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A Volterra Series Description of Crosstalk Interference in Communications Systems

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This paper studies a general description of interchannel and intrachannel crosstalk interference created in a communications system. This description is in the form of a Volterra series expansion of the interference signal in terms of the signal which produced the interference. From it we are able to precisely define the "intelligible" part of the crosstalk. This description also provides us with quantitative measures of the amount of crosstalk created in some communications channel by signals in another channel, as well as a measure (intelligible crosstalk ratio) of the amount of intelligible crosstalk produced. We then consider a particular model for the generation of intelligible crosstalk [or direct adjacent channel interference (DACI)] between two neighboring angle-modulated channels in which the signal in one channel adds to the signal of the second channel, the sum is filtered, and the filter output then passes through an AM-PM conversion device. Using our definition, a simple expression for the intelligible crosstalk ratio for this model is derived in terms of the filter characteristic. We observe that this crosstalk ratio exhibits a number of properties usually associated with DACI.

I. INTRODUCTION

Crosstalk interference is an important consideration in transmission system engineering.[1] It is defined[2] as the disturbance created in one (desired) communications channel† by the signals in another (interfering) communications channel. Crosstalk is classified as due to interchannel or intrachannel effects and may be of either intelligible or unintelligible type. *Interchannel* crosstalk occurs between two different communications channels as, for example, when the transmitted signals of an interfering channel pass through the channel selectivity filters of

† Here "channels" refer to different communications paths (which are distinguished by, e.g., different frequency bands or different physical transmission media) together with the receivers associated with each of these paths.

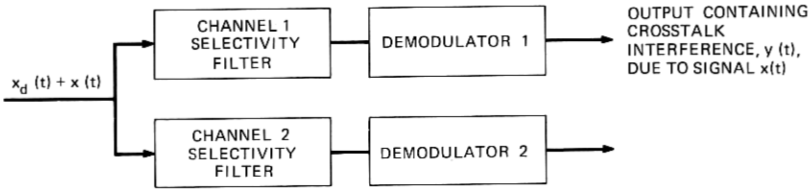

Fig. 1—Example of interchannel crosstalk. $x_d(t)$ is the desired signal in channel 1. $x(t)$, the desired signal in channel 2, creates interference in channel 1.

the desired channel and appear at its output (see Fig. 1). Another cause of interchannel crosstalk is electrical coupling between various transmission media, e.g., between wire pairs in a multipair cable. *Intrachannel* crosstalk occurs in a single communications channel and is due to nonlinearities in the receiver which act on the received signal to produce some disturbing signals in addition to the desired (linear) signal. Intrachannel crosstalk is also known as "intermodulation distortion."[3] If the signals in the channels are speech signals, crosstalk interference is described as *intelligible* or *unintelligible*, depending on whether the created interference is "understandable" or not. These terms are also applied to nonspeech signals, in which case intelligible means that the crosstalk is of "the same type as the desired signal."[2]

In this paper, we study a general mathematical technique which can be used to describe interchannel and intrachannel crosstalk created in a communications system. The description is in the form of a Volterra series expansion[4] of the interference signal in terms of the signal which produced the interference. This expansion furnishes some insight into which part of the total crosstalk interference is intelligible, and thus we will be able to precisely define what is meant by intelligible crosstalk. In this way, some of the subjectivity inherent in the earlier "definition" of intelligible crosstalk is removed. In addition, our description will provide quantitative measures of the amount of crosstalk created in some communications channel by signals in another channel, as well as a measure of the amount of intelligible crosstalk produced. The latter quantity will be called the *intelligible crosstalk ratio*. These measures may be valuable tools in systems design applications.

The Volterra series analysis of nonlinear systems with memory was first introduced by Wiener[5] and was further developed by Bedrosian and Rice.[4] In Section II, we discuss some definitions and results of this theory which will be needed in our analysis. A general description of crosstalk interference and a definition of intelligible crosstalk are given in Section III. We also define the intelligible crosstalk ratio in this section and compare it with previous measures of intelligible crosstalk. As an application of these results, we consider an example in Section IV of a model for the generation of intelligible crosstalk [or direct adjacent channel interference (DACI)[6]] between two neighboring angle-modulated channels in which the signal in one channel adds to the signal of the second channel, the sum is filtered, and the filter output then passes through an AM-PM conversion device. Using our definition, a simple expression for the intelligible crosstalk ratio for this model is derived in terms of the filter characteristic. We will see that this crosstalk ratio exhibits a number of properties usually associated with DACI. We conclude by calculating the crosstalk ratio for the case of a k-pole filter.

II. VOLTERRA SERIES ANALYSIS

In this section, we will discuss some definitions and results in the Volterra series analysis of nonlinear systems with memory. These results will be needed in the sequel. The reader is referred to Bedrosian and Rice[4] for a complete account of the theory of Volterra series as well as their application to the analysis of PM and other nonlinear systems.

For any two signals $y(t)$ and $x(t)$, possibly complex-valued, we will say that *$y(t)$ has a generalized Volterra series (GVS) expansion in terms of $x(t)$ with Volterra kernels (functions)* $\{g_n^{y,x}\}$ if and only if we can write:

$$y(t) = g_0^{y,x} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} du_1 \cdots du_n \, g_n^{y,x}(u_1, \cdots, u_n) \cdot \prod_{r=1}^{n} x(t - u_r) \quad (1)$$

$$= g_0^{y,x} + \int_{-\infty}^{\infty} du_1 \, g_1^{y,x}(u_1) x(t - u_1)$$

$$+ \frac{1}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du_1 \, du_2 \, g_2^{y,x}(u_1, u_2) x(t - u_1) x(t - u_2)$$

$$+ \frac{1}{3!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du_1 \, du_2 \, du_3 \, g_3^{y,x}(u_1, u_2, u_3) x(t - u_1) \cdot x(t - u_2) x(t - u_3) + \cdots$$

where the functions $g_n^{y,x}$, $n \geqq 1$, are symmetric functions of n variables and $g_0^{y,x}$ is a constant. For convenience, we denote this fact by the notation $y(t) = \mathrm{GVS}[x(t); \{g_n^{y,x}\}]$. If $x(t)$ is the input to a system and $y(t)$ is its output, then the Volterra kernels $\{g_n^{y,x}\}$ completely characterize the system. If $g_0^{y,x} = a_0$ and $g_n^{y,x}(u_1, \cdots, u_n) = a_n \, \delta(u_1) \cdots \delta(u_n)$ for $n \geqq 1$ where $\delta(u)$ is the delta function, then

$$y(t) = \sum_{n=0}^{\infty} a_n \frac{[x(t)]^n}{n!}$$

represents the input-output relationship of a memoryless system.

