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Computing Distortion in Analog FM Communication Systems

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This paper describes a method for computing baseband distortion in analog FM communication systems; the method is based on recent theoretical work available in the literature. The input baseband signal is taken to be a zero-mean, stationary Gaussian process having an arbitrary power spectral density. A variety of graphical results are presented in order to demonstrate the utility of this method of computing FM distortion. It is shown that the often-used noise loading test does not necessarily represent a worst-case test.

I. INTRODUCTION

Theoretically, analog FM signals generally possess an infinite bandwidth. Thus, when such signals are passed through a linear system having a finite bandwidth, some FM distortion must occur. The measurement of such distortion is costly and very time consuming. Accordingly, the development of methods for the computation of FM distortion is of practical interest.

The purpose of this paper is to describe how we used the theoretical results derived by A. Mircea,[1] E. Bedrosian[2] and S. O. Rice[3] to develop a computer program to compute the FM distortion resulting from linear time-invariant-filter structures. The input baseband modulation is taken to be a zero-mean, stationary Gaussian process having an arbitrary baseband power spectral density.

II. SERIES EXPANSION UNDERLYING THE COMPUTATION

Consider the analog FM communication system presented in Fig. 1a. The associated mathematical problem for studying FM distortion is illustrated in Fig. 1b. The problem is to deduce the double-sided power spectral density, $W_\theta(f)$, of the output random process, $\theta(t)$, given $\Gamma(f)$ and the double-sided power spectral density, $W_\phi(f)$, of the input Gaussian modulation. Once this mathematical problem is solved satisfactorily, we can then compute the FM distortion at baseband. In the FM case, $W_{\phi'}(f)$, the power spectral density of $\phi'(t)$, is given and $W_{\theta'}(f)$, the power spectral density of $\theta'(t)$, is desired. In the PM case, $W_\phi(f)$ is given and $W_\theta(f)$ is desired. However, the two problems are closely related because of the following relations:

$$W_{\phi'}(f) = \omega^2 W_\phi(f) \tag{1}$$

and

$$W_{\theta'}(f) = \omega^2 W_\theta(f). \tag{2}$$

In fact, an FM communication system can be designed by using only PM equipment, as is illustrated in Fig. 1a.

Using Rice's[3] notation, a series expansion of $W_\theta(f)$ is given by:

$$\begin{aligned} W_\theta(f) = {} & \theta_{\mathrm{dc}}^2\delta(f) + \tfrac{1}{4}W_\phi(f)\,|U(f) + U^*(-f)|^2 \\ & + \frac{1}{8}\int_{-\infty}^{\infty} d\rho W_\phi(\rho)W_\phi(f-\rho)\,|T(\rho, f-\rho) - T^*(-\rho, -f+\rho)|^2 \\ & + \frac{1}{24}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma W_\phi(\rho)W_\phi(\sigma)W_\phi(f-\rho-\sigma) \\ & \times |S(\rho, \sigma, f-\rho-\sigma) + S^*(-\rho, -\sigma, -f+\rho+\sigma)|^2 \\ & + 0(\phi^6 W_\phi) \end{aligned} \tag{3}$$

where

$$\begin{aligned} * &= \text{complex conjugate} \\ \theta_{\mathrm{dc}} &= \text{dc part of } \theta(t) \end{aligned}$$

$$\begin{aligned} T(\rho, f-\rho) = {} & S(\rho, f-\rho) + \int_{-\infty}^{\infty} d\sigma W_\phi(\sigma)[2S(\sigma,\rho)S(-\sigma, f-\rho) \\ & - S(\sigma, f-\sigma) - \Gamma(\sigma)\Gamma(-\sigma)S(\rho, f-\rho) \\ & + S(\rho+\sigma, f-\rho-\sigma)] \end{aligned}$$

$$\begin{aligned} U(f) = {} & \Gamma(f) + \int_{-\infty}^{\infty} d\rho W_\phi(\rho)\Gamma(\rho)S(-\rho, f) \\ & + \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma W_\phi(\rho)W_\phi(\sigma)\{-\tfrac{1}{2}\Gamma(\rho+\sigma)S(-\rho-\sigma, f) \\ & + \Gamma(\sigma)[3S(-\sigma,\rho)S(-\rho, f) - S(\rho, f-\rho-\sigma) \\ & + S(\rho-\sigma, f-\rho)]\} \end{aligned}$$

$$S(\rho, \sigma) = \Gamma(\rho+\sigma) - \Gamma(\rho)\Gamma(\sigma)$$

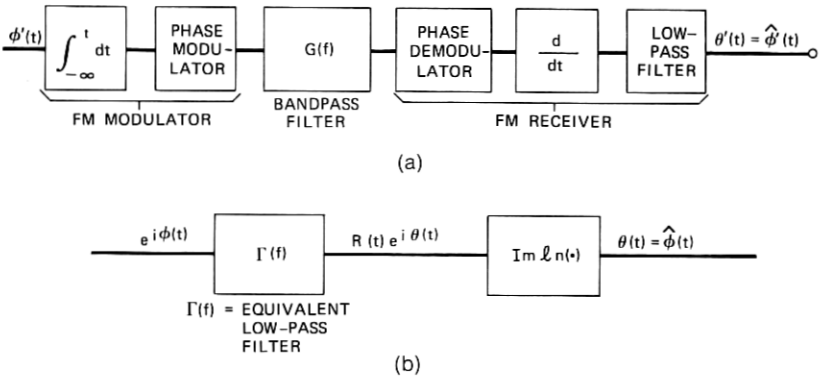

Fig. 1—(a) Analog FM communication system. $\hat{\phi}'(t)$ is a distorted version of $\phi'(t)$. (b) Associated mathematical model for studying FM distortion.

and

$$S(\rho, \sigma, \nu) = \Gamma(\rho + \sigma + \nu) - \Gamma(\rho + \sigma)\Gamma(\nu) - \Gamma(\rho + \nu)\Gamma(\sigma) - \Gamma(\sigma + \nu)\Gamma(\rho) + 2\Gamma(\rho)\Gamma(\sigma)\Gamma(\nu).$$

We shall neglect the dc term, θ_{dc}^2, since we are mainly interested in the continuous part of $W_\theta(f)$. In addition, for the range of parameters of interest to us, we have found that the double integral associated with the $U(f)$ function can also be neglected.

