THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 52, No. 5, May–June, 1973
Printed in U.S.A.

Semilattice Characterization of Nonblocking Networks

By V. E. BENEŠ

(Manuscript received August 30, 1972)

A connecting network is called strictly nonblocking if no call is blocked in any state; it is nonblocking in the wide sense if there exists a rule for routing calls through the network so as to avoid all states in which calls are blocked, and yet still satisfy all demands for connection as they arise, without disturbing calls already present. Characterizations of both senses of nonblocking have been given in previous work, using simple metric and closure topologies defined on the set of states. We give new characterizations based on the natural map $\gamma(\cdot)$ *that carries each state into the assignment it satisfies. This map is a semilattice homomorphism, such that* $\gamma(x) \cap \gamma(y) \geqq \gamma(x \cap y)$. *It turns out that the case of equality in this inequality is very relevant to nonblocking performance. In particular, let a subset* X *of states be said to have the intersection property if for every* x *in* X *and every assignment* a *there exists* y *in* X *such that* y *realizes* a *(i.e.,* $\gamma(y) = a$*) and* $\gamma(x \cap y) = \gamma(x) \cap \gamma(y)$. *Then a network is nonblocking in the wide sense if and only if some subset of its states has the intersection property, and it is strictly nonblocking if and only if the entire set of states has the intersection property.*

I. INTRODUCTION

In a nonblocking network, no call need be lost because of link mismatch or junctor unavailability. Efficient nonblocking networks were invented by Charles Clos and, although they are not in common use at the present time, they are distinct possibilities for practical applications in the future, and they have substantial theoretical interest as outer limits on possible designs.

Two degrees or strengths of the nonblocking property have been distinguished.[1,2] A connecting network is called *strictly* nonblocking if no call is blocked in any state; it is nonblocking *in the wide sense* if there exists a rule for routing calls through the network so as to avoid all states in which calls are blocked, and yet still satisfy all demands for connection as they arise, without disturbing calls already in progress. These properties have been given[1,2] topological characterizations, and examples of each are known, although it must be said that examples of *efficient* wide-sense nonblocking networks are yet to be found.

Our aim in this paper is to give new alternative characterizations of the nonblocking properties in terms of the semilattice structures of the set of network states and of the set of assignments the states realize; a key role is played by the homomorphism $\gamma(\cdot)$ that carries each state into the assignment it realizes.

The property of being nonblocking in the wide sense lies between two other properties: that of being strictly nonblocking (nonblocking in the strict sense) and that of being rearrangeable. In a strictly nonblocking network, no call is blocked in any state; in a rearrangeable network, calls can always be given new routes (rearranged) so as to unblock any blocked call. The three properties (along, doubtless, with others not yet studied) form a spectrum of possible ways of operating switching equipment that exhibits or summarizes the tradeoff obtainable between efficient usage of switches and amount of calculation: the richer the network is in crosspoints, the less one has to do to use it so as to achieve desired load and loss. In a strictly nonblocking network, any path for an idle call will do; there always is one, and no traffic advantage is gained by use of one rather than another. In a wide-sense nonblocking network, the right choice of a path may mean the difference between zero loss and blocking some calls. By calculation, though, one can always find a route that will result in no blocking. In a rearrangeable network, finally, nonblocking behavior is again attainable, but, in general, only at the cost of constantly recalculating new routes for all the desired calls simultaneously, and reswitching them as necessary.

II. PRELIMINARIES

We shall use a model for the combinatorial aspects of a connecting network. This model is called a semilattice,[3] or partially ordered system with intersections, and it can be thought of as arising as follows: a connecting network ν is a quadruple $\nu = (G, I, \Omega, S)$ where G is a graph depicting network structure, I is the set of nodes of G which are inlets, Ω is the set of nodes of G that are outlets, and S is the set of permitted states. Variables w, x, y, and z at the end of the alphabet denote states, while u and v denote a typical inlet and a typical outlet, respectively. A state x can be thought of as a set of disjoint chains on G, each chain joining I to Ω. Not every such set of chains represents a state: sets with wastefully circuitous chains may be excluded from S. It is possible that $I = \Omega$ (one-sided network), that $I \cap \Omega = \phi$ (two-sided network), or that some intermediate condition obtain, depending on the "community of interest" aspects of the network ν.