The n-fold ($n \geqq 1$) Fourier transform of $g_n^{y,x}(u_1, \cdots, u_n)$ is denoted by:

$$G_n^{y,x}(f_1, \cdots, f_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} du_1 \cdots du_n \, g_n^{y,x}(u_1, \cdots, u_n) \cdot \exp\left[-j(\omega_1 u_1 + \cdots + \omega_n u_n)\right] \quad (2)$$

where $\omega_i = 2\pi f_i$, $i = 1, 2, \cdots$. Observe that if $g_0^{y,x} = 0$ and $g_n^{y,x} \equiv 0$ for $n \geqq 2$, then $g_1^{y,x}(u_1)$ is the familiar impulse response of a linear time-invariant system and $G_1^{y,x}(f_1)$ is its transfer function. By analogy, we will call $G_n^{y,x}(f_1, \cdots, f_n)$ the *nth order Volterra transfer function*. Since $\{g_n^{y,x}\}$ are symmetric functions, then so are $\{G_n^{y,x}\}$.

If $x(t)$ has Fourier transform $X(f)$, i.e.,

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \qquad \omega = 2\pi f,$$

then it is easy to see[4] that $y(t)$ and its Fourier transform $Y(f)$ are given by:

$$y(t) = g_0^{y,x} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} df_1 \cdots df_n \, G_n^{y,x}(f_1, \cdots, f_n) \cdot e^{j(\omega_1 + \cdots + \omega_n)t} \prod_{r=1}^{n} X(f_r) \quad (3)$$

and

$$\begin{aligned} Y(f) = {} & g_0^{y,x}\delta(f) + \frac{1}{1!} G_1^{y,x}(f) X(f) \\ & + \frac{1}{2!} \int_{-\infty}^{\infty} df_1 \, G_2^{y,x}(f_1, f - f_1) X(f_1) X(f - f_1) \\ & + \frac{1}{3!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} df_1 \, df_2 \, G_3^{y,x}(f_1, f_2, f - f_1 - f_2) X(f_1) X(f_2) \\ & \cdot X(f - f_1 - f_2) + \cdots . \end{aligned} \quad (4)$$

Next, suppose we apply a harmonic input of the form $\sum_{i=1}^{n} e^{j\omega_i t}$ to a system whose input and output are related by a generalized Volterra series expansion. Then the output of the system is an infinite series of harmonic terms. The following property, which is easy to demonstrate,[4] shows that the coefficients of these harmonic terms are the Volterra transfer functions of various orders.

Property 1: Suppose $y(t) = \text{GVS}[x(t); \{g_n^{y,x}\}]$. If $x(t) = \sum_{i=1}^{n} e^{j\omega_i t}$ where $\omega_i = 2\pi f_i$, $i = 1, \cdots, n$, and $\{f_i\}$ are incommensurable,† then for

† Frequencies $f_1, \cdots, f_n$ are said to be *incommensurable* if for any integers $m_1, \cdots, m_n$, not all zero, $m_1 f_1 + \cdots + m_n f_n \neq 0$.

$n \geqq 1$ and for $k \leqq n$:

$G_k^{y,x}(f_1, \cdots, f_k)$ = the coefficient of the $\exp[j(\omega_1 + \cdots + \omega_k)t]$ term in the expansion of $y(t)$.

Methods of measuring the Volterra kernels and transfer functions of a system having a Volterra series representation have been studied by George,[7] Schetzen,[8] and Lee and Schetzen.[9] These methods rely on the use of realizable input probing signals.

Bedrosian and Rice[4] have also shown the following:

Property 2: Suppose $y(t) = \text{GVS}[x(t); \{g_n^{y,x}\}]$. If $x(t) = P\cos\omega t$, $\omega = 2\pi f$, then

$$y(t) = g_0^{y,x} + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \left(\frac{P}{2}\right)^n \frac{\exp[j(2k-n)\omega t]}{k!(n-k)!} G_{k,n-k}^{y,x}(f) \tag{5}$$

where $G_{k,n-k}^{y,x}(f)$ denotes $G_n^{y,x}(f_1, \cdots, f_n)$ with the first k of the f_i's equal to f and the remaining $n - k$ equal to $-f$. The leading terms in (5) are:

$$\begin{aligned} y(t) = & \left[\frac{P^2}{4} G_2^{y,x}(f, -f) + \cdots\right] \\ & + e^{j\omega t}\left[\frac{P}{2} G_1^{y,x}(f) + \frac{P^3}{16} G_3^{y,x}(f, f, -f) + \cdots\right] \\ & + e^{-j\omega t}\left[\frac{P}{2} G_1^{y,x}(-f) + \frac{P^3}{16} G_3^{y,x}(-f, -f, f) + \cdots\right] \\ & + e^{j2\omega t}\left[\frac{P^2}{8} G_2^{y,x}(f, f) + \cdots\right] \\ & + e^{-j2\omega t}\left[\frac{P^2}{8} G_2^{y,x}(-f, -f) + \cdots\right] \\ & + e^{j3\omega t}\left[\frac{P^3}{48} G_3^{y,x}(f, f, f) + \cdots\right] \\ & + e^{-j3\omega t}\left[\frac{P^3}{48} G_3^{y,x}(-f, -f, -f) + \cdots\right] + \cdots . \end{aligned} \tag{6}$$

When $x(t) = P\cos\omega_1 t + Q\cos\omega_2 t$, then $y(t)$ is a sum of complex exponentials, the $\exp[j(N\omega_1 + M\omega_2)t]$ component of $y(t)$ being, for $M \geqq 0$ and $N \geqq 0$,

$$e^{j(N\omega_1 + M\omega_2)t} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(P/2)^{2l+N}(Q/2)^{2k+M}}{(N+l)!\,l!\,(M+k)!\,k!} G_{N+l,l;M+k,k}^{y,x}(f_1, f_2) \tag{7}$$

where $\omega_i = 2\pi f_i$, $i = 1, 2$, and $G_{N+l,l;M+k,k}^{y,x}(f_1, f_2)$ denotes $G_n^{y,x}(f_1, \cdots, f_n)$

with $n = N + 2l + M + 2k$ and the first $N + l$ of the f_i's equal to f_1, the next l equal to $-f_1$, the next $M + k$ equal to f_2, and the last k equal to $-f_2$.

In Appendix A we show that the input-output pairs of a certain class of nonlinear systems can be related by a Volterra series expansion with certain Volterra kernels. The result is a slight generalization of one proved in Ref. 4.