Notice that $W_\phi(f_1) = 0$ does not imply that $W_\theta(f_1) = 0$. This is contrary to the case of a linear system. That is, even if we apply no input power in the frequency interval $(f_1, f_1 + df)$, we generally get some "intermodulation noise" at the output in this frequency interval. Actually, eq. (3) is a truncated form of an infinite series of functionals of $\Gamma(f)$ and $W_\phi(f)$. However, we shall see that it is possible to select system parameters which are of practical importance and which allow us to neglect all of the terms represented by $0(\phi^6 W_\phi)$. Accordingly, we shall define the signal power $S(f)df$ in the frequency interval $(f, f + df)$ at the output to be

$$S(f)df = \tfrac{1}{4}W_\phi(f)\,|\,U(f) + U^*(-f)\,|^2 df. \tag{4}$$

For the range of parameters which are of practical importance, it turns out that $S(f) \doteq \frac{1}{4}W_\phi(f)\,|\,\Gamma(f) + \Gamma^*(-f)\,|^2$. $S(f)$ represents the spectral contribution at the output which is free of intermodulation noise. We also define the FM distortion power $D(f)df$ appearing in the output frequency interval $(f, f + df)$ to be

$$D(f)df = \frac{1}{8}\int_{-\infty}^{\infty} d\rho W_\phi(\rho)W_\phi(f-\rho) \times |T(\rho, f-\rho) - T^*(-\rho, -f+\rho)|^2 df + \frac{1}{24}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma W_\phi(\rho)W_\phi(\sigma)W_\phi(f-\rho-\sigma) \times |S(\rho, \sigma, f-\rho-\sigma) + S^*(-\rho, -\sigma, -f+\rho+\sigma)|^2 df. \quad (5)$$

$D(f)$ represents the spectral contribution at the output which results from intermodulation noise.

The two quantities of prime interest in this section are the signal-to-distortion ratio, $S(f)/D(f)$, and the ratio of the average signal power, σ_S^2, to the average distortion power, σ_D^2, in the output baseband. The latter quantities are defined by

$$\sigma_S^2 = \begin{cases} 2\int_0^B S(f)df & \text{PM Case} \\ 2\int_0^B \omega^2 S(f)df & \text{FM Case} \end{cases} \quad (6)$$

$$\sigma_D^2 = \begin{cases} 2\int_0^B D(f)df & \text{PM Case} \\ 2\int_0^B \omega^2 D(f)df & \text{FM Case} \end{cases} \quad (7)$$

where B = baseband bandwidth of the modulation. When the ratio of $\sigma_S^2/\sigma_D^2 \geqq 10$, one is usually safe in disregarding the terms labeled $0(\phi^6 W_\phi)$ in eq. (3).†

III. NUMERICAL METHODS EMPLOYED

An input power spectral density, $W_{\phi'}(f)$, which is often used when measuring FM distortion is the uniform spectrum, given by

$$W_{\phi'}(f) = \begin{cases} \dfrac{(2\pi D)^2}{2B}, & |f| \leqq B \\ 0, & |f| > B \end{cases} \quad (8)$$

† Equation (3) is a special case of a much more general equation which was recently reported by E. Bedrosian and S. O. Rice in the Proc. IEEE, *59*, No. 12, pp. 1688–1707, eq. (14) and Section IVc, December 1971.

where D = RMS frequency deviation and B = baseband bandwidth. From eq. (1), we have

$$W_{\phi}(f) = \begin{cases} \dfrac{D^2}{2Bf^2}, & |f| \leqq B \\ 0 \quad , & |f| > B \end{cases} \tag{9}$$

When such a uniform $W_{\phi'}(f)$ is used to measure FM distortion, the measurement is referred to as a "noise loading test." The noise loading test is used, for example, to estimate the FM distortion in microwave relay systems resulting when thousands of telephone channels are multiplexed to form a composite baseband signal.

A bandlimited form of $W_{\phi'}(f)$ is very convenient numerically, since it serves to convert the infinite limits of integration in eqs. (4) and (5) into finite limits of integration. However, if we attempt to evaluate equations (4) and (5) using a bandlimited $W_{\phi}(f)$ such as is given in eq. (9), we would run into difficulty whenever the argument of $W_{\phi}(\cdot)$ is equal to zero. In order to circumvent this apparent difficulty, we have selected an integration grid such that the argument of $W_{\phi}(\cdot)$ is never allowed to be zero. Equations (4) and (5) are then numerically evaluated by using a combination of Simpson's rule and the trapezoidal rule.

The particular integration grid used was obtained by setting

$$\rho = (2i + 1)\Delta \tag{10}$$

$$\sigma = (2l + 1)\Delta \tag{11}$$

$$f = (2n + 1)\Delta \tag{12}$$

$$\Delta = (2k + 1)^{-1} \tag{13}$$

$$B = 1.0 \tag{14}$$

where i, l, n, k are integers. In most of our numerical evaluations, $k = 20$.

IV. NUMERICAL RESULTS

4.1 *Test Cases*

In order to test the operation of the computer program, we evaluated $D(f)$ for the case when $W_{\phi'}(f)$ is uniform and $\Gamma(f)$ is the transfer

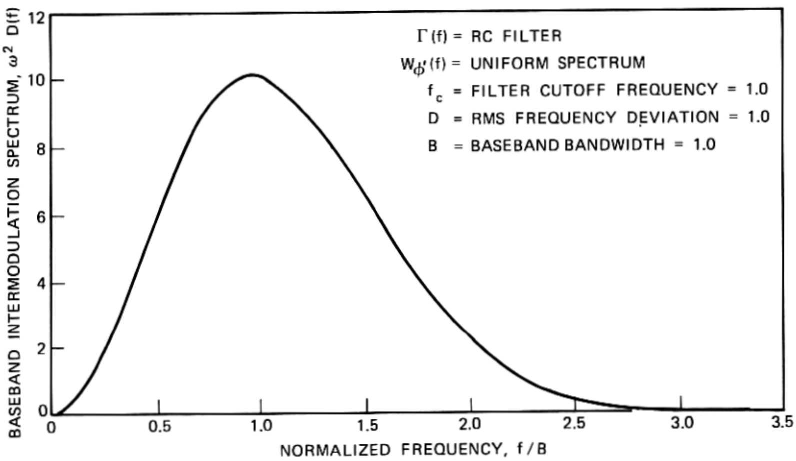

Fig. 2—Baseband intermodulation spectrum resulting from FM distortion.

function of a simple RC filter. That is, $W_\phi(f)$ is defined by eq. (9) and $\Gamma(f)$ is defined as

$$\Gamma(f) = \frac{1}{1 + i\left(\frac{f}{f_c}\right)} \tag{15}$$

where

$$f_c = 1.0$$

$$D = 1.0$$

$$B = 1.0.$$

Figure 2 shows the resulting computer plot of $\omega^2 D(f)$. Figure 3 shows a plot of $10 \log [S(f)/D(f)]$ for $f = 0.084B$, $0.36B$, and B, as a function of the RMS frequency deviation D. These results compare well with both the theoretical and experimental results which are presented in Refs. 2, 4, and 5.