The set S of states is partially ordered by inclusion $\leqq$, where $x \leqq y$ means that state x can be obtained from state y by removing zero or more calls. If x and y satisfy the same assignment of inlets to outlets, i.e., are such that all and only those inlets $u \epsilon I$ are connected in x to outlets $v \epsilon \Omega$ which are connected to the same v in y (though possibly by different routes), then we say that x and y are equivalent, written $x \sim y$.

We denote by A_x the set of states that are immediately above x in the partial ordering $\leqq$, and by B_x the set of those that are immediately below. Thus

$$A_x = \{\text{states accessible from } x \text{ by adding a call}\}$$
$$B_x = \{\text{states accessible from } x \text{ by a hangup}\}.$$

It can be seen, further, that the set S of states is not merely partially ordered by $\leqq$, but also forms a semilattice, or a partially ordered system with intersections,[3] with $x \cap y$ defined to be the state consisting of those calls and their respective routes which are common to both x and y.

An *assignment* is a specification of what inlets should be connected to what outlets. The set A of assignments can be represented as the set of all fixed-point-free correspondences from subsets of I to Ω. The set A is partially ordered by inclusion, and there is a natural map $\gamma(\cdot): S \rightarrow A$ which takes each state $x \epsilon S$ into the assignment it realizes; the map $\gamma(\cdot)$ is a semilattice homomorphism of S into A, with the properties

$$x \geqq y \Rightarrow \gamma(x) \geqq \gamma(y),$$
$$x \geqq y \Rightarrow \gamma(x - y) = \gamma(x) - \gamma(y),$$
$$\gamma(x \cap y) \leqq \gamma(x) \cap \gamma(y),$$
$$\gamma(x) = \phi \Rightarrow x = 0 = \text{zero state, with no calls up.}$$

Variables a, b are used for members of A.

A *unit* assignment is, naturally, one that assigns exactly one inlet to some one outlet, and it corresponds to having just one call in progress. It is convenient to identify new calls c and unit assignments, and to write $\gamma(x) \cup c$ for the larger assignment consisting of $\gamma(x)$ and the call c together, with the understanding of course that none of the terminals of c is busy in $\gamma(x)$. Not every assignment need be realizable by some state of S. Indeed, it is common for practical networks to realize only a small fraction of the possible assignments.

A simple pseudometric topology on S is defined by the "distance" formula

$$d(x,y) = |\gamma(x)\Delta\gamma(y)|$$

where Δ denotes the symmetric difference, and $|\cdot|$ cardinality, of sets. The distance between states $d(x,y)$ is the number of pairs $(u,v) \in I \times \Omega$ that are either connected in x and not in y, or connected in y and not in x. Clearly, $d(x,y) = 0$ if and only if $x \sim y$, and the d-closure of a set X is just

$$X^d = \{y: y \sim x \text{ for some } x \in X\}.$$

A set X is dense in a set Y in the d-topology iff

$$Y \subseteq X^d.$$

III. INTERSECTION PROPERTY

We shall introduce a property of subsets X of the set S of states, called the *intersection property*, and then show that a network ν is nonblocking in the wide sense if and only if some subset of S has the intersection property. We call it the intersection property because it involves the equality case

$$\gamma(x \cap y) = \gamma(x) \cap \gamma(y) \tag{1}$$

of the semilattice homomorphism inequality

$$\gamma(x \cap y) \leqq \gamma(x) \cap \gamma(y); \tag{2}$$

the latter is *always* true. Our result therefore says roughly that if equality in (2) holds for enough states, then ν is wide-sense nonblocking and this condition is necessary.

