarization. We have again for $r < a(\mathcal{E}_y = \mathcal{H}_x = 0)$

$$\mathcal{E}_z = \frac{iA\kappa}{2\beta_\nu}\left[J_{\nu+1}(\kappa r)\begin{Bmatrix}\cos(\nu+1)\phi\\ \sin(\nu+1)\phi\end{Bmatrix} - J_{\nu-1}(\kappa r)\begin{Bmatrix}\cos(\nu-1)\phi\\ \sin(\nu-1)\phi\end{Bmatrix}\right] \quad (39)$$

$$\mathcal{E}_x = AJ_\nu(\kappa r)\begin{Bmatrix}\cos\nu\phi\\ \sin\nu\phi\end{Bmatrix} \quad (40)$$

$$\mathcal{H}_z = \frac{iA\kappa}{2k}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[J_{\nu+1}(\kappa r)\begin{Bmatrix}\sin(\nu+1)\phi\\ -\cos(\nu+1)\phi\end{Bmatrix} + J_{\nu-1}(\kappa r)\begin{Bmatrix}\sin(\nu-1)\phi\\ -\cos(\nu-1)\phi\end{Bmatrix}\right] \quad (41)$$

$$\mathcal{H}_y = nA\frac{\beta_\nu}{|\beta_\nu|}\sqrt{\frac{\epsilon_o}{\mu_o}}J_\nu(\kappa r)\begin{Bmatrix}\cos\nu\phi\\ \sin\nu\phi\end{Bmatrix}. \quad (42)$$

The field in the cladding is again obtained by the replacements mentioned above.

The mode amplitudes must be related to the power P (that is the same for all the modes). We have for the fields (33) through (36) and for the orthogonally polarized field (39) through (42)

$$A = \left\{ \frac{4\sqrt{\frac{\mu_o}{\epsilon_o}}\,\gamma^2 P}{e_\nu \pi a^2 n(n_1^2 - n_2^2)k^2 |J_{\nu-1}(\kappa a)J_{\nu+1}(\kappa a)|} \right\}^{\frac{1}{2}} \tag{43}$$

with

$$e_\nu = \begin{cases} 2 & \text{for} \quad \nu = 0 \\ 1 & \text{for} \quad \nu \neq 0 \end{cases}. \tag{44}$$

The eigenvalue equation of these simplified fiber modes is

$$\kappa \frac{J_{\nu-1}(\kappa a)}{J_\nu(\kappa a)} = i\gamma \frac{H_{\nu-1}^{(1)}(i\gamma a)}{H_\nu^{(1)}(i\gamma a)}. \tag{45}$$

With the help of the functional relations of the cylinder functions, it is easy to show that (45) is also valid if $\nu - 1$ is replaced by $\nu + 1$.

The simplified guided modes listed here are not the same as the usual HE and EH modes of the round fiber. It can be shown[9,15] that the simplified modes listed in this paper result from the usual fiber modes as superpositions of an HE and an EH mode. The $HE_{\nu+1,\mu}$ and $EH_{\nu-1,\mu}$ modes have very nearly the same propagation constant, they are almost degenerate. Since this degeneracy is not perfect, our simplified modes are not modes in the true sense of the word. They decompose into the $HE_{\nu+1,\mu}$ and $EH_{\nu-1,\mu}$ modes of the round fiber as they travel along the waveguide thus changing their shape. A true mode is defined by the fact that only its phase changes (in the lossless case) as it travels down the guide. However, the approximate modes do form a complete orthogonal set of modes and can thus be used to express any field that can exist in the fiber. Even after one of the approximate modes has decomposed into HE and EH modes, it can again be expressed in terms of approximate modes at this point. The fact that the approximate modes are not true modes in the usual sense does not limit their usefulness for studying mode conversion and radiation problems.

The important fundamental HE_{11} mode of the fiber corresponds to the lowest-order approximate mode with $\nu = 0$. This is a true mode that does not decompose as it travels along the waveguide.

IV. SIMPLIFIED RADIATION MODES OF THE FIBER

The radiation modes of the fiber can again be simplified by using $(n_1/n_2 - 1) \ll 1$.[16] There is a slight complication, however. The simplified description of the guided modes was made possible by the fact that they are very nearly transverse modes, their transverse components being much larger than their longitudinal components. The radiation modes are nearly transverse only if their propagation constants are nearly $\beta = n_2k$. Since the continuous spectrum of radiation modes extends from $\beta = -n_2k$ to $\beta = n_2k$, only the modes in the immediate vicinity of the two end points of this interval are also nearly transverse modes. Throughout most of the spectral region of β values, the approximation corresponding to that for the guided modes does not work. However, we can still use the fact that the refractive indices of core and cladding are nearly identical and use the radiation modes of free space in the region where our mode approximation technique fails. A simplified treatment of all the radiation modes is thus also possible. The two approximations complement each other. In the region near $\beta = \pm n_2k$, where we use the approximate radiation modes of the guide, the free-space radiation modes do not work very well because reflections at the core-cladding interface at grazing angles are important. Inside of the β range, where the waveguide mode approximation method fails, we can use the free-space radiation modes with confidence since the interface does not cause much reflection for waves passing through it at reasonably steep angles.