As a final test case, we present the results for the case when $W_\phi(f)$ is still defined by eq. (9), but $\Gamma(f)$ now represents a 3-pole Butterworth filter with some mistuning, in which case

$$\Gamma(f) = \frac{1 + 2(i\xi_0) + 2(i\xi_0)^2 + (i\xi_0)^3}{1 + 2(i\xi) + 2(i\xi)^2 + (i\xi)^3} \tag{16}$$

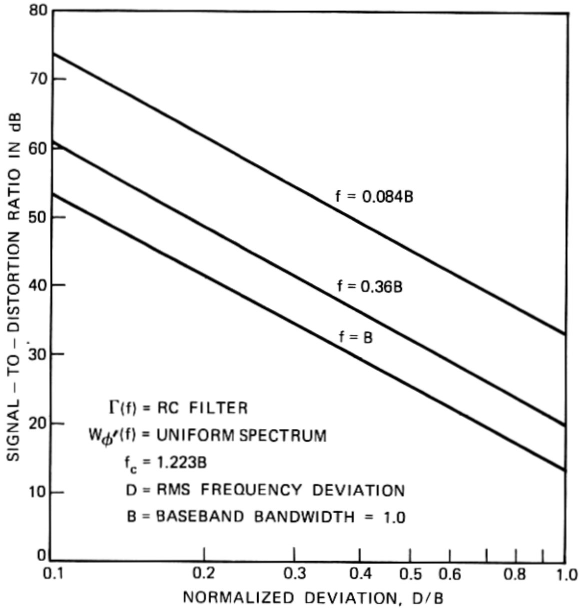

Fig. 3—Signal-to-distortion ratio at particular frequencies resulting from $\Gamma(f)$ and $W_{\phi'}(f)$.

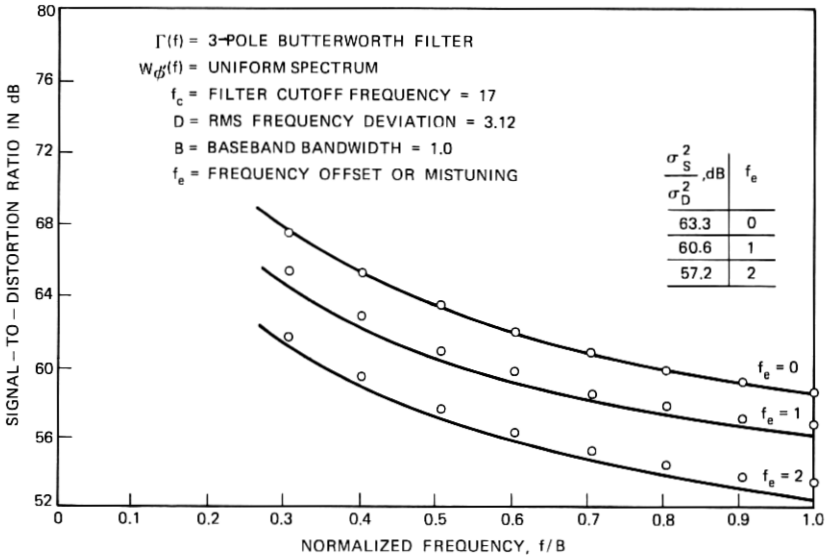

Fig. 4—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_{\phi'}(f)$ for particular values of frequency offset or mistuning. The points are from the theoretical approximation given as eqs. (17) and (18).

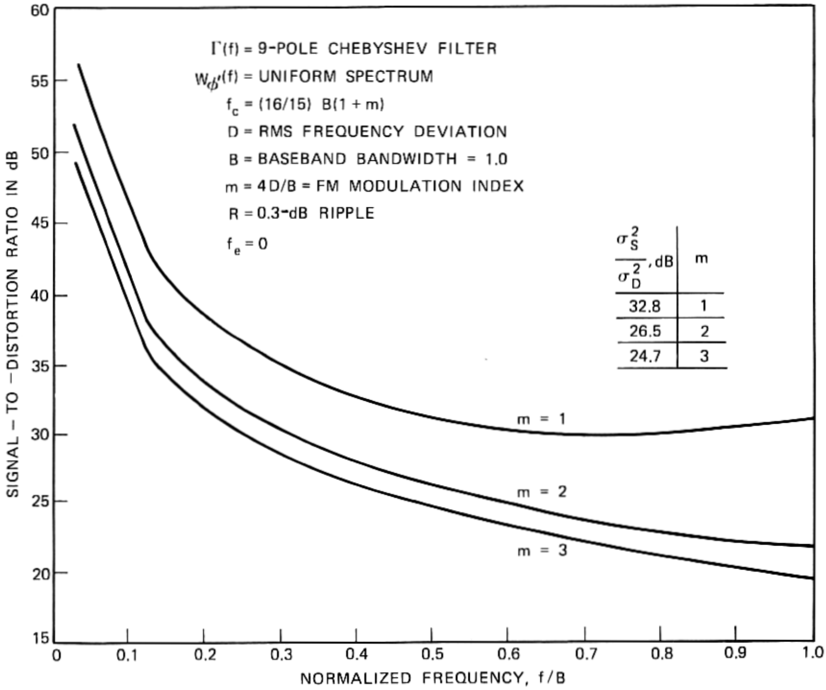

Fig. 5—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_{\phi'}(f)$.

where

$$\xi = \frac{f - f_e}{f_c}$$

$$\xi_0 = -\frac{f_e}{f_c}$$

$$B = 1.0$$

$$D = 3.12$$

$$f_c = 17.0$$

$$f_e = 0,\ 1,\ 2.$$

Figure 4 presents the computer plot for this case. The results compare very well with experimental and Monte Carlo results presented in Ref. 4.

A theoretical approximation for the above case, with $0 \leqq |f| \leqq B$,

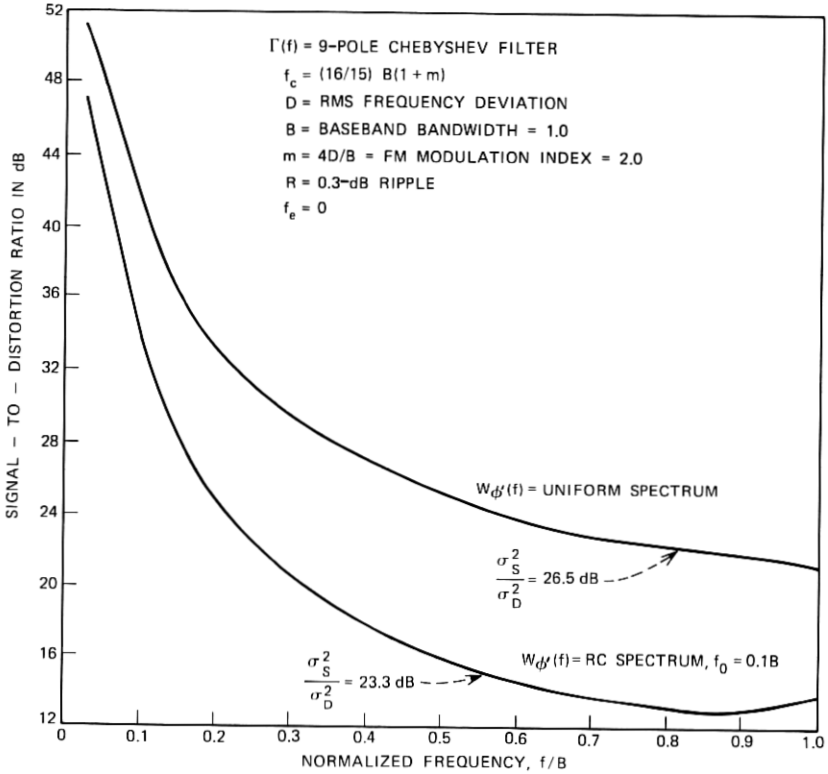

Fig. 6—Signal-to-distortion ratio resulting from $\Gamma(f)$ and two forms of $W_{\phi'}(f)$.