Property 3: Suppose

$$y(t) = F\left[\int_{-\infty}^{\infty} g(u)\hat{h}[x(t-u)]du\right] \tag{8}$$

where F and $\hat{h}$ are functions of a complex variable having series expansions:

$$\hat{h}(z) = \sum_{\nu=0}^{\infty} \hat{h}_\nu \frac{z^\nu}{\nu!} \tag{9}$$

$$F(z) = \sum_{l=0}^{\infty} F_l \frac{(z - z_0)^l}{l!} \tag{10}$$

with

$$z_0 \triangleq \hat{h}_0 \int_{-\infty}^{\infty} g(u)du.$$

Let

$$G(f) \triangleq \int_{-\infty}^{\infty} g(u)e^{-j\omega u}du, \qquad \omega = 2\pi f.$$

Then $y(t) = \text{GVS}[x(t); \{g_n^{y,x}\}]$ with $g_0^{y,x} = F_0$ and

$$\begin{aligned}
G_1^{y,x}(f_1) &= F_1 \hat{h}_1 G(f_1) \\
G_2^{y,x}(f_1, f_2) &= F_1 \hat{h}_2 G(f_1 + f_2) + F_2 \hat{h}_1^2 G(f_1)G(f_2) \\
G_3^{y,x}(f_1, f_2, f_3) &= F_1 \hat{h}_3 G(f_1 + f_2 + f_3) + F_2 \hat{h}_1 \hat{h}_2[G(f_1)G(f_2 + f_3) \\
&\quad + G(f_2)G(f_1 + f_3) + G(f_3)G(f_1 + f_2)] \\
&\quad + F_3 \hat{h}_1^3 G(f_1)G(f_2)G(f_3).
\end{aligned}$$

Expressions for the higher-order Volterra transfer functions are given in eq. (49) of Ref. 4.

Finally, suppose that $y(t)$ and $x(t)$ are related by a Volterra series expansion, and that $y(t)$ is transformed by some function $\hat{F}(\cdot)$ to produce a signal $w(t) = \hat{F}[y(t)]$. Then, for a certain class of functions $\hat{F}(\cdot)$, the following result, which is proved in Appendix B, shows that $w(t)$ also has a Volterra series expansion in terms of $x(t)$ with specific kernels.

Property 4: Suppose $y(t) = \text{GVS}[x(t); \{g_n^{y,x}\}]$ and $w(t) = \hat{F}[y(t)]$ where $\hat{F}$ is a function of a complex variable having series expansion:

$$\hat{F}(z) = \sum_{l=0}^{\infty} \hat{F}_l \frac{(z - g_0^{y,x})^l}{l!}.$$

Then $w(t) = \text{GVS}[x(t); \{g_n^{w,x}\}]$ where $g_0^{w,x} = \hat{F}_0$ and

$$\begin{aligned} G_1^{w,x}(f_1) &= \hat{F}_1 G_1^{y,x}(f_1) \\ G_2^{w,x}(f_1, f_2) &= \hat{F}_1 G_2^{y,x}(f_1, f_2) + \hat{F}_2 G_1^{y,x}(f_1) G_1^{y,x}(f_2) \\ G_3^{w,x}(f_1, f_2, f_3) &= \hat{F}_1 G_3^{y,x}(f_1, f_2, f_3) + \hat{F}_2 [G_1^{y,x}(f_1) G_2^{y,x}(f_2, f_3) \\ &\quad + G_1^{y,x}(f_2) G_2^{y,x}(f_1, f_3) + G_1^{y,x}(f_3) G_2^{y,x}(f_1, f_2)] \\ &\quad + \hat{F}_3 G_1^{y,x}(f_1) G_1^{y,x}(f_2) G_1^{y,x}(f_3). \end{aligned} \tag{11}$$

Expressions for the higher-order kernels can be obtained from the method discussed in Appendix B.

III. MATHEMATICAL DESCRIPTION OF CROSSTALK INTERFERENCE

With the Volterra series analysis discussed in the last section, we can now give a mathematical description of interchannel and intrachannel crosstalk. Consider interchannel crosstalk first. Suppose $x(t)$ is some signal in one communications channel which enters a second channel as a signal $\hat{x}(t)$, where $\hat{x}(t)$ is $x(t)$ (possibly) transformed by some operation. Assume that the second channel contains some devices which operate on $\hat{x}(t)$ to produce a signal, $y(t)$, at the output of the channel. If the operations which transformed $x(t)$ into $\hat{x}(t)$ and $\hat{x}(t)$ into $y(t)$ consist of, for instance, nonlinear operations described by power series in cascade with time-invariant linear operations, then it is clear from Properties 3 and 4 that $y(t)$ will have a generalized Volterra series expansion in $x(t)$:

$$y(t) = g_0^{y,x} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} du_1 \cdots du_n \, g_n^{y,x}(u_1, \cdots, u_n) \cdot \prod_{r=1}^{n} x(t - u_r). \tag{1}$$

That is, the crosstalk interference, $y(t)$, appearing at the output of the second channel, can be expressed in terms of the signal in the first channel, $x(t)$, which created it. The first term in the summation in (1) will be denoted by

$$y_L(t) = \int_{-\infty}^{\infty} du_1 \, g_1^{y,x}(u_1) x(t - u_1). \tag{12}$$

It is that part of $y(t)$ which is linear in $x(\cdot)$; $y_L(t)$ can be obtained by passing $x(\cdot)$ through a time-invariant linear filter with impulse response $g_1^{y,x}(\cdot)$. If $Y_L(f)$ is the Fourier transform of $y_L(t)$, then:

$$Y_L(f) = G_1^{y,x}(f)X(f). \tag{13}$$

The higher-order terms in the summation in eq. (1) represent greater nonlinear distortions of the signal $x(t)$ than do the lower-order terms. This can be seen from eq. (4), where we observe that each term in (1) has a spectrum which contributes to the spectrum of $y(t)$, the higher-order terms distorting the spectrum of $x(t)$ to a greater degree. The spectrum of the linear part of $y(t)$ given by (13), however, is simply $X(f)$ multiplied by a weight function. Because of this, we might expect $y_L(t)$ to be more intelligible than the other terms in (1), in the case when $x(t)$ is a speech-like signal. In fact, we will define $y_L(t)$ to be the *intelligible* part of the crosstalk, and $y(t) - y_L(t)$ will be called the *unintelligible* part.