We begin by listing the approximate radiation modes of the fiber for β near $\pm n_2k$. The field equations are very similar to those of the guided modes. In the fiber core at $r < a$ we have

$$\mathcal{E}_z = \frac{iB\sigma}{2\beta}\left[J_{\nu+1}(\sigma r)\begin{Bmatrix}\sin(\nu+1)\phi\\ -\cos(\nu+1)\phi\end{Bmatrix} + J_{\nu-1}(\sigma r)\begin{Bmatrix}\sin(\nu-1)\phi\\ -\cos(\nu-1)\phi\end{Bmatrix}\right] \tag{46}$$

$$\mathcal{E}_y = BJ_\nu(\sigma r)\begin{Bmatrix}\cos\nu\phi\\ \sin\nu\phi\end{Bmatrix} \tag{47}$$

$$\mathcal{H}_z = -\frac{iB\sigma}{2k}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[J_{\nu+1}(\sigma r)\begin{Bmatrix}\cos(\nu+1)\phi\\ \sin(\nu+1)\phi\end{Bmatrix} - J_{\nu-1}(\sigma r)\begin{Bmatrix}\cos(\nu-1)\phi\\ \sin(\nu-1)\phi\end{Bmatrix}\right] \tag{48}$$

$$\mathcal{H}_x = -nB\frac{\beta}{|\beta|}\sqrt{\frac{\epsilon_o}{\mu_o}}J_\nu(\sigma r)\begin{Bmatrix}\cos\nu\phi\\ \sin\nu\phi\end{Bmatrix}. \tag{49}$$

The remaining field components vanish. The propagation constant β is a continuous variable for radiation modes unrestricted by an eigenvalue equation. The parameter σ is defined as

$$\sigma = (n_1^2 k^2 - \beta^2)^{\frac{1}{2}} \tag{50}$$

Instead of specifying the modifications that are required to transform the expression of the field inside of the core into the expression for the cladding field, we state the field in the region $r > a$ in detail.

$$\mathcal{E}_z = \frac{iC\rho}{2\beta}\left[(H_{\nu+1}^{(1)}(\rho r) + DH_{\nu+1}^{(2)}(\rho r)) \begin{Bmatrix} \sin (\nu+1)\phi \\ -\cos (\nu+1)\phi \end{Bmatrix} + (H_{\nu-1}^{(1)}(\rho r) + DH_{\nu-1}^{(2)}(\rho r)) \begin{Bmatrix} \sin (\nu-1)\phi \\ -\cos (\nu-1)\phi \end{Bmatrix}\right] \tag{51}$$

$$\mathcal{E}_y = C(H_\nu^{(1)}(\rho r) + DH_\nu^{(2)}(\rho r)) \begin{Bmatrix} \cos \nu\phi \\ \sin \nu\phi \end{Bmatrix} \tag{52}$$

$$\mathcal{H}_z = -\frac{iC\rho}{2k}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[(H_{\nu+1}^{(1)}(\rho r) + DH_{\nu+1}^{(2)}(\rho r)) \begin{Bmatrix} \cos (\nu+1)\phi \\ \sin (\nu+1)\phi \end{Bmatrix} - (H_{\nu-1}^{(1)}(\rho r) + DH_{\nu-1}^{(2)}(\rho r)) \begin{Bmatrix} \cos (\nu-1)\phi \\ \sin (\nu-1)\phi \end{Bmatrix}\right] \tag{53}$$

$$\mathcal{H}_x = -nC\frac{\beta}{|\beta|}\sqrt{\frac{\epsilon_o}{\mu_o}}(H_\nu^{(1)}(\rho r) + DH_\nu^{(2)}(\rho r)) \begin{Bmatrix} \cos \nu\phi \\ \sin \nu\phi \end{Bmatrix}. \tag{54}$$

$H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Hankel functions of the first and second kind. The parameter ρ is defined as

$$\rho = (n_2^2 k^2 - \beta^2)^{\frac{1}{2}}. \tag{55}$$

The amplitude coefficients are

$$C = \frac{i\pi a}{4}[\sigma J_{\nu+1}(\sigma a)H_\nu^{(2)}(\rho a) - \rho J_\nu(\sigma a)H_{\nu+1}^{(2)}(\rho a)]B \tag{56}$$

and

$$D = -\frac{\sigma J_{\nu+1}(\sigma a)H_\nu^{(1)}(\rho a) - \rho J_\nu(\sigma a)H_{\nu+1}^{(1)}(\rho a)}{\sigma J_{\nu+1}(\sigma a)H_\nu^{(2)}(\rho a) - \rho J_\nu(\sigma a)H_{\nu+1}^{(2)}(\rho a)}. \tag{57}$$

For the field with the orthogonal polarization, we simply state the field expressions inside of the core. It should be apparent from inspection of (46) through (57) how the field expression in the cladding is obtained from that of the core. The relations between the amplitude coefficient are the same in either case. We have for $r < a$

$$\mathcal{E}_z = \frac{iB\sigma}{2\beta}\left[J_{\nu+1}(\sigma r) \begin{Bmatrix} \cos (\nu+1)\phi \\ \sin (\nu+1)\phi \end{Bmatrix} - J_{\nu-1}(\sigma r) \begin{Bmatrix} \cos (\nu-1)\phi \\ \sin (\nu-1)\phi \end{Bmatrix}\right] \tag{58}$$

$$\mathcal{E}_x = BJ_\nu(\sigma r)\begin{Bmatrix}\cos \nu\phi \\ \sin \nu\phi\end{Bmatrix} \tag{59}$$

$$\mathcal{H}_z = \frac{iB\sigma}{2k}\sqrt{\frac{\epsilon_o}{\mu_o}}\left\{J_{\nu+1}(\sigma r)\begin{bmatrix}\sin (\nu+1)\phi \\ -\cos (\nu+1)\phi\end{bmatrix} + J_{\nu-1}(\sigma r)\begin{bmatrix}\sin (\nu-1)\phi \\ -\cos (\nu-1)\phi\end{bmatrix}\right\} \tag{60}$$

$$\mathcal{H}_y = nB\frac{\beta}{|\beta|}\sqrt{\frac{\epsilon_o}{\mu_o}}\,J_\nu(\sigma r)\begin{Bmatrix}\cos \nu\phi \\ \sin \nu\phi\end{Bmatrix}. \tag{61}$$

It remains to relate the amplitude coefficient to the power P. Because of the continuous mode spectrum, it is not possible to normalize the radiation modes with a finite amount of power. The parameter P is defined by the relation