A subset $X \subseteq S$ is said to have the *intersection property* if and only if for every $x \in X$ and every $a \in A$, there exists $y \in X$ such that $\gamma(y) = a$ and

$$\gamma(y) \cap \gamma(x) = \gamma(x \cap y).$$

A subset $X \subseteq S$ is *closed below* if $x \in X$ and $y \leqq x$ imply $y \in X$. The *lower closure* of a subset X is the set $\mathbf{X} = \{y: y \in S \text{ and } y \leqq x$ for some

$x \in X\}$; this is just all the states reachable from a member of X by hangups.

Our first result is an important lemma to the effect that the intersection property is preserved by lower closure.

Lemma 1: If X has the intersection property, then so does $\mathbf{X}$.

Proof: Take $x \in \mathbf{X}$, $a \in A$. We are to find $y \in \mathbf{X}$ such that $\gamma(y) = a$ and $\gamma(y \cap x) = \gamma(y) \cap \gamma(x)$. Since $x \in \mathbf{X}$, there is a $z \in X$ such that $x \leqq z$. X has the intersection property, so there is a $w \in X$ such that $\gamma(w) = a$ and

$$\gamma(w \cap z) = \gamma(w) \cap \gamma(z). \tag{3}$$

We show that we can choose y to be w. Obviously $w \in \mathbf{X}$ and $\gamma(w) = a$. Also, intersecting (3) with $\gamma(x)$ we find

$$\gamma(x) \cap \gamma(w \cap z) = \gamma(x) \cap \gamma(w) \cap \gamma(z).$$

Since $x \leqq z$, we have $\gamma(x) \leqq \gamma(z)$, $\gamma(x) \cap \gamma(z) = \gamma(x)$, and so the right-hand side is just $\gamma(w) \cap \gamma(x)$. The left-hand side consists of calls which are in progress in x, and are also in progress in both w and z, on the same routes in each. Since $x \leqq z$, these must use the same routes in x as they do in z and w. Thus the left-hand side comprises exactly those calls which are in progress in each of z, w, and x, on the same routes in each, namely $\gamma(z \cap w \cap x)$. This equals $\gamma(x \cap w)$ because $x \leqq z$. Thus

$$\begin{aligned}\gamma(x \cap w) &= \gamma(z \cap w \cap x)\\ &= \gamma(x) \cap \gamma(z \cap w)\\ &= \gamma(x) \cap \gamma(z) \cap \gamma(w)\\ &= \gamma(x) \cap \gamma(w),\end{aligned}$$

and this proves the Lemma 1. Our next result notes that a subset X having the intersection property must lie entirely in the set N of states in which no call is blocked.

Lemma 2: If X has the intersection property and $x \in X$, then no call idle in x is blocked in x, i.e., $X \subseteq N$.

Proof: Let $x \in X$, c idle in x, $a = \gamma(x) \cup c$. Then there is a $y \in X$ such that $\gamma(y) = a$ and $\gamma(x \cap y) = \gamma(x) \cap \gamma(y)$. Thus the calls in progress in both x and y, and on the same routes in each, are all and only the calls up in x. Hence $x \cap y = x$, or $x \leqq y$, so that $y \in A_x$, and c is not blocked in x. Thus $X \subseteq N$.

IV. WIDE-SENSE NONBLOCKING NETWORKS

We shall need a lemma that identifies the intersection of two states:

Lemma 3: If $z \leqq x$, $z \leqq y$, *and* $\gamma(z) = \gamma(x) \cap \gamma(y)$, *then* $z = x \cap y$.

