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A New Approach to Optimum Pulse Shaping in Sampled Systems Using Time-Domain Filtering

By K. H. MUELLER

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A new approach to time-domain pulse shaping in digital sampled systems is described. The proposed method allows time-limited impulse responses with optimum specified energy distribution in the frequency domain to be generated. Additional constraints to guarantee zero intersymbol interference are easily taken into account. Nyquist-type pulses which have the maximum possible amount of their total energy concentrated below some given frequency are one particularly important application. An example of such an impulse response with only 6 percent excess bandwidth is presented which shows that 99.96 percent of the energy can be concentrated in the desired bandwidth with a pulse 16 baud intervals long that can be generated using a read-only memory (ROM) with only 256 bits of storage. This new class of signals can be used advantageously for waveform generation and processing in digital data systems.

I. INTRODUCTION

The joint optimization of functions in both time and frequency domain is a classical problem in communication theory. Hilberg and Rothe[1] have recently found the lowest possible product of pulse and one-sided spectral widths and have numerically evaluated the impulse and frequency response–which is not Gaussian–that corresponds to this minimum. Landau, Pollak, and Slepian[2–4] in their classical papers have derived the pulse-form of given duration that has a maximum of its energy concentrated below a certain frequency and vice versa; the solutions to this problem are given by the now well-known prolate spheroidal wave functions. Additional comments on this problem have recently been given by Hilberg.[5] A widespread opinion is that pulses with minimum energy at high frequencies should have a rounded form with many continuous derivatives. This is not true; in fact, the optimum pulses based on the prolate spheroidal wave functions are usually not continuous at the limits of their truncation interval. Hilberg[6] has shown that constraints of continuous derivatives tend to increase substantially the total out-of-band energy.

Steep spectral roll-off above the Nyquist frequency and small residual out-of-band energy are desirable properties for signals in data transmission systems to achieve maximum signaling rate over band-limited channels and to avoid fold-over distortion in modulation and demodulation. We have here, however, a very important additional constraint: The generated signal must also have negligible intersymbol interference. One method of deriving shaping filters which simultaneously minimize intersymbol interference and stopband response was proposed by Spaulding.[7] His procedure generates better results than the traditional approach of approximation to the raised cosine roll-off in the frequency domain only.

In this paper we will again carry out optimization in the frequency domain only; but we constrain the intersymbol interference to be exactly zero and we truncate the pulse duration to a chosen number of baud intervals. The impulse response is represented in sampled form. This new class of signals will have particular application in digital modem design.

II. THE SAMPLED APPROACH

A sampled Nyquist-type impulse response with samples a_i is shown in Fig. 1. For convenience, we will assume even symmetry, an integral

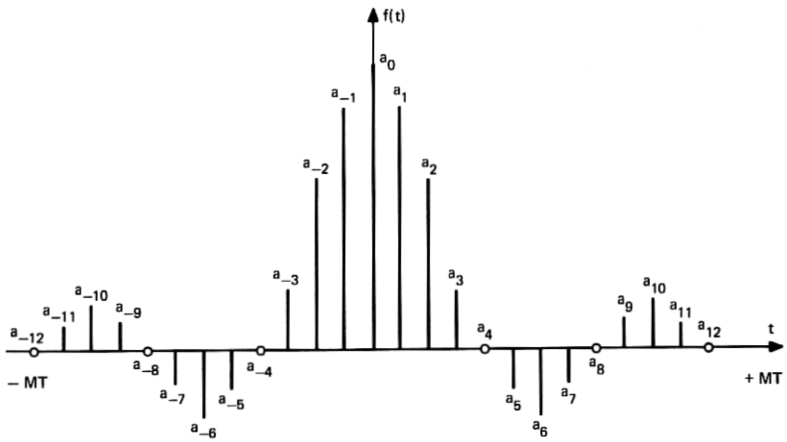

Fig. 1—Impulse response given by samples a_k. Truncation at $t = \pm MT$; sample spacing $\Delta t = T/\mu$.