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\boldsymbol{\mathcal{E}}_\rho \times \boldsymbol{\mathcal{H}}_{\rho'}^*)\cdot \mathbf{e}_z dx dy = \frac{\beta^*}{|\beta|}P\delta(\rho-\rho'). \tag{62}$$

The amplitude coefficient B is thus [e_ν is defined by (36)]

$$B = \frac{\left(\dfrac{\mu_o}{\epsilon_o}\right)^{\frac{1}{4}}(8\rho P)^{\frac{1}{2}}}{\sqrt{e_\nu n}\,a\pi^{\frac{3}{2}}\left|\sigma J_{\nu-1}(\sigma a)H_\nu^{(1)}(\rho a) - \rho J_\nu(\sigma a)H_{\nu-1}^{(1)}(\rho a)\right|}. \tag{63}$$

It is important to remember that the radiation modes listed so far are valid only in the immediate vicinity of $\beta = \pm n_2 k$. Inside the β range, we use the radiation modes of homogeneous space with refractive index n_2. These modes can be expressed in a number of different ways. The simplest expressions would result from a plane-wave representation. However, for our present purposes, it seems advisable to use field expressions that resemble most closely the radiation modes (46) through (54) and (58) through (61) in order to achieve continuity of the field expressions throughout the entire β range. The modes of the homogeneous medium (in vacuum we would say free-space modes) are simpler than the radiation modes of the fiber, since one expression applies throughout all of space. There is no need to treat the fields inside and outside of the core separately. The field expressions that satisfy our requirements are [ρ is defined by (55), we use $n = n_2$]

$$\mathcal{E}_z = \frac{iC\rho}{2\beta}\left[J_{\nu+1}(\rho r)\begin{Bmatrix}\sin (\nu+1)\phi \\ -\cos (\nu+1)\phi\end{Bmatrix} + J_{\nu-1}(\rho r)\begin{Bmatrix}\sin (\nu-1)\phi \\ -\cos (\nu-1)\phi\end{Bmatrix}\right] \tag{64}$$

$$\mathcal{E}_x = 0 \tag{65}$$

$$\mathcal{E}_y = CJ_\nu(\rho r) \begin{Bmatrix} \cos \nu\phi \\ \sin \nu\phi \end{Bmatrix} \tag{66}$$

$$\mathcal{H}_z = -\frac{iC\rho}{2k}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[J_{\nu+1}(\rho r)\begin{Bmatrix} \cos (\nu+1)\phi \\ \sin (\nu+1)\phi \end{Bmatrix} - J_{\nu-1}(\rho r)\begin{Bmatrix} \cos (\nu-1)\phi \\ \sin (\nu-1)\phi \end{Bmatrix}\right] \tag{67}$$

$$\mathcal{H}_x = -\frac{C}{4}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[2\left(\frac{\beta}{k}+\frac{n^2k}{\beta}\right)J_\nu(\rho r)\begin{Bmatrix} \cos \nu\phi \\ \sin \nu\phi \end{Bmatrix} + \left(\frac{n^2k}{\beta}-\frac{\beta}{k}\right)\left(J_{\nu+2}(\rho r)\begin{Bmatrix} \cos (\nu+2)\phi \\ \sin (\nu+2)\phi \end{Bmatrix} + J_{\nu-2}(\rho r)\begin{Bmatrix} \cos (\nu-2)\phi \\ \sin (\nu-2)\phi \end{Bmatrix}\right)\right] \tag{68}$$

$$\mathcal{H}_y = \frac{C}{4}\sqrt{\frac{\epsilon_o}{\mu_o}}\left(\frac{\beta}{k}-\frac{n^2k}{\beta}\right)\left[J_{\nu+2}(\rho r)\begin{Bmatrix} \sin (\nu+2)\phi \\ -\cos (\nu+2)\phi \end{Bmatrix} - J_{\nu-2}(\rho r)\begin{Bmatrix} \sin (\nu-2)\phi \\ -\cos (\nu-2)\phi \end{Bmatrix}\right]. \tag{69}$$

The orthogonally polarized modes are

$$\mathcal{E}_z = \frac{iC\rho}{2\beta}\left[J_{\nu+1}(\rho r)\begin{Bmatrix} \cos (\nu+1)\phi \\ \sin (\nu+1)\phi \end{Bmatrix} - J_{\nu-1}(\rho r)\begin{Bmatrix} \cos (\nu-1)\phi \\ \sin (\nu-1)\phi \end{Bmatrix}\right] \tag{70}$$

$$\mathcal{E}_x = CJ_\nu(\rho r) \begin{Bmatrix} \cos \nu\phi \\ \sin \nu\phi \end{Bmatrix} \tag{71}$$

$$\mathcal{E}_y = 0 \tag{72}$$

$$\mathcal{H}_z = \frac{iC\rho}{2k}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[J_{\nu+1}(\rho r)\begin{Bmatrix} \sin (\nu+1)\phi \\ -\cos (\nu+1)\phi \end{Bmatrix} + J_{\nu-1}(\rho r)\begin{Bmatrix} \sin (\nu-1)\phi \\ -\cos (\nu-1)\phi \end{Bmatrix}\right] \tag{73}$$

$$\mathcal{H}_x = -\frac{C}{4}\sqrt{\frac{\epsilon_o}{\mu_o}}\left(\frac{\beta}{k}-\frac{n^2k}{\beta}\right)\left[J_{\nu+2}(\rho r)\begin{Bmatrix} \sin (\nu+2)\phi \\ -\cos (\nu+2)\phi \end{Bmatrix} - J_{\nu-2}(\rho r)\begin{Bmatrix} \sin (\nu-2)\phi \\ -\cos (\nu-2)\phi \end{Bmatrix}\right] \tag{74}$$

$$\mathcal{H}_y = \frac{C}{4}\sqrt{\frac{\epsilon_o}{\mu_o}}\left[2\left(\frac{\beta}{k}+\frac{n^2k}{\beta}\right)J_\nu(\rho r)\begin{Bmatrix} \cos \nu\phi \\ \sin \nu\phi \end{Bmatrix} + \left(\frac{\beta}{k}-\frac{n^2k}{\beta}\right)\left(J_{\nu+2}(\rho r)\begin{Bmatrix} \cos (\nu+2)\phi \\ \sin (\nu+2)\phi \end{Bmatrix} + J_{\nu-2}(\rho r)\begin{Bmatrix} \cos (\nu-2)\phi \\ \sin (\nu-2)\phi \end{Bmatrix}\right)\right]. \tag{75}$$