The Volterra kernels $\{g_n^{y,x}\}$ and especially the Volterra transfer functions $\{G_n^{y,x}\}$ can be used as a *measure* of the degree of nonlinearity of each of the terms in (1). Moreover, since by Property 1 the Volterra transfer functions are the responses at certain frequencies to a harmonic sum input, they have further intuitive appeal as appropriate measures of system performance. In particular, as a measure of the intelligible crosstalk created in one channel by signals in the other channel, we will define the *intelligible crosstalk ratio at frequency* f, $R(f)$, to be

$$R(f) \triangleq \frac{|Y_L(f)|^2}{|X(f)|^2} = |G_1^{y,x}(f)|^2. \tag{14}$$

Previous authors followed two different approaches in defining intelligible crosstalk and intelligible crosstalk ratio. One idea, followed by Ruthroff,[6] Bennett,[10] Curtis,[11] and Hatch[12] was to assume that the signal, $x(t)$, in one channel is a constant amplitude sinusoid at frequency f and having power P_1. Then, for certain models, they were able to show that $y(t)$, the resulting interference in the second channel, contained a sinusoid at frequency f with power P_2. They defined the intelligible crosstalk ratio at frequency f to be P_2/P_1. Extending this idea a little further, one might let $x(t)$ be a sum of sinusoids at incommensurable frequencies $f_1, \cdots, f_n$ ($\omega_i \triangleq 2\pi f_i$), i.e., $x(t) = \sum_{i=1}^{n} \sin \omega_i t$. If, for some problem, we can express the resulting interference $y(t)$ as a sum of sinusoids, with b the coefficient of $\sin \omega_1 t$ in this sum, then the intelligible crosstalk ratio at frequency f_1 would be taken to be $|b|^2$.

Our definition of intelligible crosstalk ratio in (14) is similar to this except that we use complex exponentials instead of sinusoids. But Property 2 shows that $G_1^{y,x}(f_1)$ is in fact the leading term of the coefficient of $\sin \omega_1 t$ (when $x(t)$ is a sum of sinusoids), and thus the two definitions may in some cases yield approximately the same numerical result. Lundquist[13] followed another approach. He assumed that $x(t)$ was arbitrary and, for a certain model, was able to express the interference $y(t)$ as a series of products of powers and derivatives of $x(t)$. He took the intelligible crosstalk to be that part of $y(t)$ which was "linear in $x(t)$." Expressing this part as a linear filtering operation on $x(t)$, having transfer function $H_L(f)$, he then defined the intelligible crosstalk ratio to be $|H_L(f)|^2$. The intelligible crosstalk ratio given in (14) is identical with that of Lundquist once the part of $y(t)$ linear in $x(t)$ is identified.

The preceding discussion is also applicable to the problem of intrachannel crosstalk. Earlier Volterra series techniques[4,14] had been applied to one such problem, namely, distortion in angle-modulated systems. In the intrachannel crosstalk problem, $x(t)$, the signal at the input of a channel, is transformed by some nonlinear devices into the output signal $y(t)$. If these devices consist of, for example, nonlinear operations described by power series in cascade with time-invariant linear filtering, then $y(t)$ has a generalized Volterra series expansion in terms of $x(t)$ as in (1). Assume that the *desired* output signal $y_0(t)$ in the absence of the (parasitic) nonlinear devices should be a time-invariant linear operation on $x(t)$ with impulse response $k(\cdot)$ and transfer function $K(\cdot)$, i.e.,

$$y_0(t) = \int_{-\infty}^{\infty} k(u_1)x(t - u_1)du_1. \tag{15}$$

Then the distortion or crosstalk at the channel output is

$$\begin{aligned} y_D(t) &= y(t) - y_0(t) \\ &= g_0^{y,x} + \int_{-\infty}^{\infty} du_1[g_1^{y,x}(u_1) - k(u_1)]x(t - u_1) \\ &\quad + \sum_{n=2}^{\infty} \frac{1}{n!}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} du_1 \cdots du_n\, g_n^{y,x}(u_1, \cdots, u_n) \\ &\qquad \cdot \prod_{r=1}^{n} x(t - u_r). \end{aligned} \tag{16}$$

The intelligible crosstalk is:

$$\int_{-\infty}^{\infty} du_1\, \hat{g}_1^{y,x}(u_1)x(t - u_1) \tag{17}$$

with

$$\hat{g}_1^{y,x}(u_1) \triangleq g_1^{y,x}(u_1) - k(u_1). \tag{18}$$

The intelligible crosstalk ratio is:

$$R(f) = |\hat{G}_1^{y,x}(f)|^2 \tag{19}$$

where

$$\hat{G}_1^{y,x}(f) \triangleq G_1^{y,x}(f) - K(f). \tag{20}$$

The remainder of our preceding discussion for interchannel crosstalk is also valid for intrachannel crosstalk. The Volterra transfer functions may be used as measures of system performance. They are similar to (generalized) "intermodulation coefficients"[2] except that they are the response to complex exponentials and not to sinusoids.

IV. INTELLIGIBLE CROSSTALK RATIO FOR A PARTICULAR MODEL

In this section we look at a model for the generation of intelligible crosstalk [or direct adjacent channel interference (DACI)] between two neighboring angle-modulated channels in which the signal in one channel adds to the signal of the second channel, the sum is filtered, and the filter output then passes through an AM-PM conversion device. An example of such a situation occurs in the TD-2 microwave radio relay system[15,16] where the principal channel discrimination is provided by IF filters. The main AM-PM conversion in this system occurs in the transmitter amplifier. This model will illustrate the ideas and techniques of the previous sections. While we seek only the first Volterra transfer function (for intelligible crosstalk), the higher-order transfer functions can be found in a similar way.

Consider, in general, two neighboring phase-modulated† communications channels (labeled "1" and "2"). (See Fig. 2.) In channel 1, the received "desired" signal or carrier is taken to be:

$$v_{i1}(t) = \cos(\omega_1 t + \phi_1(t)) \tag{21}$$

where $\phi_1(t)$ is the phase modulation and the amplitude of $v_{i1}(t)$ has been normalized to unity. We assume that $v_{i1}(t)$ passes through a linear, time-invariant filter in channel 1 without distortion so that at the filter output the signal is:

$$v_{o1}(t) = \cos(\omega_1 t + \phi_1(t)). \tag{22}$$

In channel 2, the received "undesired" or interfering signal is assumed to be:

$$v_{i2}(t) = \kappa \cos(\omega_2 t + \phi(t)) \tag{23}$$

so that the signal (or carrier)-to-interference ratio is

$$\mu \triangleq \frac{1}{\kappa^2} \tag{24}$$

† Frequency-modulated channels can be treated in a similar way, and the results are the same.