number μ of samples per baud interval, and coincidence of every μth sample with the zero crossings; thus

$$\left.\begin{aligned} a_{\pm k\mu} &= 0 \qquad \text{for} \quad k = \pm 1, \pm 2, \cdots, \pm M \\ a_i &= a_{-i} \\ a_i &= 0 \qquad \text{for} \quad |i| \geqq \mu M \end{aligned}\right\}. \tag{1}$$

The resulting spectrum is

$$S(\omega) = W(\omega) \sum_{i=-\mu M}^{\mu M} a_i e^{-j\omega i T/\mu} \tag{2}$$

where $W(\omega)$ is an associated weighting function which may take into account the conversion from impulses to a staircase waveform or any other form of interpolation network. Let us define the $(2\mu M + 1)$-sample vector

$$\mathbf{a}^T = (a_{-\mu M}, \cdots, a_0, \cdots, a_{\mu M}) \tag{3}$$

and the transformation vector

$$\mathbf{p}^T = \{p_i\}, \qquad \text{with} \quad p_i = e^{-j\omega i T/\mu}, \tag{4}$$

so that we can write the spectrum in the simple form

$$S(\omega) = W(\omega)\mathbf{a}^T\mathbf{p}. \tag{5}$$

The power density spectrum is given by*

$$|S(\omega)|^2 = |W(\omega)|^2\mathbf{a}^T\mathbf{p}\mathbf{p}^\dagger\mathbf{a}. \tag{6}$$

If we assume that the function $w(t)$ has energy E_w and is nonoverlapping (width $\leqq T/\mu$), the total energy $E(\infty)$ is simply

$$E(\infty) = \mathbf{a}^T\mathbf{a}E_w. \tag{7}$$

The energy below ω_0 is of course

$$E(\omega_0) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} |W(\omega)|^2\mathbf{a}^T\mathbf{p}\mathbf{p}^\dagger\mathbf{a}\, d\omega. \tag{8}$$

Our goal is again to find $\mathbf{a}$, so that

$$\lambda = \frac{E(\omega_0)}{E(\infty)} = \max \tag{9}$$

or, by combining the last three equations,

$$\lambda\mathbf{a}^T\mathbf{a} = \mathbf{a}^T R\mathbf{a} \tag{10}$$

where the elements of the symmetric matrix R are defined by

$$r_{ik} = \frac{1}{\pi E_w}\int_0^{\omega_0} |W(\omega)|^2 \cos\left(\omega T\,\frac{i-k}{\mu}\right) d\omega \tag{11}$$

and $\mathbf{a}$ has to satisfy the constraints (1). This constraint reduces the

* A dagger is used to indicate the conjugate transpose, $\mathbf{a}^\dagger = \mathbf{a}^{*T}$.

degree of the quadratic form (10), since $a_{\mu k} = \delta_{0k}$ and thus all terms $r_{\mu i,\mu k}$ are immaterial. After elimination of the zero elements and concentration of the remaining elements, the new form

$$\lambda \hat{\mathbf{a}}^T \hat{\mathbf{a}} = \hat{\mathbf{a}}^T \hat{R} \hat{\mathbf{a}} \tag{12}$$

evolves, which is now free of constraints. The desired solution is then simply given by the eigenvector of $\hat{R}$ which corresponds to the largest eigenvalue $\lambda_{\max}$ (since no other choice of $\hat{a}$ will give a larger λ). The original vector **a** can easily be obtained by inserting zeros in the correct positions of $\hat{\mathbf{a}}$.

The elements r_{ik} in (11) will of course depend on the choice of ω_0 and numerical integration generally will be necessary to evaluate them. Because of the Toeplitz and symmetric nature of R, only a small number of terms need really be calculated. Numerical integration and determination of eigenvectors are available as subroutines with most computers, so that no complex programs need be written for the proposed optimization method. We also would like to emphasize that the described method is very flexible. One might, for example, try to minimize the energy contribution within some given frequency range $\omega_1 < \omega < \omega_2$; this may easily be achieved by changing the integration limits in (11).