The amplitude coefficient C is related to P

$$C = \left\{ \frac{4\sqrt{\frac{\mu_o}{\epsilon_o}}\, k\rho\beta P}{e_\nu \pi(\beta^2 + n^2k^2)} \right\}^{\frac{1}{2}}. \tag{76}$$

The modes of the homogeneous medium, like all the other modes, are mutually orthogonal among each other. With the help of the relations

$$J_\nu(x)H^{(1)}_{\nu+1}(x) - J_{\nu+1}(x)H^{(1)}_\nu(x) = \frac{2}{i\pi x} \tag{77a}$$

and

$$J_\nu(x)H^{(2)}_{\nu+1}(x) - J_{\nu+1}(x)H^{(2)}_\nu(x) = -\frac{2}{i\pi x} \tag{77b}$$

it is easy to show that the radiation modes of the homogeneous medium and the radiation modes of the round fiber reduce to the same expressions in the limit $|\beta| = n_2k$, $n_1 = n_2 = n$.

V. COUPLING COEFFICIENTS FOR CORE-CLADDING IMPERFECTIONS

We consider a fiber whose core-cladding interface is described by the function

$$r(x, y, z) = a + f(z) \cos (m\phi + \psi). \tag{78}$$

If we choose a different function $f(z)$ and different phase ψ for each integer m and sum over the second term on the right side, we generate a Fourier series which allows us to describe core-cladding imperfections of the most general kind.

We assume that a given mode labeled ν is traveling in the waveguide and ask for the coupling from this mode to all other guided and radiation modes. The function $f(z)$ can be separated out from the coupling coefficient by defining

$$K_{\mu\nu} = \bar{K}_{\mu\nu} f(z). \tag{79}$$

We have mentioned earlier that the longitudinal field components of the guided modes are much smaller than their transverse components. The same statement is true for the radiation modes only if their propagation constant β is very nearly equal to $\pm n_2k$. For $|\beta|$ values much smaller than n_2k, the longitudinal field components of the radiation modes can be as large as or larger than the transverse components. The coupling coefficients contain scalar products of the two fields that are coupled together. Coupling coefficients that involve at least one guided mode are thus determined primarily by the transverse component of both fields, since the product of the longitudinal components is small over most of the range of β values. We thus neglect the contribution of the longitudinal components and gain the advantage of much simpler expressions for the coupling coefficients. The only region of the β range where the longitudinal components could make a significant contribution to the coupling process is near $\beta = 0$. Outside of a small region near $\beta = 0$, our approximation is reasonably accurate. To this approximation, modes with orthogonal (transverse) polarization are not coupled by core-cladding imperfections of the form (78).

For coupling between two guided modes ν and μ we obtain from (30) and (79) with the help of the field expressions

$$\bar{K}_{\mu\nu}^{(p,q)} = \frac{e_{\mu\nu m}}{\sqrt{e_\mu e_\nu}} \frac{p\gamma_\nu\gamma_\mu J_\nu(\kappa_\nu a)J_\mu(\kappa_\mu a)}{2iank[|J_{\nu-1}(\kappa_\nu a)J_{\nu+1}(\kappa_\nu a)J_{\mu-1}(\kappa_\mu a)J_{\mu+1}(\kappa_\mu a)|]^{\frac{1}{2}}}. \tag{80}$$

The factor $e_{\mu\nu m}$ can assume the values 4, 2, 1, or 0. It is zero unless $\mu = \nu \pm m$. Table I shows the values of $e_{\mu\nu m}$ for all possible cases. The factor e_ν is defined by (44), κ_ν and γ_ν are determined by (37) and (38).

Coupling between the guided mode ν and radiation modes μ must be described by two different coupling coefficients depending on the value of the propagation constant β of the radiation mode. For values of $|\beta|$ close to n_2k, we use the radiation modes of the fiber and obtain

$$\bar{K}_{\mu\nu}^{(p,q)} = \frac{e_{\mu\nu m}}{\sqrt{e_\nu e_\mu}} \times \frac{p\left(\frac{n_1}{n_2} - 1\right)^{\frac{1}{2}}\gamma_\nu\sqrt{\rho}J_\nu(\kappa_\nu a)J_\mu(\sigma a)}{i\pi a|J_{\nu-1}(\kappa_\nu a)J_{\nu+1}(\kappa_\nu a)|^{\frac{1}{2}}|\sigma J_{\mu-1}(\sigma a)H_\mu^{(1)}(\rho a) - \rho J_\mu(\sigma a)H_{\mu-1}^{(1)}(\rho a)|}. \tag{81}$$

For β values inside the range $-n_2k < \beta < n_2k$ excluding the end points, we use the radiation modes of homogeneous space and find

$$\bar{K}_{\mu\nu}^{(p,q)} = \frac{e_{\mu\nu m}}{\sqrt{e_\nu e_\mu}} \frac{p\left[n\left(\frac{n_1}{n_2} - 1\right)k\rho\beta\right]^{\frac{1}{2}}\gamma_\nu J_\nu(\kappa_\nu a)J_\mu(\rho a)}{i[2(\beta^2 + n^2k^2)|J_{\nu-1}(\kappa_\nu a)J_{\nu+1}(\kappa_\nu a)|]^{\frac{1}{2}}}. \tag{82}$$

Use of (77) allows us again to see that (81) and (82) become identical in the limit $n_1 = n_2 = n$, $|\beta| = n_2k$.