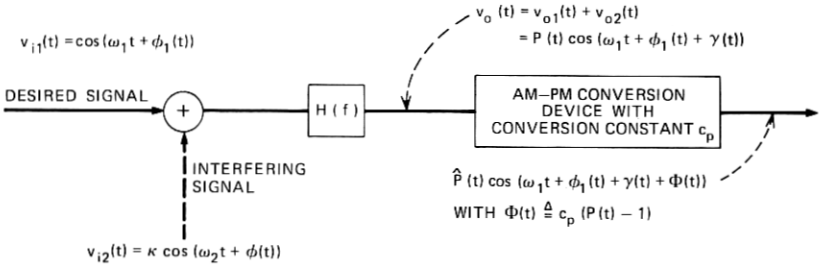

Fig. 2—Model for generation of intelligible crosstalk between two neighboring phase-modulated channels.

and in decibels:

$$\text{CIR} = 10 \log_{10} \mu \text{ (dB)}. \tag{25}$$

The signal $v_{i2}(t)$ is presumed to pass through the filter of channel 1 and produce the filter output:

$$v_{o2}(t) = \int_{-\infty}^{\infty} du\, h(u) v_{i2}(t-u)$$

$$= \kappa \int_{-\infty}^{\infty} du\, h(u) \cos[\omega_2(t-u) + \phi(t-u)] \tag{26}$$

where $h(\cdot)$ is the filter impulse response. We will denote the filter's transfer function by:

$$H(f) = \int_{-\infty}^{\infty} h(u) e^{-j\omega u} du, \qquad \omega = 2\pi f. \tag{27}$$

Using the relation

$$\cos \alpha = \tfrac{1}{2}[e^{j\alpha} + e^{-j\alpha}] \tag{28}$$

and setting

$$A(t) \triangleq \int_{-\infty}^{\infty} du\, h(u) e^{-j\omega_2 u} e^{j\phi(t-u)}, \tag{29}$$

$$B(t) \triangleq \int_{-\infty}^{\infty} du\, h(u) e^{j\omega_2 u} e^{-j\phi(t-u)}, \tag{30}$$

we can rewrite (26) as:

$$v_{o2}(t) = \kappa[\tfrac{1}{2}A(t)e^{j\omega_2 t} + \tfrac{1}{2}B(t)e^{-j\omega_2 t}]. \tag{31}$$

It is easy to see that with

$$V(t) \triangleq [A(t)B(t)]^{\frac{1}{2}} \tag{32}$$

and

$$\phi_2(t) \triangleq \frac{1}{2j} \ln \frac{A(t)}{B(t)}, \tag{33}$$

$v_{o2}(t)$ equals:

$$v_{o2}(t) = \kappa V(t) \cos(\omega_2 t + \phi_2(t)). \tag{34}$$

We will assume that $|\kappa V(t)| < 1$. The output, $v_o(t)$, of the filter in channel 1 is:

$$\begin{aligned} v_o(t) &= v_{o1}(t) + v_{o2}(t) \\ &= \cos(\omega_1 t + \phi_1(t)) + \kappa V(t) \cos(\omega_2 t + \phi_2(t)) \\ &= [1 + \kappa^2 V^2(t) + 2\kappa V(t) \cos\theta(t)]^{\frac{1}{2}} \cos(\omega_1 t + \phi_1(t) + \gamma(t)) \\ &= P(t) \cos(\omega_1 t + \phi_1(t) + \gamma(t)) \end{aligned} \tag{35}$$

where

$$P(t) \triangleq [1 + \kappa^2 V^2(t) + 2\kappa V(t) \cos\theta(t)]^{\frac{1}{2}}, \tag{36}$$

$$\theta(t) \triangleq (\omega_2 - \omega_1)t + \phi_2(t) - \phi_1(t),$$

and

$$\gamma(t) \triangleq \tan^{-1}\left[\frac{\kappa V(t) \sin\theta(t)}{1 + \kappa V(t) \cos\theta(t)}\right].$$

The amplitude function, $P(t)$, can be expanded in the power series:[17]

$$\begin{aligned} P(t) &= \sum_{n=0}^{\infty} C_n^{-\frac{1}{2}}(\cos\theta(t))(-1)^n(\kappa V(t))^n \\ &= 1 + \sum_{n=1}^{\infty} C_n^{-\frac{1}{2}}(\cos\theta(t))(-1)^n(\kappa V(t))^n \end{aligned}$$

where $\{C_n^{-\frac{1}{2}}(\cdot)\}$ are the Gegenbauer polynomials of degree n and order $-\frac{1}{2}$.

By definition,[18] if $a(t) \cos(\omega_c t + \psi(t))$ is the input to an AM-PM conversion device with conversion constant c_p (radians), then its output is $\hat{a}(t) \cos[\omega_c t + \psi(t) + c_p(a(t) - 1)]$. So if $v_o(t)$ passes through such a device, the undesired output phase in channel 1 is $\gamma(t) + \Phi(t)$ where

$$\Phi(t) \triangleq c_p(P(t) - 1) = c_p \sum_{n=1}^{\infty} C_n^{-\frac{1}{2}}(\cos\theta(t))(-1)^n(\kappa V(t))^n. \tag{37}$$

From Ref. 17, we also have:

$$C_n^{-\frac{1}{2}}(\cos\theta(t)) = \sum_{m=0}^{n} \frac{\Gamma(m - \frac{1}{2})\Gamma(n - m - \frac{1}{2})}{m!(n - m)![\Gamma(-\frac{1}{2})]^2} \cdot \cos[(n - 2m)\theta(t)]$$

where $\Gamma(\cdot)$ is the gamma function. Then,

$$\Phi(t) = c_p \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{\Gamma(m - \frac{1}{2})\Gamma(n - m - \frac{1}{2})}{m!(n - m)![\Gamma(-\frac{1}{2})]^2} \cdot \cos[(n - 2m)\theta(t)] \cdot (-1)^n(\kappa V(t))^n. \tag{38}$$

Assuming that $f_2 - f_1$ ($f_i = \omega_i/2\pi$) is greater than the baseband frequencies of channel 1, we see from (36) that terms of the form $\cos[p\theta(t)]$, $p \neq 0$, do not contribute to the baseband interference in channel 1. In addition, it can be shown that $\gamma(t)$ is outside the baseband. Thus, retaining only the terms for n even and $m = n/2$ in (38), the undesired output phase or crosstalk interference is just:

$$y(t) \triangleq c_p \sum_{n=1}^{\infty} \left[\frac{\Gamma(n - \frac{1}{2})}{\Gamma(-\frac{1}{2})n!} \right]^2 (\kappa V(t))^{2n}$$
$$= c_p\left[F\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; (\kappa V(t))^2\right) - 1\right] \tag{39}$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function[19] defined by:

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)} \frac{z^n}{n!}.$$

We next show that the crosstalk interference $y(t)$ has a generalized Volterra series in $\phi(t)$, the signal creating the interference, and we find the Volterra transfer function $G_1^{y,\phi}(f)$. We begin by rewriting (32) as:

$$V^2(t) = \exp\left[\ln A(t) + \ln B(t)\right]$$
$$= \exp\left[A_1(t) + B_1(t)\right]$$

where

$$A_1(t) \triangleq \ln A(t) \quad \text{and} \quad B_1(t) \triangleq \ln B(t).$$