Two cases of $W(\omega)$ are of practical interest. The first one is the zero-order-hold function which generates a staircase waveform (this is the usual output of D/A converters). In this case, we have*

$$W(\omega) = \frac{T}{\mu} \operatorname{sinc}\left(\frac{\omega T}{2\mu}\right) \tag{13}$$

and therefore

$$r_{ik} = \frac{1}{\mu\pi} \int_0^{\pi(1+\beta)} \operatorname{sinc}^2\left(\frac{x}{2\mu}\right) \cos\left[\frac{x}{\mu}(i-k)\right] dx \tag{14}$$

where we have expressed ω_o in terms of the normalized Nyquist excess bandwidth β.

In the second case we will assume $w(t) = \delta(t)$, so that the spectrum $W(\omega)$ is flat. Due to the periodicity of the resulting spectrum, it is reasonable to consider the energy distribution within one period only. The resulting elements of the matrix R can then be expressed in closed form

$$\left.\begin{aligned} r_{ik} &= \frac{1+\beta}{\mu} \operatorname{sinc} \pi(1+\beta)\left[\frac{i-k}{\mu}\right] \qquad && \text{if} \quad i \neq k \\ r_{ii} &= \frac{1+\beta}{\mu} && \text{if} \quad i = k \end{aligned}\right\}, \tag{15}$$

which further simplifies the optimization procedure.

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* We define $\operatorname{sinc}(x) = \sin(x)/x$ for convenience.

III. GENERALIZATION FOR ARBITRARY SPECTRAL BANDS

Equation (10) is a special case of the more general problem of maximizing the energy in one or more specified frequency bands with respect to the energy in some other frequency bands. Taking into account the desired integrating limits and the constraints (1), a quadratic form

$$\lambda \hat{\mathbf{a}}^T Q \hat{\mathbf{a}} = \hat{\mathbf{a}}^T \hat{R} \hat{\mathbf{a}} \tag{16}$$

will then evolve, containing (12) as a special case with $Q = I$. By substituting

$$\mathbf{b} = \sqrt{Q}\hat{\mathbf{a}}, \tag{17}$$

we have now to deal with the new form

$$\lambda \mathbf{b}^T \mathbf{b} = \mathbf{b}^T \sqrt{Q}^{-1} \hat{R} \sqrt{Q}^{-1} \mathbf{b}, \tag{18}$$

which is identical to (12). We are looking for the particular $\mathbf{b}$ satisfying

$$\sqrt{Q}^{-1} \hat{R} Q^{-1} \mathbf{b} = \lambda \mathbf{b}. \tag{19}$$

By premultiplying both sides with $\sqrt{Q}^{-1}$, we get

$$Q^{-1} \hat{R} \hat{\mathbf{a}} = \lambda \hat{\mathbf{a}}, \tag{20}$$

so that the desired $\hat{\mathbf{a}}$ is simply the eigenvector of $Q^{-1}R$ which corresponds to the largest eigenvalue $\lambda_{\max}$.* The matrix Q is guaranteed to be nonsingular since it is not possible to have zero energy in a finite frequency interval with a time-truncated impulse response.

IV. EXAMPLE

To get some feeling for the capabilities of the described optimization procedure, the samples of a Nyquist-type impulse response were calculated using the following parameters:

Excess bandwidth factor $\beta = 0.06$
Truncation for $|t| > 8T$ ($M = 8$)
$\mu = 4$ samples per baud interval.

The unusually tight roll-off would allow full 4800-baud operation over voice-grade telephone channels with a QAM or a VSB system. If the sample values are coded into 8 bits plus sign, the chosen resolution will bring down the quantization noise to a negligible level of −65 dB. The storage requirement is still only 256 bits, so a rather small bipolar ROM may be used.