Proof: The hypothesis implies that

$$\gamma(x - z) = \gamma(x) - \gamma(z) = \gamma(x) - [\gamma(x) \cap \gamma(y)]$$
$$\gamma(y - z) = \gamma(y) - \gamma(z) = \gamma(y) - [\gamma(x) \cap \gamma(y)].$$

The right-hand sides are disjoint, so $\gamma(x - z) \cap \gamma(y - z) = \phi$, and the homomorphism inequality for $\gamma(\cdot)$ gives $\gamma[(x - z) \cap (y - z)] = \phi$, whence $(x - z) \cap (y - z) = 0$. Since z is included in each of x and y, we have

$$x = z \cup (x - z), \qquad y = z \cup (y - z)$$
$$x \cap y = z \cup z(y - z) \cup (x - z)z \cup (x - z)(y - z)$$

(here we have used a more convenient notation for intersection on the right-hand side). The last three terms on the right vanish, so $x \cap y = z$.

The following characterization of wide-sense nonblocking was given in an earlier work:[1]

Theorem 1: ν *is nonblocking in the wide sense* iff *there exists a subset* $X \subseteq N$ *with* $X = \mathbf{X}$, *and such that for every* $x \in X$, $A_x \cap X$ *is d-dense in* A_x, *i.e.,* $A_x \subseteq (A_x \cap X)^d$.

The principal new result is now proved. It is

Theorem 2: ν *is nonblocking in the wide sense* iff *some subset* $X \subseteq S$ *has the intersection property.*

Proof (sufficiency): By Lemmas 1 and 2 we can assume that X is closed below, and that $X \subseteq N$. By Theorem 1 it is enough to prove that for every $x \in X$, $A_x \cap X$ is d-dense in A_x, i.e.,

$$A_x \subseteq (A_x \cap X)^d, \quad x \in X. \tag{4}$$

Let $x \in X$ and $z \in A_x$. There exists then $y \in X$ such that $\gamma(y) = \gamma(z)$ and $\gamma(x \cap y) = \gamma(x) \cap \gamma(y)$; thus also

$$\gamma(x \cap y) = \gamma(x) \cap \gamma(z) = \gamma(x),$$

the second equality following from $x \leqq z$. As in Lemma 2, we conclude from $\gamma(x \cap y) = \gamma(x)$ that $y \in A_x$. Then $y \in A_x \cap X$ and $y \sim z$, or

$z \in (A_x \cap X)^d$. Since z was an arbitrary state in A_x, we have shown (4), and so the sufficiency.

Proof (necessity): Since ν is nonblocking in the wide sense, there exists by Theorem 1 a subset X of states which is closed below, is contained in N, and is such that any call new in a state of X can be put up *salva* staying in X. We show that X has the intersection property. Let then $x \in X$ and $a \in A$. Obtain a state $z \leqq x$ by removing from x all the calls that are not part of the assignment $\gamma(x) \cap a$. Next, starting at z, put up the (additional) calls comprising $a - \gamma(z)$ so as to reach a state $y \in X$ with $\gamma(y) = a$. This is possible because any call new in a state of X can be put up so as to keep the system in X. We now claim that

$$\gamma(x \cap y) = \gamma(x) \cap \gamma(y).$$

Since $z \leqq x$, $z \leqq y$, this follows from Lemma 3 as soon as we prove that $\gamma(z) = \gamma(x) \cap \gamma(y)$. To see this, note that $\gamma(z) \leqq \gamma(x)$ and $\gamma(z) \leqq \gamma(y)$, so that $\gamma(z) \leqq \gamma(x) \cap \gamma(y)$. Conversely, by construction, any call up in both x and y is either up in z (never having been disturbed), or else was taken out to reach z and then put back up. However, only calls not up in a were taken down, and only calls up in a were put back. Thus the second alternative is ruled out, and any call up in both x and y is up in z, i.e., $\gamma(x) \cap \gamma(y) = \gamma(z)$. Lemma 3 now implies that $z = x \cap y$, so that

$$\begin{aligned} \gamma(x) \cap \gamma(y) &= \gamma(z) \\ &= \gamma(x \cap y). \end{aligned}$$

Hence X has the intersection property, as claimed.