Recalling the definition of $A(t)$ in (29), we apply Property 3 [with $g(u) = h(u)e^{-j\omega_2 u}$, $\hat{h}(x) = e^{jx}$, and $F(z) = \ln z$] to get that $A_1(t) = \text{GVS}[\phi(t); \{g_n^{A_1,\phi}\}]$ with

$$g_0^{A_1,\phi} = F_0 = \ln z_0 = \ln H(f_2)$$
$$G_1^{A_1,\phi}(f) = F_1\hat{h}_1 G(f) = jH(f_2 + f)/H(f_2).$$

Similarly, for $B_1(t)$:

$$g_0^{B_1,\phi} = \ln H(-f_2)$$
$$G_1^{B_1,\phi}(f) = -jH(f - f_2)/H(-f_2).$$

Setting $D(t) = A_1(t) + B_1(t)$, we have $D(t) = \text{GVS}[\phi(t); \{g_n^{D,\phi}\}]$ and clearly $g_n^{D,\phi} = g_n^{A_1,\phi} + g_n^{B_1,\phi}$. Since $h(u)$ is real and $H(-f) = H^*(f)$ we have:

$$g_0^{D,\phi} = \ln H(f_2) + \ln H(-f_2) = \ln |H(f_2)|^2$$
$$G_1^{D,\phi}(f) = j\,\frac{H(f_2 + f)}{H(f_2)} - j\,\frac{H(f - f_2)}{H(-f_2)}.$$

Next we apply Property 4 to $V^2(t) = \exp[D(t)]$ with $\hat{F}(z) = e^z$ and $\hat{F}_0 = \hat{F}_1 = \cdots$ to get:

$$g_0^{V^2,\phi} = \hat{F}_0 = \exp\left[g_0^{D,\phi}\right] = |H(f_2)|^2$$
$$G_1^{V^2,\phi}(f) = \hat{F}_1 G_1^{D,\phi}(f) = jH^*(f_2)H(f_2 + f) - jH(f_2)H^*(f_2 - f).$$

Finally, we apply Property 4 to (39) with

$$\hat{F}(z) = c_p[F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \kappa^2 z) - 1]$$

to get that $y(t) = \text{GVS}[\phi(t); \{g_n^{y,\phi}\}]$. Also

$$\begin{aligned}\hat{F}_0 &= \hat{F}(z) \text{ evaluated at } z = g_0^{V^2,\phi} \\ &= c_p[F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; |\kappa H(f_2)|^2) - 1]\end{aligned}$$

and

$$\hat{F}_1 = \frac{d}{dz}\hat{F}(z)\Big|_{z=g_0^{V^2,\phi}} = \frac{c_p}{4}\kappa^2 F(\tfrac{1}{2}, \tfrac{1}{2}; 2; |\kappa H(f_2)|^2).$$

Hence,

$$\begin{aligned}g_0^{y,\phi} &= \hat{F}_0 = c_p[F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; |\kappa H(f_2)|^2) - 1] \\ G_1^{y,\phi}(f) &= \hat{F}_1 G_1^{V^2,\phi}(f) \\ &= \frac{c_p}{4}\kappa^2\, Q(\kappa|H(f_2)|)\cdot j[H^*(f_2)H(f_2+f) \\ &\qquad - H(f_2)H^*(f_2-f)]\end{aligned} \tag{40}$$

where

$$\begin{aligned}Q(z) &\triangleq F(\tfrac{1}{2}, \tfrac{1}{2}; 2; z^2) \\ &= \frac{4}{\pi z^2}[E(z) - (1-z^2)K(z)]\end{aligned} \tag{41}$$

and E and K denote complete elliptic integrals of modulus z (Ref. 19, pp. 47 and 358).

Then the intelligible crosstalk ratio equals:

$$\begin{aligned}R(f) &= |G_1^{y,\phi}(f)|^2 \\ &= \frac{c_p^2}{16}\kappa^4 Q^2(\kappa|H(f_2)|)\Big|H(f_2+f)H^*(f_2) \\ &\qquad - H^*(f_2-f)H(f_2)\Big|^2\end{aligned} \tag{42}$$

where

$$\begin{aligned}Q(\kappa|H(f_2)|) &= 1 + \tfrac{1}{8}(\kappa|H(f_2)|)^2 \\ &\quad + \frac{3}{64}(\kappa|H(f_2)|)^4 + \frac{25}{1024}(\kappa|H(f_2)|)^6 + \cdots.\end{aligned} \tag{43}$$

For a given value of κ (or CIR), we need only calculate the value of $Q(\kappa|H(f_2)|)$ once for any filter transfer function having attenuation $|H(f_2)|^2$. When $\kappa \leqq 1$ (or CIR $\geqq$ 0 dB) and $10\log_{10}|H(f_2)|^2 \leqq -10$ dB, we can approximate, with very good accuracy, $Q(\kappa|H(f_2)|) \cong 1$, and then:

$$R(f) \cong \frac{c_p^2}{16}\kappa^4|H(f_2+f)H^*(f_2) - H^*(f_2-f)H(f_2)|^2. \tag{44}$$

If $C(f)$ and $\Theta(f)$ are the magnitude and phase of $H(f)$,

$$H(f) = C(f)e^{j\Theta(f)}, \tag{45}$$

then

$$|H(f_2+f)H^*(f_2) - H^*(f_2-f)H(f_2)|^2 = [C^2(f_2+f) + C^2(f_2-f)]C^2(f_2) - 2C(f_2+f)C(f_2-f)C^2(f_2) \cdot \cos[\Theta(f_2+f) + \Theta(f_2-f) - 2\Theta(f_2)]. \tag{46}$$

The last expression together with either (42) or (44) is well suited for computational purposes requiring only the values of the amplitude and phase of $H(\cdot)$ at frequencies f_2, $f_2 + f$, and $f_2 - f$.

One should note that in this analysis we have assumed that the filter gain at f_1 was unity. It is easy to see that, if the gain is not unity, the only difference in eqs. (42) to (46) is that $H(f)$ is replaced by the normalized transfer function $H(f)/|H(f_1)|$.

The expression for the intelligible crosstalk ratio given in (44) exhibits a number of properties usually associated with DACI.[1,11,16] For example, noting that $\mu = 1/\kappa^2$ and CIR $= 10 \log_{10} \mu$ (dB) and expressing the intelligible crosstalk ratio in decibels as $10 \log_{10} R(f)$ (dB), we see from (44) that if CIR decreases 1 dB then the crosstalk ratio increases 2 dB. We observe that the way in which we have defined $R(f)$ also makes $R(f)$ independent of the power of the input (phase). Moreover, by assuming that the amplitude of the desired signal in (21) is arbitrary (instead of unity), it is easy to check that, for fixed CIR, $R(f)$ is independent of the desired signal power.