was developed by Rice[3] and is given by

$$D(f) \doteq \frac{D^4}{8B^2}(2B - |f|)(\lambda_{2i} + 2^{-1}D^2\lambda_{4i})^2 + \frac{D^6}{48B^3}(3B^2 - f^2)(\lambda_{3i})^2 \quad (17)$$

and

$$S(f) \doteq W_\phi(f) \quad (18)$$

where λ_{ni} is the imaginary part of λ_n and $\lambda_n/n!$ is the coefficient of f^n in the power series expansion of $\ln \Gamma(f)$

$$\lambda_{2i} = 2f_e f_c^{-3}$$

$$\lambda_{3i} = -2f_c^{-3}$$

$$\lambda_{4i} = 48 f_e f_c^{-5}.$$

Some points which were computed from this theoretical approximation are indicated in Fig. 4.

Having verified the soundness of the computer program with the foregoing test cases, let us present some new results.

4.2 *New Results*

In this section, we shall present some new results which were obtained by using the above methods. These results will also help to demonstrate the general types of FM distortion problems that can be analyzed.

4.2.1 *n-pole Chebyshev filter*

Let $W_{\phi'}(f)$ be uniform as given by eq. (8) with $\Gamma(f)$ given by

$$\Gamma(f) = \frac{\prod_{k=1}^{n} (i\xi_0 - s_k)}{\prod_{k=1}^{n} (i\xi - s_k)} \tag{19}$$

where

$$s_k = -\sin\left[(2k-1)\frac{\pi}{2n}\right]\sinh\left[\frac{1}{n}\sinh^{-1}\left(\frac{1}{b}\right)\right]$$
$$+ i\cos\left[(2k-1)\frac{\pi}{2n}\right]\cosh\left[\frac{1}{n}\sinh^{-1}\left(\frac{1}{b}\right)\right], \qquad k = 1, \cdots, n$$

$$\xi = \frac{f - f_e}{f_c}, \qquad \xi_0 = -\frac{f_e}{f_c}$$

$$B = 1 = \text{baseband bandwidth}$$

$$2f_c = K2B(1 + m) = K \text{ times Carson's rule}$$
$$= \text{filter bandwidth}$$

$$m = \frac{4D}{B} = \text{FM modulation index}$$

$$R = 10 \log (1 + b^2) = \text{in-band ripple}$$

$$f_e = \text{offset frequency or mistuning.}$$

$\Gamma(f)$, defined by eq. (19), represents an n-pole Chebyshev filter. Some results for this case are presented in Figs. 5, 6 and 7. Also, a computer

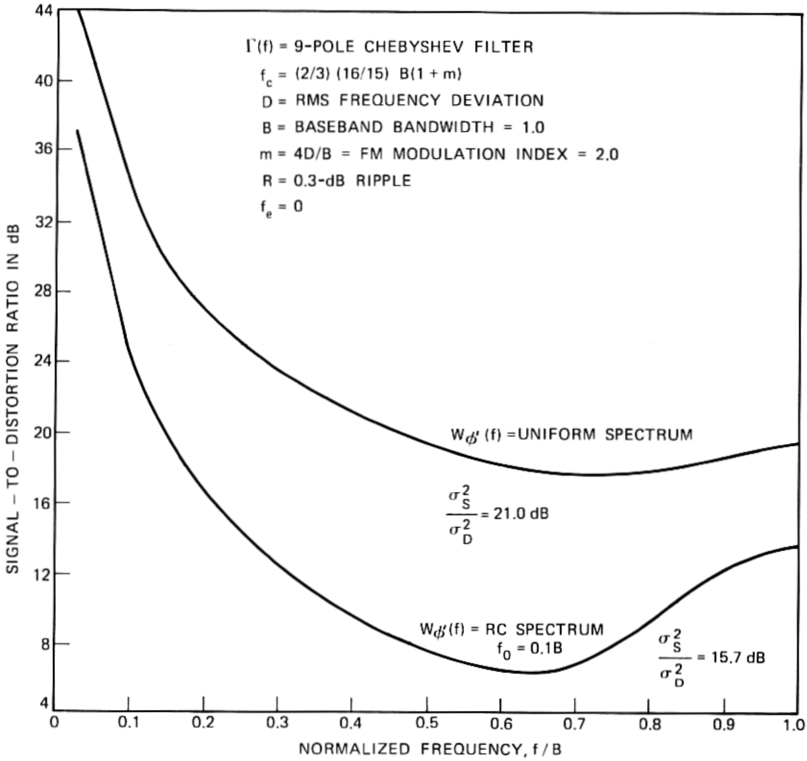

Fig. 7—Signal-to-distortion ratio resulting from $\Gamma(f)$ and two forms of $W_{\phi'}(f)$.

plot is presented in Figs. 6 and 7 for the case when $W_{\phi'}(f)$ is an RC spectrum† given by

$$W_{\phi'}(f) = \begin{cases} \dfrac{(2\pi D)^2}{2f_0 \tan^{-1}\left(\dfrac{B}{f_0}\right)} \left[1 + \left(\dfrac{f}{f_0}\right)^2\right]^{-1}, & |f| \leqq B \\ 0 \qquad , & |f| > B \end{cases} \tag{20}$$

where

$$f_0 = 3 \text{ dB bandwidth.}$$

An RC spectrum is often used to model a video signal. Notice that, as $f_0 \to \infty$, the RC spectrum approaches the uniform spectrum as given

† $W_{\phi'}(f)$ is the spectrum produced by passing bandlimited "white" noise through an RC filter.