V. STRICTLY NONBLOCKING NETWORKS

Because of Lemma 2, the intersection property can also be used to characterize the property of being strictly nonblocking, as is shown by the following result:

Theorem 3: ν is strictly nonblocking iff *(the set of states) S has the intersection property.*

Proof: Sufficiency is obvious, by Lemma 2. Conversely, if ν is strictly nonblocking, then $\gamma(S) = A$ and

$$A_x \subseteq (A_x \cap S)^d, \quad \text{for every } x \in S.$$

Thus ν is nonblocking in the wide sense; indeed, trivially, S has the property that any call new in a state of S can be put up *salva* staying in S. The necessity argument of Theorem 2 now shows that S has the intersection property.

VI. COMPARISON, EMBEDDING, AND ISOMORPHISM

We next relate the intersection property to a certain partial ordering $\leqq$ of *networks*, introduced in an earlier work,[4] and used there for clarifying some problems of comparison of networks. This partial ordering was defined over the set $N(I,\Omega)$ of all networks $\nu = (G, I, \Omega, S)$ for which the set I of inlets and the set Ω of outlets are fixed, while the graph G and the set S of states may vary in any way consistent with their defining a network in the sense of Ref. 2.

$N(I,\Omega)$ is partially ordered by the following relation $\leqq$: $\nu_1 \leqq \nu_2$ iff $\exists$ domain $D \subseteq S(\nu_1)$ and an onto map μ: $D \to S(\nu_2)$ such that D is closed below and

(*i*) μ preserves assignments: $\gamma(\mu x) = \gamma(x)$

(*ii*) $x, y \in D, \mu x \geqq \mu y \Rightarrow x \geqq y$.

The relationship $\nu_1 \leqq \nu_2$ means intuitively that one can mimic ν_2 within ν_1. That this is so is not obvious. Indeed, using the notion of isomorphism as a precision of the mimicry in question, it has been proved[4] that $\nu_1 \leqq \nu_2$ if and only if there is an isomorph of ν_2 in ν_1. Roughly, in the definition, μ maps the states of ν_1 doing the mimicking *onto* $S(\nu_2)$; it tells what state mimics what. Condition (*i*) then naturally states that the mimicked state satisfies the same assignment. Condition (*ii*), finally, insists that mimicry preserve inclusion, in the sense that only states x, y with $x \geqq y$ can mimic similarly related states μx, μy.

Remark: In the definition of the partial ordering $\leqq$ for the set $N(I,\Omega)$, the condition $D = \mathbf{D}$, that the domain of the map μ be closed below, may be dropped, because it is implied by the other conditions. To see this, let D, μ be as in the definition of $\leqq$ except omit $D = \mathbf{D}$, and take $x \in D$, $y \leqq x$. We show $y \in D$. Clearly, $\gamma(y) \leqq \gamma(x) = \gamma(\mu x)$, so there is a state $z \in \mu(D)$ with $z \leqq \mu x$ and $\gamma(z) = \gamma(y)$, because $\mu(D)$ is closed below, since μ is onto. Hence there exists $w \in D$ with $z = \mu w$. Thus $\mu w \leqq \mu x$, so by the second property of μ, $w \leqq x$. We now have $y \leqq x$, $w \leqq x$, $\gamma(y) = \gamma(w)$. This implies $y = w$ and so $y \in D$, because a state x can have below it at most one state satisfying a given assignment.

Theorem 4: ν is nonblocking in the wide sense iff $\exists \nu_1$, $\nu \leqq \nu_1$ *and* ν_1 *is strictly nonblocking.*