V. EXAMPLE

The intelligible crosstalk ratio was calculated for the example considered by Lundquist[13] with CIR = 0 dB. The crosstalk ratio for other values of CIR can be found by adding 2 dB to the crosstalk ratio for each dB decrease in CIR. We assumed an AM-PM conversion constant of 5 degrees/dB[18] or $c_p = 5(0.1516) = 0.758$ radians, and a k-pole filter having transfer function:

$$H(f) = \frac{1}{\left[1 + j\left(\frac{f - f_1}{f_o}\right)\right]^k}. \tag{47}$$

Given the number of poles k, the frequency separation $\Delta f = f_2 - f_1$, and the value of the "attenuation at the adjacent channel" defined as $-10 \log_{10} |H(f_2)|^2$ (dB), we can determine f_o from (47). Equations (44) and (46) were used to compute $R(f)$ for various values of k, baseband frequency f, frequency separation Δf, and adjacent channel attenuation. The results are given in Figs. 3 to 5. Figure 3 shows the de-

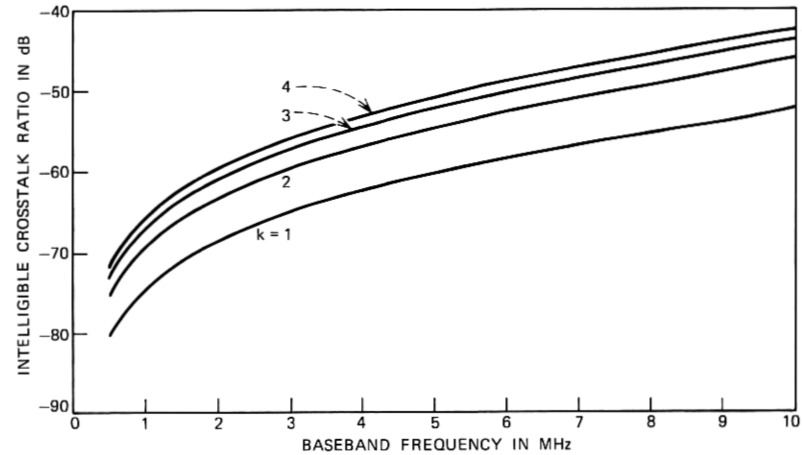

Fig. 3—Intelligible crosstalk ratio versus baseband frequency, for $\Delta f = 20$ MHz and adjacent channel attenuation = 20 dB.

pendence of the intelligible crosstalk ratio on the baseband frequency f, for fixed frequency separation $\Delta f = 20$ MHz and adjacent channel attenuation of 20 dB. We see from Fig. 3 that DACI is greater at higher

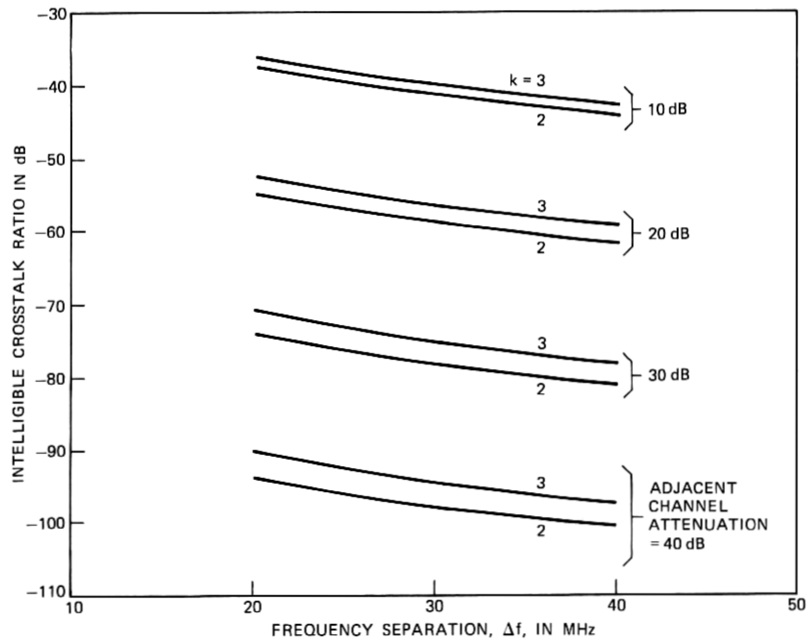

Fig. 4—Intelligible crosstalk ratio versus frequency separation, for baseband frequency = 5 MHz.

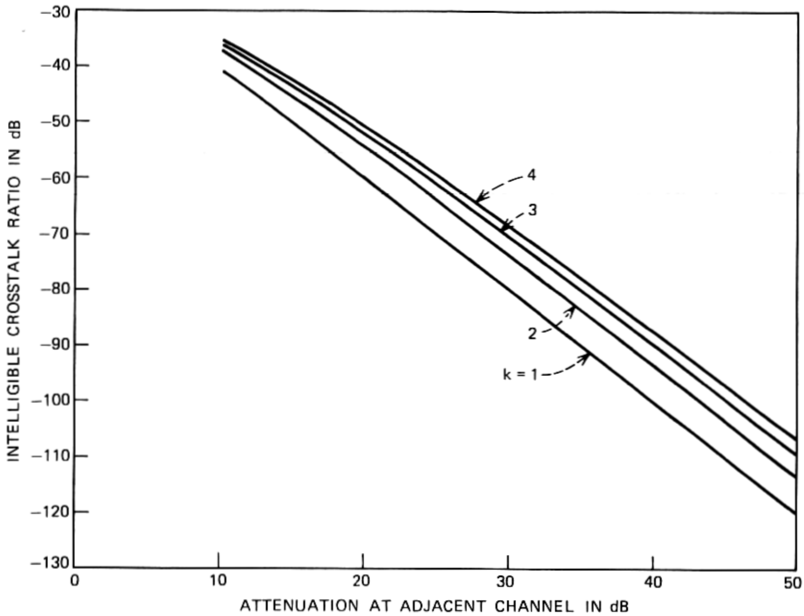

Fig. 5—Intelligible crosstalk ratio versus attenuation at adjacent channel, for $\Delta f = 20$ MHz and baseband frequency = 5 MHz.

baseband frequencies, increasing approximately 6 dB when f is doubled. For a fixed baseband frequency of 5 MHz, Fig. 4 shows the relation between $R(f)$ and the frequency separation Δf. We observe that there is not much variation of $R(f)$ with Δf for a given adjacent channel attenuation. In Fig. 5 we show the effect of increasing the adjacent channel attenuation for a fixed baseband frequency of 5 MHz and a frequency separation of 20 MHz. Here a 1-dB increase in attenuation produces about a 2-dB decrease in crosstalk ratio.