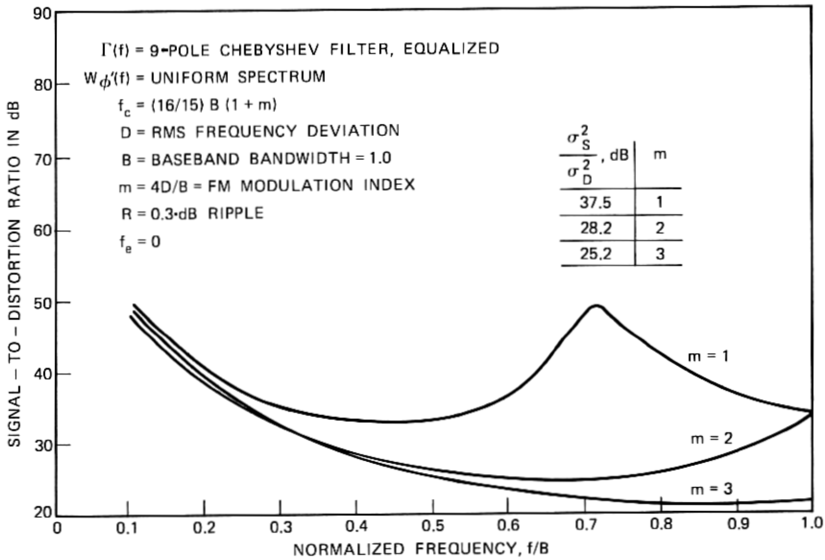

Fig. 8—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_{\phi'}(f)$.

by eq. (8). From Figs. 6 and 7, we see that more FM distortion results when $W_{\phi'}(f)$ is an RC or video spectrum than when $W_{\phi'}(f)$ is uniform.

4.2.2. *n-pole, equalized Chebyshev filter*

If the phase function associated with eq. (19) is taken to be zero,† $\Gamma(f)$ can be written as

$$\Gamma(f) = \left[\frac{1 + b^2 T_n^2(\xi_0)}{1 + b^2 T_n^2(\xi)}\right]^{\frac{1}{2}} \tag{21}$$

where $T_n(\xi)$ is a Chebyshev polynomial given by

$$T_n(\xi) = \begin{cases} \cos\,[n \cos^{-1}(\xi)], & |\xi| \leqq 1 \\ \cosh\,[n \cosh^{-1}(\xi)], & |\xi| > 1 \end{cases}.$$

$\Gamma(f)$, defined by eq. (21), represents an n-pole, equalized Chebyshev filter. Some results for this case are presented in Fig. 8. By comparing Figs. 5 and 8, we can determine the effect of equalization on FM distortion. In this case, equalization does not reduce the FM distortion significantly.

† Or linear in frequency since time delay is unimportant here.

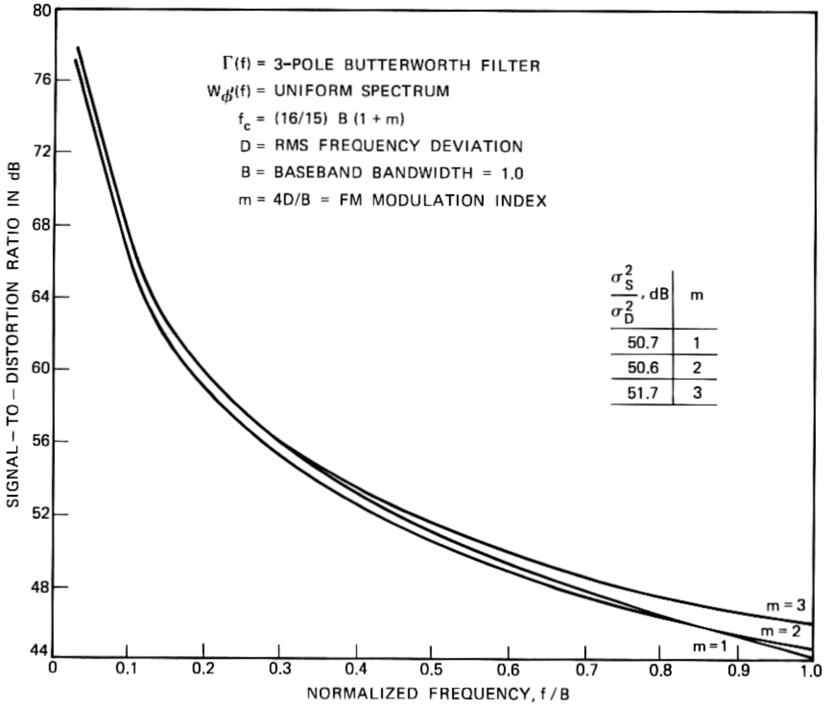

Fig. 9—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_{\phi'}(f)$.

4.2.3 *n-pole Butterworth filter*

Let $W_{\phi'}(f)$ be uniform as given by eq. (8) with $\Gamma(f)$ given by

$$\Gamma(f) = \frac{\prod_{k=1}^{n} (p_k)}{\prod_{k=1}^{n} (i\omega + p_k)} \tag{22}$$

where

$$p_k = (2\pi f_c) \exp\left[i\frac{\pi}{2}\left(\frac{2k-1}{n} - 1\right)\right], \qquad k = 1, 2, \cdots, n$$

$$B = 1 = \text{baseband bandwidth}$$

$$2f_c = K2B(1 + m)$$

$$m = \frac{4D}{B}.$$

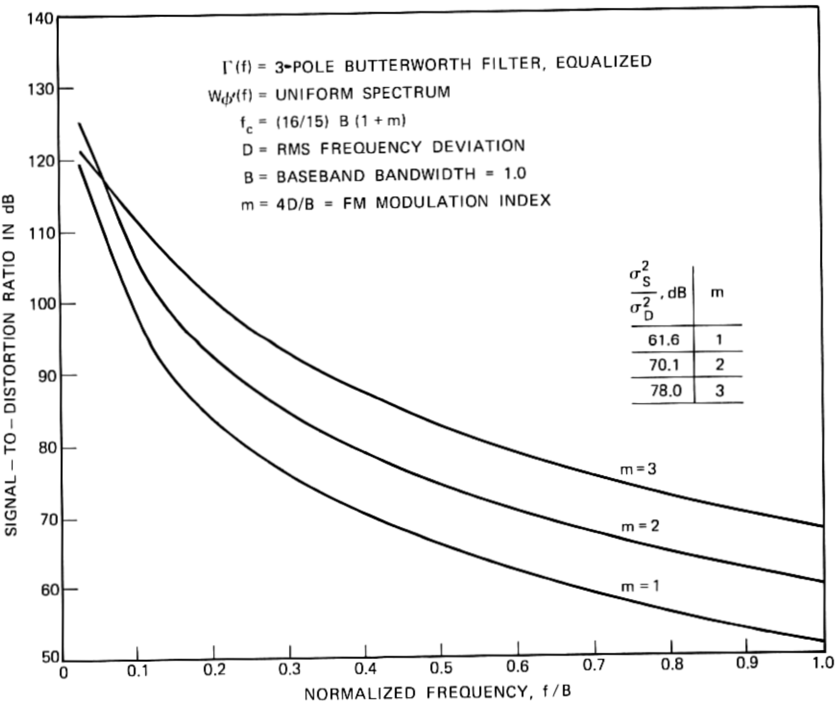

Fig. 10—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_{\phi'}(f)$.