Proof: If $\nu \leqq \nu_1$, there is a domain $D \subseteq S(\nu)$ and an onto map $\mu: D \to S(\nu_1)$ such that $\gamma(\mu x) = \gamma(x)$, and $\mu x \geqq \mu y$ implies $x \geqq y$. We show that D has the intersection property. Take $x \in D$ and $a \in A$ and focus on $\mu x \in S(\nu_1)$. Clearly, since ν_1 is strictly nonblocking, there exists a state y of ν_1 with $\gamma(y) = a$ and

$$\gamma(\mu x) \cap \gamma(y) = \gamma(\mu x \cap y).$$

(It suffices to take down the calls in x not up in a, and then put up the ones in a not up in x.) Since μ is onto, $y = \mu z$ for some $z \in D$, with $a = \gamma(y) = \gamma(z)$. Since $\gamma(\mu x) = \gamma(x)$, we have

$$\gamma(x) \cap \gamma(z) = \gamma(\mu x \cap \mu z).$$

Since μx and μz are both states of ν_1, so is $\mu x \cap \mu z$; there is a state $w \in D$ with $\mu w = \mu x \cap \mu z$, since μ is onto. Now note that $\mu x \geqq \mu w$ and $\mu y \geqq \mu w$, so that the second property of μ implies $x \geqq w$ and $y \geqq w$. Together with $\gamma(x) \cap \gamma(z) = \gamma(w)$ this implies by Lemma 3 that $w = x \cap z$, and so

$$\gamma(x) \cap \gamma(z) = \gamma(x \cap z).$$

Thus D has the intersection property, and so ν is wide-sense nonblocking, by Theorem 2. Conversely, if ν is nonblocking in the wide sense, there is a subset X of S with the intersection property. Define ν_1 by

$$\nu_1 = (G, I, \Omega, X).$$

Taking $D = X$ and $\mu =$ identity, we conclude $\nu \leqq \nu_1$; Lemma 2 implies that ν_1 is nonblocking, and Theorem 3 is proved.

Our intuitive feeling is that a wide-sense nonblocking network has embedded in it a largest strictly nonblocking network, to whose states the system is restricted by any rule for routing that guarantees no blocking. An appropriate sense of "embedded" is provided by the concept of isomorphism.[3] An isomorphism between two partially ordered systems is a one-to-one correspondence that preserves order in both directions. An isomorph of ν_2 within ν_1 would be a subset $M \subseteq S(\nu_1)$ and a correspondence $i: M \leftrightarrow S(\nu_2)$ such that $x \geqq y$ iff $ix \geqq iy$. In Ref. 4, the existence of an isomorph was related to the partial ordering $\leqq$ for networks, by this result:

Theorem 5: $\nu_1 \leqq \nu_2$ iff $\exists M \subseteq S(\nu_1)$, $\exists$ *correspondence* $i\colon M \leftrightarrow S(\nu_2)$ *such that*

(*i*) $\gamma(ix) = \gamma(x)$
(*ii*) $x \geqq y$ iff $ix \geqq iy$.

Therefore Theorem 4 can be rephrased as

Theorem 6: ν *is nonblocking in the wide sense if and only if there is an isomorph of a nonblocking network embedded in* $S(\nu)$, *and the isomorphism preserves* $\gamma(\cdot)$.

This is a precise form of the intuitive feeling voiced above.

Note added in proof: It should be noticed that Theorems 4 and 6 imply that the quest (mentioned at the top of p. 698) for *efficient* wide-sense nonblocking networks is in a sense vain: there is no "intermediate" amount of switching equipment that will give wide-sense nonblocking behavior but is not so expensive as (it would have to be to give) a strictly nonblocking network; as soon as you have a wide-sense nonblocking network, you have at most to throw away some states to obtain a strictly nonblocking one.

REFERENCES

1. Beneš, V. E., "Algebraic and Topological Properties of Connecting Networks," B.S.T.J., *41*, No. 4 (July 1962), pp. 1249–1274.
2. Beneš, V. E., *Mathematical Theory of Connecting Networks and Telephone Traffic*, New York: Academic Press, 1965.
3. Birkhoff, G., *Lattice Theory*, Am. Math. Soc. Colloq. Publ. (rev. ed.) *XXV* (1948).
4. Beneš, V. E., "On Comparing Connecting Networks," Journal of Combinatorial Theory, *2*, No. 4 (June 1967), pp. 507–513.