The coupling coefficients for the approximate guided and radiation modes of the round optical fiber allow us to solve a large number of problems involving fibers with core-cladding interface irregularities.

VI. FAR-FROM-CUTOFF APPROXIMATIONS

For purposes of multimode operation it is often desirable to have simple expressions for the coupling coefficients which are valid far from

TABLE I—TABULATION OF THE FACTOR $e_{\mu\nu m}$ FOR ALL POSSIBLE COMBINATIONS OF ANGULAR FIELD DEPENDENCE OF THE MODES AND THE CORE-CLADDING INTERFACE DISTORTION

$e_{\mu\nu m} = 0$ unless specified otherwise

incident mode	spurious mode	distortion
$\cos \nu\phi$	$\cos \mu\phi$	$\cos m\phi$
	$e_{\mu\nu m} = \begin{cases} 4 & \nu = \mu = m = 0 \\ 2 & \begin{cases}\nu = 0, & \mu = m \\ \mu = 0, & \nu = m\end{cases} \\ 1 & \begin{cases}0 < \mu = \nu \pm m \\ 0 < \mu = m - \nu\end{cases} \end{cases}$	
incident mode	spurious mode	distortion
$\sin \nu\phi$	$\sin \mu\phi$	$\cos m\phi$
	$e_{\mu\nu m} = \begin{cases} 0 & \nu \text{ or } \mu = 0 \\ +1 & 0 < \mu = \nu \pm m \\ -1 & 0 < \mu = m - \nu \end{cases}$	
incident mode	spurious mode	distortion
$\cos \nu\phi$	$\sin \mu\phi$	$\sin m\phi$
	$e_{\mu\nu m} = \begin{cases} 0 & \mu = 0 \\ 2 & \nu = 0, \quad \mu = m \\ 1 & \mu = \nu + m \\ -1 & 0 < \mu = \nu - m \\ 1 & 0 < \mu = m - \nu \end{cases}$	
incident mode	spurious mode	distortion
$\cos \nu\phi$	$\sin \mu\phi$	$\cos m\phi$
	$e_{\mu\nu m} = 0$	
incident mode	spurious mode	distortion
$\cos \nu\phi$	$\cos \mu\phi$	$\sin m\phi$
	$e_{\mu\nu m} = 0$	
incident mode	spurious mode	distortion
$\sin \nu\phi$	$\sin \mu\phi$	$\sin m\phi$
	$e_{\mu\nu m} = 0$	

cutoff. We obtain such approximations by using the approximation for large argument of the Hankel function ($\gamma a \gg 1$)

$$H_\nu^{(1)}(i\gamma a) \approx \sqrt{\frac{2}{i\pi\gamma a}}\, e^{-i[\pi/4+\nu(\pi/2)]}e^{-\gamma a}. \tag{83}$$

With (83), we obtain from (45) for $\gamma a \gg 1$

$$\gamma J_\nu(\kappa a) = -\kappa J_{\nu-1}(\kappa a). \tag{84}$$

We remarked earlier that (45) is also valid if $\nu - 1$ is replaced with $\nu + 1$. We thus also have

$$\gamma J_\nu(\kappa a) = \kappa J_{\nu+1}(\kappa a). \tag{85}$$

Multiplying (84) with (85) and taking the square root results in

$$\begin{aligned}\gamma J_\nu(\kappa a) &= \kappa[-J_{\nu-1}(\kappa a)J_{\nu+1}(\kappa a)]^{\frac{1}{2}} \\ &= \kappa[|J_{\nu-1}(\kappa a)J_{\nu+1}(\kappa a)|]^{\frac{1}{2}}.\end{aligned} \tag{86}$$

Far from cutoff, (86) allows us to write the coupling coefficient (80) between guided modes in the simple form

$$\bar{K}_{\mu\nu}^{(p,q)} = \frac{pe_{\mu\nu m}}{\sqrt{e_\nu e_\mu}} \frac{\kappa_\mu \kappa_\nu}{2iank}. \tag{87}$$

Very far from cutoff, $\gamma a \to \infty$, we obtain the approximate eigenvalue equation

$$J_\nu(\kappa_\nu a) = 0 \tag{88}$$

from (84) or (85). The roots of the Bessel function $J_\nu(\kappa_\nu a)$ thus determine the values of $\kappa_\nu a$ that appear in (87). For higher-order modes $\kappa_\nu a$ becomes large so that we can approximate

$$J_\nu(\kappa_\nu a) \approx \sqrt{\frac{2}{\pi\kappa_\nu a}} \cos\left[\kappa_\nu a - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]. \tag{89}$$

Equation (88) requires that the argument of the cosine function equal $(2N + 1)\pi/2$. This leads to a direct determination of

$$\kappa_\nu a \approx \left(\nu + 2N + \frac{3}{2}\right)\frac{\pi}{2} \tag{90}$$

with $N = 0, 1, 2 \cdots$. The equations (87) and (90) provide us with an approximate determination of the coupling coefficient between two guided modes without the need for solving a transcendental eigenvalue equation. In a strict sense, we would have to label κ and the coupling coefficient with N as well as ν. We refrain from burdening the symbols with too many indices.