* Note that $Q^{-1}\hat{R}$ need not be symmetric, but its eigenvalues are the same as those of the symmetric matrix in (19).

The resulting spectrum is shown in Fig. 2. Attenuation is 6 dB at the Nyquist frequency and 17.6 dB at the 6-percent edge. The sinc(·) weighting caused by the staircase output is not included; it would produce additional attenuation at higher frequencies. The resulting eigenvalue was $\lambda_{max} = 0.99963$, showing that in fact the residual out-of-band energy is very small.

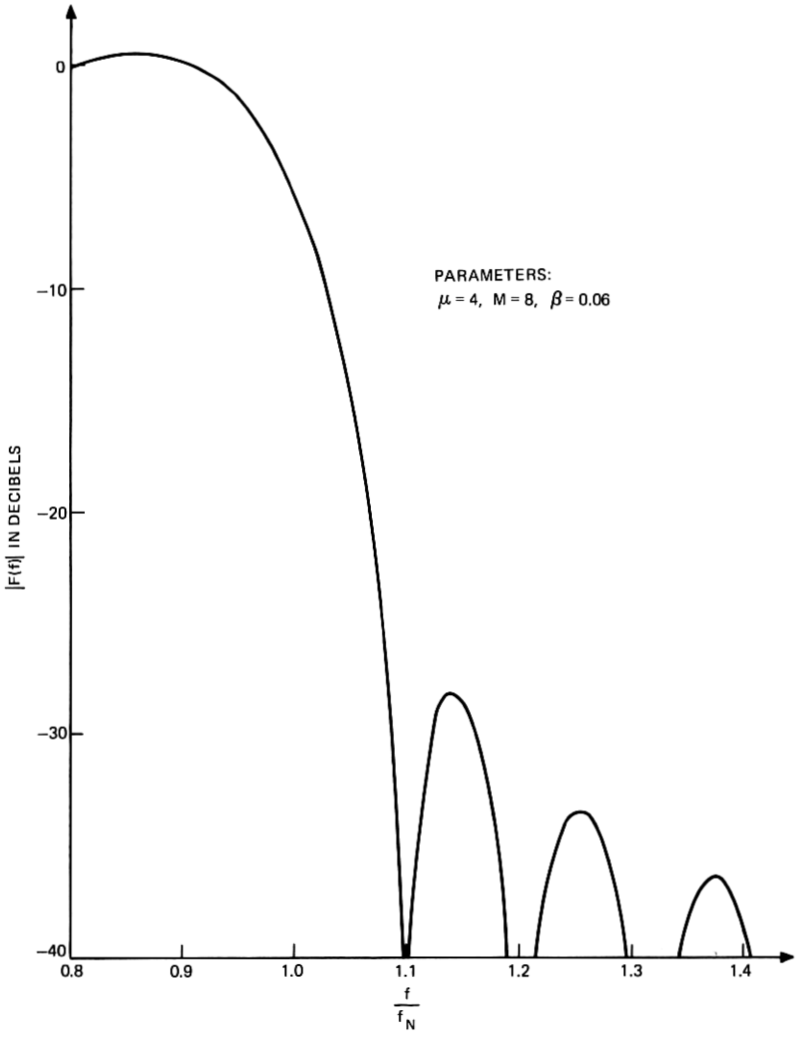

Fig. 2—Spectrum of optimized impulse response with $\lambda = 0.99963$.

V. CONCLUSIONS

A new optimization method for sampled Nyquist-type impulse responses has been proposed. Minimum energy in one frequency band as compared to the energy in any other frequency band is achieved. The computation is straightforward and involves the determination of eigenvectors of a symmetric matrix. It is shown how the constraint for zero intersymbol interference can easily be included. Applications of this method are numerous in digital signal synthesis and processing. Storage can be achieved with high accuracy using ROM's of moderate size. Any desired scaling of time and frequency response is possible with such a system and the well-known disadvantages of traditional filters, namely aging and tuning, are nonexistent.

VI. ACKNOWLEDGMENTS

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