VI. CONCLUSION

By use of Volterra series analysis, we have presented a general mathematical description of the crosstalk interference created in a communications system. From this description, we were able to isolate the part of the crosstalk that was intelligible and to define the intelligible crosstalk ratio as a measure of the intelligible crosstalk created in the system. We then looked at a model in which intelligible crosstalk was generated between two neighboring PM channels. Using our results, we derived an expression for the intelligible crosstalk ratio for this model. This expression exhibited a number of properties usually associated with direct adjacent channel interference. The crosstalk ratio was computed for the case of a k-pole filter as a function of various parameters.

APPENDIX A

In this appendix we sketch the proof of Property 3. Following Bedrosian and Rice[4] we define the function†

$$\hat{H}(\xi) = \int_{-\infty}^{\infty} g(u)\hat{h}[\xi x(t-u)]du \tag{48}$$

so that from (9):

$$\hat{H}(0) = \hat{h}_0 \int_{-\infty}^{\infty} g(u)du = z_0. \tag{49}$$

From (10) we see that:

$$F[\hat{H}(0)] = F(z_0) = F_0. \tag{50}$$

Expanding the function $F[\hat{H}(\xi)]$ in a Maclaurin series we obtain:

$$F[\hat{H}(\xi)] = \sum_{n=0}^{\infty} \frac{\xi^n}{n!}\left[\frac{d^n}{d\xi^n} F[\hat{H}(\xi)]\right]_{\xi=0}. \tag{51}$$

Then

$$\begin{aligned} y(t) &= F[\hat{H}(1)] = F[\hat{H}(0)] + \sum_{n=1}^{\infty} \frac{1}{n!}\left[\frac{d^n}{d\xi^n} F[\hat{H}(\xi)]\right]_{\xi=0} \\ &= F_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\left[\frac{d^n}{d\xi^n} F[\hat{H}(\xi)]\right]_{\xi=0}. \end{aligned} \tag{52}$$

Applying the results in eqs. (49), (114), and (115) of Ref. 4 we get:

$$y(t) = F_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} du_1 \cdots du_n \, g_n^{y,x}(u_1, \cdots, u_n) \cdot \prod_{r=1}^{n} x(t-u_r) \tag{53}$$

for some kernels $\{g_n^{y,x}\}$ with

$$\begin{aligned} G_1^{y,x}(f) &= F_1 \hat{h}_1 G(f), \\ G_2^{y,x}(f_1, f_2) &= F_1 \hat{h}_2 G(f_1 + f_2) + F_2 \hat{h}_1^2 G(f_1)G(f_2), \end{aligned} \tag{54}$$

and

$$\begin{aligned} G_3^{y,x}(f_1, f_2, f_3) &= F_1 \hat{h}_3 G(f_1 + f_2 + f_3) \\ &+ F_2 \hat{h}_1 \hat{h}_2 [G(f_1)G(f_2 + f_3) + G(f_2)G(f_1 + f_3) + G(f_3)G(f_1 + f_2)] \\ &+ F_3 \hat{h}_1^3 G(f_1)G(f_2)G(f_3). \end{aligned}$$

† The dependence of $\hat{H}(\xi)$ on t will be suppressed.

The higher-order Volterra transfer functions are given by eq. (49) of Ref. 4. Thus, $y(t) = \mathrm{GVS}[x(t); \{g_n^{y,x}\}]$ and $g_0^{y,x} = F_0$ which is the desired result.

APPENDIX B

Here we derive Property 4. Define the function $\hat{H}(\xi)$ by:†

$$\hat{H}(\xi) = g_0^{y,x} + \sum_{n=1}^{\infty} \frac{\xi^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} du_1 \cdots du_n \, g_n^{y,x}(u_1, \cdots, u_n) \cdot \prod_{r=1}^{n} x(t - u_r). \quad (55)$$

Then,

$$\hat{H}(0) = g_0^{y,x}$$

and the νth derivative of $\hat{H}(\xi)$ evaluated at $\xi = 0$ equals:

$$\hat{H}^{(\nu)}(0) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} du_1 \cdots du_\nu \, g_\nu^{y,x}(u_1, \cdots, u_\nu) \prod_{r=1}^{\nu} x(t - u_r), \quad \nu \geqq 1. \quad (56)$$

Next,

$$y(t) = \hat{H}(1)$$

and

$$w(t) = \hat{F}[y(t)] = \hat{F}[\hat{H}(1)]. \quad (57)$$

Note that with $\hat{F}^{(l)}(z)$ denoting the lth derivative of $\hat{F}(z)$:

$$\hat{F}^{(l)}[\hat{H}(0)] = \hat{F}^{(l)}[g_0^{y,x}] = \hat{F}_l, \qquad l \geqq 0. \quad (58)$$

Expanding $\hat{F}[\hat{H}(\xi)]$ in a Maclaurin series,

$$\hat{F}[\hat{H}(\xi)] = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \left[\frac{d^n}{d\xi^n} \hat{F}[\hat{H}(\xi)] \right]_{\xi=0}, \quad (59)$$

we get:

$$w(t) = \hat{F}[\hat{H}(1)] = \hat{F}[\hat{H}(0)] + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^n}{d\xi^n} \hat{F}[\hat{H}(\xi)] \right]_{\xi=0}$$

$$= \hat{F}_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^n}{d\xi^n} \hat{F}[\hat{H}(\xi)] \right]_{\xi=0}. \quad (60)$$

Using the results in eqs. (98) and (112) through (115) of Ref. 4 we can write (60) as:

$$w(t) = \hat{F}_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} du_1 \cdots du_n \, g_n^{w,x}(u_1, \cdots, u_n) \cdot \prod_{r=1}^{n} x(t - u_r) \quad (61)$$

† The dependence of $\hat{H}(\xi)$ on t will be suppressed.

where, in particular,

$$g_1^{w,x}(u_1) = \hat{F}_1 g_1^{y,x}(u_1),$$
$$g_2^{w,x}(u_1, u_2) = \hat{F}_1 g_2^{w,x}(u_1, u_2) + \hat{F}_2[g_1^{y,x}(u_1)g_1^{y,x}(u_2)], \qquad (62)$$

and

$$g_3^{w,x}(u_1, u_2, u_3) = \hat{F}_1 g_3^{y,x}(u_1, u_2, u_3) + \hat{F}_2[g_1^{y,x}(u_1)g_2^{y,x}(u_2, u_3) + g_1^{y,x}(u_2)g_2^{y,x}(u_1, u_3) + g_1^{y,x}(u_3)g_2^{y,x}(u_1, u_2)] + \hat{F}_3[g_1^{y,x}(u_1)g_1^{y,x}(u_2)g_1^{y,x}(u_3)]. \qquad (63)$$

Therefore, $w(t) = \text{GVS}[x(t); \{g_n^{w,x}\}]$ with $g_0^{w,x} = \hat{F}_0$ and the Volterra transfer functions given in the statement of Property 4.

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