The coupling coefficients between guided and radiation modes can similarly be simplified far from cutoff of the guided modes. The far-from-cutoff approximation of (81) is

$$\bar{K}_{\mu\nu}^{(p,q)} = \frac{e_{\mu\nu m}}{\sqrt{e_\nu e_\mu}} \frac{p\left(\dfrac{n_1}{n_2} - 1\right)^{\frac{1}{2}} \sqrt{\rho}\kappa_\nu J_\mu(\sigma a)}{i\pi a|\sigma J_{\mu-1}(\sigma a)H_\mu^{(1)}(\rho a) - \rho J_\mu(\sigma a)H_{\mu-1}^{(1)}(\rho a)|}. \tag{91}$$

This equation is valid only for $|\beta|$ values very close to $n_2 k$. The coefficient that describes coupling between guided modes and the radiation modes of homogeneous space, eq. (82), leads to the far-from-cutoff approximation

$$\bar{K}_{\mu\nu}^{(p,q)} = \frac{e_{\mu\nu m}}{\sqrt{e_\nu e_\mu}} \frac{p\left[n\left(\frac{n_1}{n_2}-1\right)k\rho\beta\right]^{\frac{1}{2}\kappa_\nu}J_\mu(\rho a)}{i[2(\beta^2+n^2k^2)]^{\frac{1}{2}}}. \tag{92}$$

This expression is valid inside the range $-n_2k < \beta < n_2k$ but not near $|\beta| = n_2k$. For most scattering problems, it is sufficient to use the coupling coefficients to the radiation modes of the homogeneous medium. Only if the scattering is sharply forward or backward directed do we have to use the slightly more complicated expression (91) of the coupling coefficients to the true radiation modes of the round fiber.

In the remainder of the paper, we apply our results to special cases.

VII. COUPLING CAUSED BY WAVEGUIDE BENDS

We consider the case of a straight fiber that is connected to a fiber section that is bent with a constant radius of curvature and finally continues in a straight section. If the curved piece of waveguide causes considerable mode conversion, the system of coupled equations (27) and (28) must be solved. However, for slight mode conversion, we can use the approximation that the incident mode does not change very much while power builds up in some of the spurious modes. In this case, we obtain the following approximate solution from (27):

$$c_\mu(L) = c_\nu(0)\int_0^L K_{\mu\nu}(z)e^{i(\beta_\mu-\beta_\nu)z}dz. \tag{93}$$

We assume $z = 0$ at the beginning of the curved section of length L. The description (32) is most appropriate in this case. The second derivative assumes the constant value $1/R$, with R being the radius of curvature of the circular bend. We thus obtain from (79) [with f replaced by (32)] and (93)

$$\left|\frac{c_\mu(L)}{c_\nu(0)}\right|^2 = \frac{1}{R^2}\frac{4|\bar{K}_{\mu\nu}|^2}{(\beta_\mu-\beta_\nu)^6}\sin^2(\beta_\mu-\beta_\nu)\frac{L}{2}. \tag{94}$$

We obtain $\bar{K}_{\mu\nu}$ from (80) or in its "far-from-cutoff" approximation from (87). The integer m appearing in (78) must be set $m = 1$ in this case, since we want to describe a continuous offset of the fiber which results in a bend. It is apparent that the amount of power transfer between the incident and the spurious mode depends critically on the separation between the two propagation constants. The sine factor in (94) describes the phasing between the two modes. If the incident and spurious modes travel with equal phase velocity, power would be transferred from the incident mode to the spurious mode in proper phase so that all the power could be exchanged between the modes. If both modes have different phase velocities (and our formula holds only in this case), the two modes get out of step so that after some distance the power that is fed from the incident mode to the spurious mode tends to interfere destructively and destroys the power that has already been transferred. The power in the spurious mode thus builds up and decays. This process does not involve reconversion of power from the spurious to the incident mode, since this process is not included in our perturbation solution.

It can be shown that a guided mode of the type (34) produces a ring-shaped far-field pattern if it is allowed to radiate out of the end of the fiber. The maximum of the ring as seen from the end of the fiber appears at an angle

$$\theta_\nu = \frac{\kappa_\nu}{k}. \tag{95}$$

The circular far-field pattern on a screen is broken up into 2ν bright dots corresponding to the angular intensity maxima of the field distribution in the fiber. The angle θ_ν is useful to distinguish guided modes experimentally. It may thus be of interest to express (94) in terms of this mode angle. Using (87), (95), and

$$\beta_\nu \approx n_1 k - \frac{\kappa_\nu^2}{2n_1 k} = n_1 k - \frac{k}{2n_1}\theta_\nu^2, \tag{96}$$

we can write (94) in the following form:

$$\left|\frac{c_\mu}{c_\nu}\right|^2 = \frac{e_{\mu\nu 1}^2}{e_\nu e_\mu}\frac{2^6 n^4 \theta_\nu^2 \theta_\mu^2}{k^4 a^2 R^2 (\theta_\nu^2 - \theta_\mu^2)^6}\sin^2(\beta_\mu - \beta_\nu)\frac{L}{2}. \tag{97}$$

For all practical applications, the separation between the angles θ_ν and θ_μ is so small that we can replace them with one angle θ. According to (90) we get for the difference

$$\theta_\nu - \theta_\mu = \Delta\theta = \pm\frac{\pi}{2ka} \tag{98}$$

if $\mu = \nu \pm 1$. We restrict the discussion to coupling between modes with the same value of N but adjacent ν values. If the results of this discussion are implemented, (97) assumes the form

$$\left|\frac{c_\mu}{c_\nu}\right|^2 = \frac{e_{\mu\nu 1}^2}{e_\nu e_\mu}\frac{2^6 k^2 a^4 n^4}{\pi^6 R^6 \theta^2}\sin^2(\beta_\mu - \beta_\nu)\frac{L}{2}. \tag{99}$$

This equation shows that the amount of power transfer caused by waveguide bends decreases with increasing mode angle.