$\Gamma(f)$, defined by eq. (22), represents an n-pole, Butterworth filter. Some results for this case are presented in Fig. 9. By comparing Figs. 5 and 9, we see that the 3-pole Butterworth filter produces much less FM distortion than does the 9-pole Chebyshev filter.

4.2.4 *n-pole, equalized Butterworth filter*

If the phase function associated with eq. (22) is taken to be zero, or linear in frequency, $\Gamma(f)$ can be written as

$$\Gamma(f) = \left[1 + \left(\frac{f}{f_c}\right)^{2n}\right]^{-\frac{1}{2}}. \tag{23}$$

$\Gamma(f)$, defined by eq. (23), represents an n-pole, equalized Butterworth filter. Some results for this case are presented in Fig. 10. By comparing Figs. 9 and 10, we see that equalization reduces the FM distortion significantly in this case.

4.2.5 *Echo, envelope delay, Butterworth filter*

Let $W_{\phi'}(f)$ be uniform as given by eq. (8) with $\Gamma(f)$ given by

$$\Gamma(f) = \underset{\text{(echo)}}{\left[\frac{1 + re^{i\omega T}}{1 + r}\right]} \underset{\text{(envelope delay)}}{[\exp\{i(b_2\omega^2 + b_3\omega^3)\}]} \times \underset{\text{(3-pole Butterworth filter)}}{\left[\frac{1 + 2(i\xi_0) + 2(i\xi_0)^2 + (i\xi_0)^3}{1 + 2(i\xi) + 2(i\xi)^2 + (i\xi)^3}\right]} \quad (24)$$

where

r = amplitude of echo

T = time delay of echo

b_2 = linear envelope delay constant

b_3 = quadratic envelope delay constant

$$\xi = \frac{f - f_e}{f_c}$$

$$\xi_0 = -\frac{f_e}{f_c}$$

f_e = frequency offset or mistuning

D = RMS frequency deviation

$B = 1$ = baseband bandwidth

$2f_c = K2B(1 + m) = K$ times Carson's rule

= filter bandwidth

$m = \dfrac{4D}{B}$ = FM modulation index.

Some results for this case are presented in Fig. 11.

Results are presented in Fig. 12 for the case when $\Gamma(f)$ is given by eq. (24) and $W_\phi(f)$, rather than $W_{\phi'}(f)$, is uniform and given by

$$W_\phi(f) = \begin{cases} \dfrac{3}{2}\dfrac{D^2}{B^3}, & |f| \leqq B \\ 0, & |f| > B. \end{cases} \quad (25)$$

This is the case of a noise loading test applied to a phase modulated

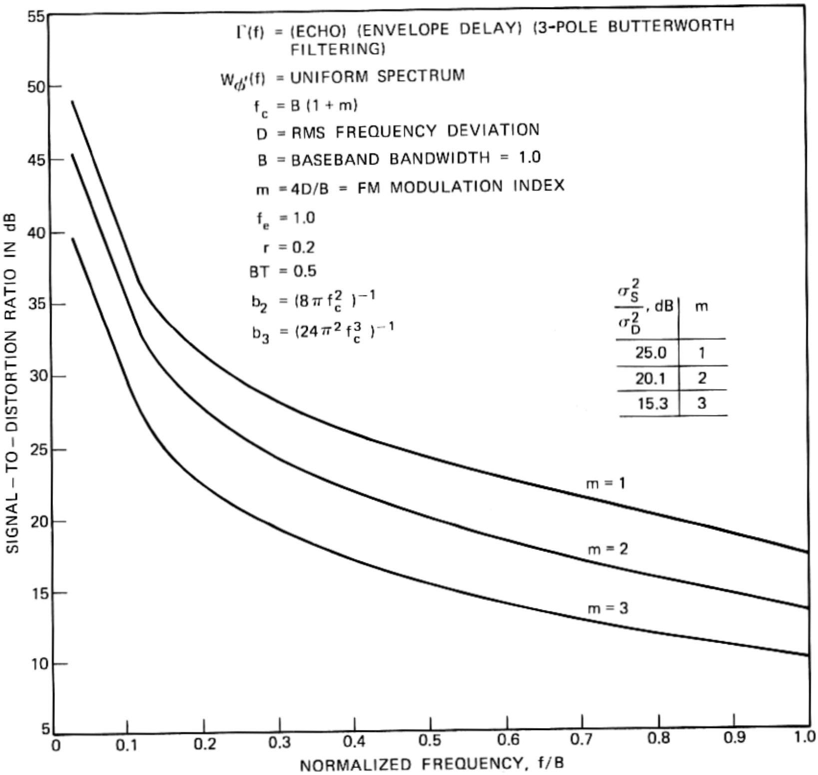

Fig. 11—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_{\phi'}(f)$.

system. From eq. (1) we have

$$W_{\phi'}(f) = \begin{cases} \dfrac{(2\pi D)^2 3 f^2}{2B^3}, & |f| \leqq B \\ 0 & |f| > B. \end{cases} \tag{26}$$

In this case, $W_{\phi'}(f)$ peaks at $f = B$ in contrast to the RC spectrum given by eq. (20), which peaks at $f = 0$. These results lead us to an interesting question. Given $W_{\phi'}(f)$ and $\Gamma(f)$, can we choose a predistortion characteristic such that the shape of $S(f)/D(f)$ is most suitable for a particular communication system? However, we have not investigated this question.

We can compare the distortion resulting from PM and FM systems by comparing Figs. 11 and 12. In fact, if the results in Fig. 12 apply to

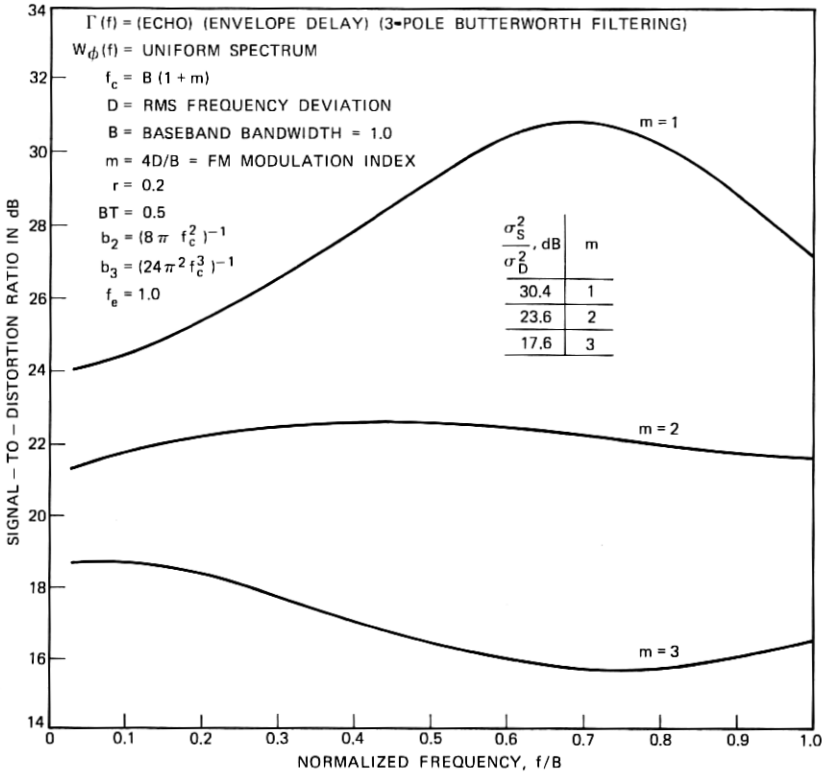

Fig. 12—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_\phi(f)$.

a Phase modulation system, then the results in Fig. 11 apply to the corresponding Frequency modulation system.