Next we consider random bends. It has been shown in Refs. 3 and 17 that the exchange of power among randomly coupled modes can be described by coupled power equations. The power coupling coefficient is given by

$$h_{\mu\nu} = |\bar{K}_{\mu\nu}|^2 F(\beta_\mu - \beta_\nu). \tag{100}$$

$F(\beta_\mu - \beta_\nu)$ is the power spectrum of the coupling function $f(z)$ or of its equivalents (31) or (32). Using (32) and writing $1/R$ instead of d^2f/dz^2, we obtain for the power coupling coefficient for random bends

$$h_{\mu\nu} = \frac{e_{\mu\nu 1}^2}{e_\nu e_\mu} \frac{\kappa_\nu^2 \kappa_\mu^2}{4a^2 n^2 k^2 (\beta_\mu - \beta_\nu)^4} C\left(\frac{1}{R}\right) \tag{101}$$

with the power spectrum of the curvature function

$$C\left(\frac{1}{R}\right) = \left\langle \left| \frac{1}{\sqrt{L}} \int_0^L \frac{1}{R(z)} e^{i(\beta_\mu - \beta_\nu)z} dz \right|^2 \right\rangle. \tag{102}$$

The symbol $\langle \ \rangle$ indicates an ensemble average.

The guided modes suffer radiation losses even in uniformly bent waveguide sections, $R(z) = \text{const.}$[18,19] These curvature losses cannot be obtained by perturbation theory and thus are not included in our discussion. However, our perturbation theory includes mode conversion losses between guided modes and radiation losses caused by changes in the waveguide curvature.

In terms of the mode angle θ, (102) can be written in the form

$$h_{\mu\nu} = \frac{e_{\mu\nu 1}^2}{e_\nu e_\mu} \frac{4n^2 k^2 a^2}{\pi^4} C\left(\frac{1}{R}\right). \tag{103}$$

The dependence on the mode angle is contained only in the power spectrum of the curvature function.

With the help of the coupling coefficients (91) and (92), radiation losses caused by random bends can be calculated. However, the explicit expression will not be given here.

VIII. MODE CONVERSION AND LOSSES DUE TO DISTORTED CORE-CLADDING INTERFACES

Instead of writing down the general formula for the power coupling coefficient between guided modes based on (80), we restrict ourselves to the far-from-cutoff approximation. From (87), (95), and (100) we obtain

$$h_{\mu\nu} = \frac{e_{\mu\nu m}^2}{e_\nu e_\mu} \frac{k^2\theta_\mu^2\theta_\nu^2}{4a^2n^2} F(\beta_\mu - \beta_\nu) \tag{104}$$

with

$$F(\beta_\mu - \beta_\nu) = \left\langle \left| \frac{1}{\sqrt{L}} \int_0^L f(z) e^{i(\beta_\mu - \beta_\nu)z} dz \right|^2 \right\rangle. \tag{105}$$

$f(z)$ is the function that appears in (78). Equation (104) is remarkably similar to eq. (37) of Ref. 3 which was derived for the slab waveguide model. For comparison of the two equations, it is necessary to remember that we have assumed that the angles θ_ν and θ_μ are much smaller than unity and that $\gamma d \gg 1$ according to our far-from-cutoff approximation. The difference in the position of n in the two equations is attributable to the different definitions of the mode angles. In our present discussion, we consider θ_ν as the angle of the far-field radiation cone outside of the fiber, while this angle was defined as the angle of the plane waves inside of the fiber core in Ref. 3. The correspondence between the two coupling coefficients requires us to consider the case of pure diameter changes, $m = 0$, and assume that both modes have no circumferential variation, $\nu = 0$ and $\mu = 0$. In this case, we have $e_{ooo}^2/(4e_oe_o) = 1$ instead of the factor 1/2 appearing in (37) of Ref. 3. The difference is accounted for if we remember that the round fiber corresponds to a slab in which both interfaces have irregularities which are perfectly correlated. The slab waveguide theory of Ref. 3 assumed, on the other hand, that the two interfaces had uncorrelated irregularities. This comparison shows that the results of the slab waveguide theory and the round fiber are in very good agreement. Our present formula (104) holds for core-cladding interface irregularities of a much more general kind. Not only pure diameter changes but elliptical deformations and deformations of even more general shapes are included.

Next, we turn to the problem of radiation losses. The power loss coefficient is defined by [compare (9.3–14) and (9.3–42) of Ref. 1]

$$\alpha_\nu = \sum_\mu \int_{-n_2k}^{n_2k} |\bar{K}_{\mu\nu}|^2 F(\beta - \beta_\nu) \frac{|\beta|}{\rho} d\beta \tag{106}$$

with $\bar{K}_{\mu\nu}$ being the coefficient for coupling between a guided mode ν and a radiation mode with angular symmetry μ and propagation constant β. The power spectrum F is defined by (105) with β_μ replaced by β. Not much can be gained from substituting (81) and (82) or their far-from-cutoff approximations (91) and (92) into (106). The integral in (106) is hard to solve and useful approximations covering the whole range of correlation lengths have not yet been found. However, using, for example, (81) and (82) in (106) simplifies the numerical integration compared to the problem discussed in Ref. 7. The radiation losses can be calculated from (81), (82), and (106) with much simpler computer programs and at considerable savings compared to the theory of Ref. 7. We use an exponential correlation function

$$R(u) = \langle f(z) f(z + u) \rangle = \bar{\sigma}^2 \exp(-|u|/B) \tag{107}$$

and obtain (Ref. 1, p. 371)

$$F(\beta - \beta_\nu) = \frac{2\bar{\sigma}^2}{B\left[(\beta - \beta_\nu)^2 + \dfrac{1}{B^2}\right]}. \tag{108}$$

The resulting radiation losses for pure diameter changes, $m = 0$, are plotted for the HE_{11} mode, $\nu = 0$, in Fig. 2 as functions of the ratio of correlation length B over core radius a for $n_1/n_2 = 1.01$. The curves were obtained by numerical integration of (106) with $\bar{K}_{\mu\nu}$ of (81) in the range $0.95n_2k \leqq |\beta| \leqq n_2k$ and with $\bar{K}_{\mu\nu}$ of (82) in the range $-0.95n_2k < \beta < 0.95n_2k$. For small index differences between core

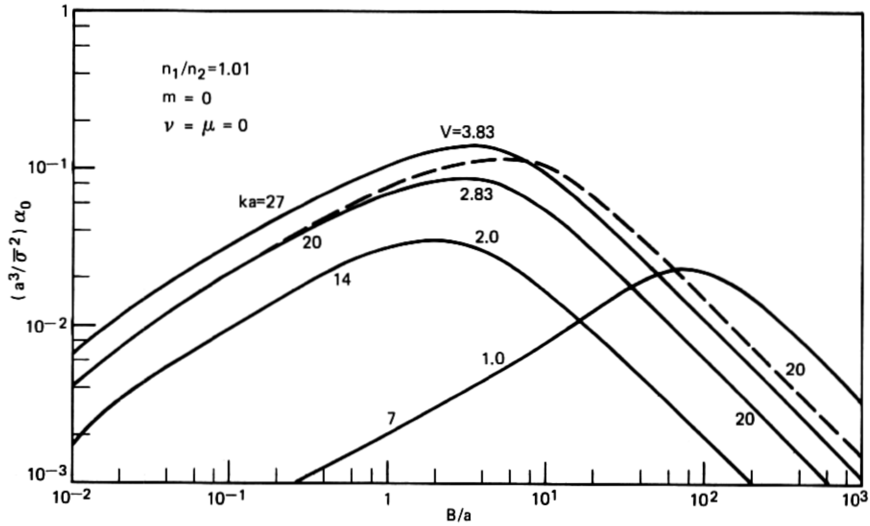

Fig. 2—Normalized radiation losses of the HE_{11} mode, $\nu = 0$, as functions of the ratio of correlation length B to core radius a for different values of $ka = 2\pi a/\lambda$ for pure diameter changes, $m = 0$. $n_1/n_2 = 1.01$. The dotted line results from using only "free space" radiation modes.

and cladding, the losses calculated from our simplified theory are in perfect agreement with the theory of Ref. 7. For larger index ratios, our approximate theory begins to fail. For $n_1/n_2 = 1.43$, the error caused by our approximation is in the order of 60 percent. The simplification gained from using (81), (82), and (106) is apparent by glancing at the complex formulas of Ref. 7.

The dotted line in Fig. 2 was computed by using the radiation modes of homogeneous space alone so that (82) instead of a combination of (81) and (82) was used in (106). It is apparent that the radiation modes of homogeneous space are not suitable to calculate the radiation losses for large values of B/a. It was pointed out in Ref. 5 that large B/a ratios lead to forward scattering. The radiation makes small angles with the core-cladding interface so that reflection at this interface becomes important. It is thus necessary to use the radiation modes of the fiber for β values near n_2k.

Our theory allows us to calculate radiation losses for more general core-cladding interface distortions. As a second example, we consider elliptical deformation, $m = 2$, and plot the result of the numerical integration of (106) in Fig. 3. The power spectrum (108) of the function $f(z)$ [defined by (78)] was used again. We also used a combination of radiation modes of the fiber and of free-space radiation modes in the same way as indicated before. The radiation losses caused by elliptical

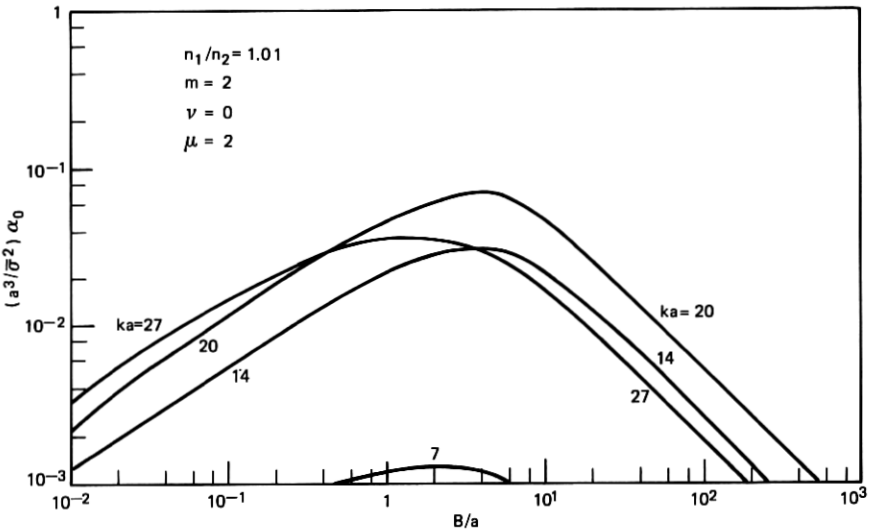

Fig. 3—HE_{11} mode radiation losses caused by elliptical deformations, $m = 2$, of the core-cladding boundary. $n_1/n_2 = 1.01$.

core-cladding interface irregularities are smaller than those of pure diameter changes.

Actual numerical values of radiation losses obtained from curves like Fig. 2 and Fig. 3 were discussed in previous publications.[1,5,7]

IX. CONCLUSIONS

We have presented a simplified theory of mode coupling in imperfect round optical fibers. The simplification was a result of restricting the discussion to fibers with small values of $n_1/n_2 - 1$. The simplified theory results in much simpler expressions for the guided and radiation modes of the fiber and consequently leads to simple expressions for the coupling coefficients. For small core-cladding index differences, the simplified theory is in excellent agreement with more general theories.

The principal contribution of this paper is a tabulation of coupling coefficients for coupling between guided and radiation modes that are necessary for solving mode coupling problems caused by general core-cladding interface imperfections. A general coupling theory based on the modes of the ideal fiber is also presented.[20]

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