4.2.6 *Filter characteristic determined experimentally*

In all of the results presented above, the filter characteristic $\Gamma(f)$ was specified mathematically. However, many situations arise for which the measured amplitude and envelope delay characteristics are available in graphic form. Of course, one may try to fit a suitable analytical expression to these experimental points and proceed as above. However, there is no need to develop such an analytical expression. As our final case, we shall present some results which illustrate this point.

Let the amplitude and phase of $\Gamma(f)$ be given by the experimental

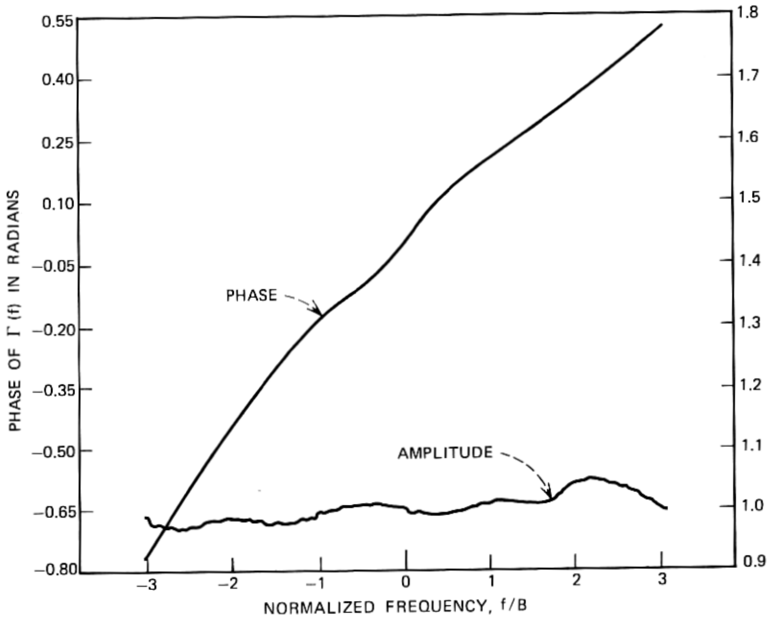

Fig. 13—Experimental amplitude and phase characteristic specifying $\Gamma(f)$.

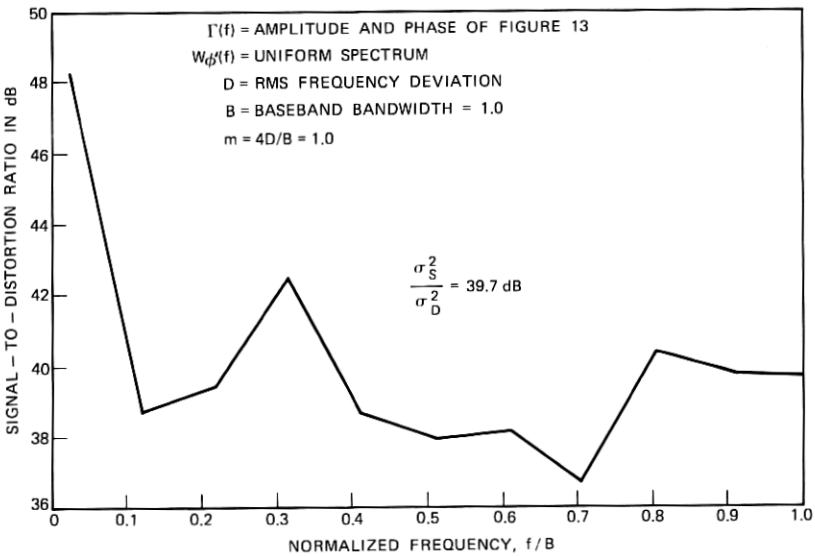

Fig. 14—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_{\phi'}(f)$.

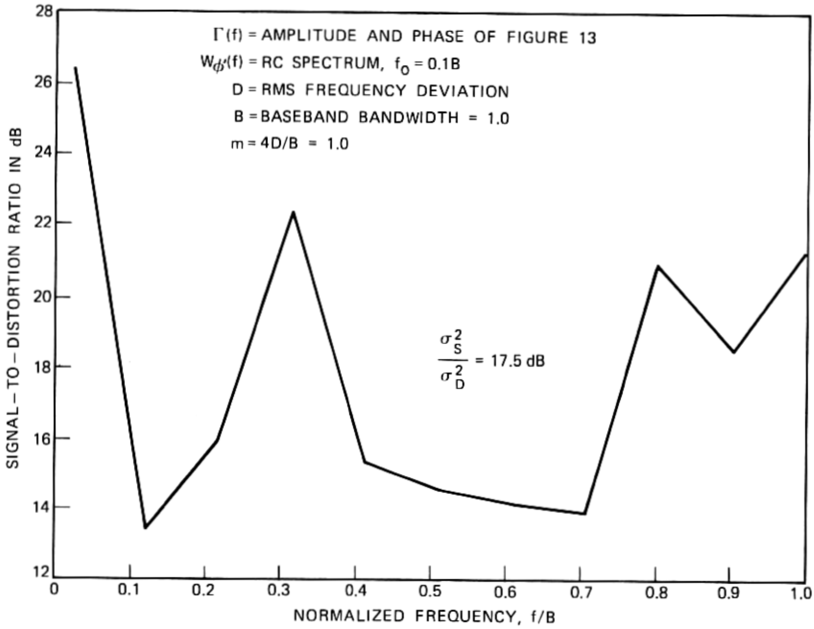

Fig. 15—Signal-to-distortion ratio resulting from $\Gamma(f)$ and $W_{\phi'}(f)$.

curves shown in Fig. 13. Let $W_{\phi'}(f)$ again be uniform and be given by eq. (8). Figure 14 presents some results for this case.

Now let $W_{\phi'}(f)$ be an RC spectrum given by eq. (20). Figure 15 presents some results for this case. We see that considerably more distortion is indicated in Fig. 15 as compared with Fig. 14.

Thus, the often-used noise loading test which makes use of a uniform spectrum may not represent a worst-case situation as far as FM distortion is concerned. A theoretical proof of this interesting fact is presented in the appendix.

V. CONCLUSIONS

Equations (4) through (7), together with a digital computer, can be used to compute the FM distortion resulting from passing FM waves through linear networks. To demonstrate the utility of the program, we have presented a variety of results in graphic form.

From this work, it is apparent that the often-used noise loading test does not necessarily represent a worst-case test. This was demonstrated for a system in which the modulating signal is a video signal. It is also apparent that predistortion may be useful in reducing FM distortion. This is in contrast to the use of conventional pre-emphasis, which is applied to combat RF noise.

Some important sources of FM distortion which were neglected in our analysis are AM-to-PM conversion, baseband and RF noise, and adjacent channel interference.

VI. ACKNOWLEDGMENTS

It gives me great pleasure to acknowledge helpful discussions with S. O. Rice. I also wish to express my thanks to J. N. Lahti and A. Yagoda for some useful comments, and to Miss R. Wright and F. L. Crawford for programming the digital computer.

APPENDIX

Theoretical Comparison of FM Distortion Resulting from Video and Uniform Spectra

The purpose of this appendix is to present a theoretical argument which shows that a video spectrum can lead to more FM distortion than a uniform spectrum. That is, a noise loading test which makes use of a uniform spectrum may not represent a worst-case test when the information source is a video signal.

Let the video signal $\phi'(t)$ be represented as a zero-mean, stationary Gaussian process having power spectral density $W_{\phi'}(f)$ given by

$$W_{\phi'}(f) = \begin{cases} \dfrac{(2\pi D)^2}{2f_0 \tan^{-1}\left(\dfrac{B}{f_0}\right)} \left[1 + \left(\dfrac{f}{f_0}\right)^2\right]^{-1}, & |f| \leqq B \\ 0, & |f| > B \end{cases} \tag{27}$$

where

$$D = \text{RMS frequency deviation}$$

$$B = \text{baseband bandwidth}$$

$$f_0 = \text{3-dB bandwidth.}$$

Let $\Gamma(f)$ be given by

$$\Gamma(f) = \exp\left[i(b_2\omega^2 + b_4\omega^4)\right] \tag{28}$$

where

$$b_2 = \text{small linear envelope delay constant}$$

$$b_4 = \text{small cubic envelope delay constant.}$$

By applying eq. (24) of Rice,[3] the leading term of $D(f)$ can be expressed as

$$D(f) = 2^{-1}(\lambda_{2i} + 2^{-1}D^2\lambda_{4i})^2 \int_{-\infty}^{\infty} d\rho W_\phi(\rho) W_\phi(f-\rho)\rho^2(f-\rho)^2 \qquad (29)$$

where

$$\lambda_{2i} = (2!)(2\pi)^2 b_2$$

$$\lambda_{4i} = (4!)(2\pi)^4 b_4.$$

By taking $S(f) = W_\phi(f) = W_{\phi'}(f)/\omega^2$ and evaluating the integral for $D(f)$, we find that

$$R(f) = \frac{\left.\dfrac{D(f)}{S(f)}\right|_{f_0}}{\left.\dfrac{D(f)}{S(f)}\right|_{f_0=\infty}} = \frac{2}{[2-|F|]\tan^{-1}\left(\dfrac{1}{F_0}\right)}\left[\frac{1+\left(\dfrac{F}{F_0}\right)^2}{4+\left(\dfrac{F}{F_0}\right)^2}\right]$$

$$\times\left\{\frac{F_0}{|F|}\ln\frac{1+F_0^2}{(|F|-1)^2+F_0^2} + \tan^{-1}\left(\frac{1}{F_0}\right) + \tan^{-1}\left(\frac{1-|F|}{F_0}\right)\right\} \qquad (30)$$

where

$$F = \frac{f}{B}$$

$$F_0 = \frac{f_0}{B}$$

$$|f| \leqq B$$

$$-\frac{\pi}{2} \leqq \tan^{-1}(\cdot) \leqq \frac{\pi}{2}.$$

$R(f)$ represents the distortion-to-signal ratio for a video spectrum divided by the distortion-to-signal ratio for a uniform spectrum. If we can show that $R(f) > 1$ for particular values of f_0, the 3-dB bandwidth of $W_{\phi'}(f)$, then we can conclude that a video spectrum can produce more FM distortion than a uniform spectrum.

In order to show that $R(f)$ can be greater than unity, consider the important frequency range $0 < |F| < 1$. For this baseband frequency range, eq. (30) yields

$$\lim_{F_0 \to 0} R(f) = \frac{4}{2-|F|} > 1. \qquad (31)$$

A plot of $R(f)$ for various values of F_0 is shown in Fig. 16.

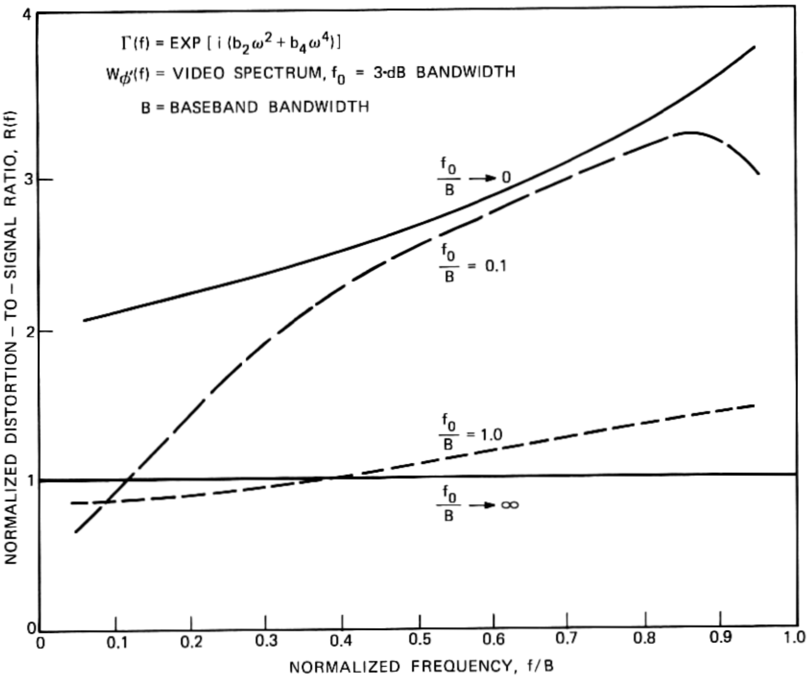

Fig. 16—$R(f)$ denotes the distortion-to-signal ratio for a video spectrum divided by distortion-to-signal ratio for a uniform spectrum.

Accordingly, we conclude that a video spectrum can produce more FM distortion in the baseband frequency range than a uniform spectrum.

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