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Passband Equalization of Differentially Phase-Modulated Data Signals

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A new approach to the automatic equalization of differentially phase-modulated data signals passing through a linear distorting medium is presented. The equalizer, which is of the transversal filter type, operates in the frequency passband and contains two sets of taps–in-phase and quadrature. A tap-rotation property of the equalizer is used to establish an absolute phase reference at the equalizer output, and once this crucial step is taken, the passband equalizer output can be used to automatically (as well as adaptively) adjust the tap weights so as to minimize a mean-square distortion function. The resulting algorithm requires correlating an error signal with the tap voltages, and thus it is possible to use a structure similar to that employed when equalizing baseband PAM.

The generality of the approach makes it applicable to the equalization of any double-sideband data signal.

I. INTRODUCTION

Recently much attention has been given to high-speed synchronous data transmission via differential phase modulation and comparison detection. High-speed (above 2400 bits per second) transmission usually requires equalization[1] to compensate for the linear distortion introduced by the channel. This study will be concerned with the automatic and adaptive equalization of such data signals. At first glance, the nonlinear nature of the modulated signal would seem to preclude linear compensation. Upon closer examination it becomes apparent that a digitally phase-modulated signal, when resolved into in-phase and quadrature components, is linear in each component. Since this property is preserved after transmission through a linear medium, simultaneous linear equalization of each signal component becomes feasible. Due, however, to the purely quadratic nature of the channel-dependent terms present in the differential detector output, a linear equalizer should† precede the detector–hence the equalizer operates in the frequency passband.

Resolving the channel output into in-phase and quadrature components suggests that satisfactory performance will be obtained if the equalized in-phase signal is approximately Nyquist and the equalized quadrature signal has most of its samples close to zero. This is accomplished by choosing an appropriate cost function (of the equalized samples) and using an appropriately structured equalizer capable of minimizing this function. We will focus our attention on a mean-square cost function and on a synchronous tapped delay line (TDL) equalizer. The equalizer has two branches–an in-phase and quadrature branch–which together perform the passband compensation. The delays are separated by a symbol interval, and each delay output is first multiplied by a tap weight (each signal in the quadrature branch is also shifted by 90 degrees) and then added to the other delayed tap signals. The generality of the approach makes it applicable to the equalization of any double-sideband modulated data signal (e.g., combined amplitude and phase modulation, quadrature amplitude modulation).

Though the differential detector output is independent of any absolute phase reference, it is convenient to select a cost function which depends on such a quantity. The awkwardness of this situation is resolved by noting the tap-rotation property of the equalizer. Simply put, this means that any assumed reference phase angle can be "matched" by a rotation of the equalizer taps. This, in effect, permits the establishment of an arbitrary phase reference at the equalizer output. Equalization is accomplished with respect to this arbitrary reference while detection is done in an incoherent fashion. The tap adjustment algorithm is similar to that done in baseband PAM, i.e., tap signals are correlated with error signals. In fact, once the notion of a tap rotation is introduced, there is a convenient analogy to baseband PAM equalization.

The basic system equations are indicated in Section II, and the equalizer cost function and structure are described in Section III. Section IV describes methods for adjusting the equalizer taps when an ideal data reference sequence is available and also when the adjustments are made using random data. In Section V we consider the effects of frequency offset and phase jitter on the equalizer.

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† In order that the mean-square distortion be a quadratic, and thus easily minimized, function of the in-phase and quadrature pulse samples.

II. BASIC SYSTEM EQUATIONS

In this section we indicate the response of a linear passband channel to a phase-modulated data signal. We also make some observations concerning the choice of phase reference that will be useful in our discussion of passband equalization.

The channel input is a phase-modulated (PM) data signal. This signal is given by

$$s_i(t) = \sum_n p(t - nT) \cos(\omega_c t + \theta_n) \tag{1a}$$

$$= u_{\text{in}}(t) \cos \omega_c t - v_{\text{in}}(t) \sin \omega_c t \tag{1b}$$

$$= r_{\text{in}}(t) \cos(\omega_c t + \Psi_{\text{in}}(t)), \tag{1c}$$

where θ_n is the information symbol, $1/T$ is the symbol rate, ω_c is the carrier frequency, $p(t)$ is the impulse response of the transmitter shaping filter,† and

$$u_{\text{in}}(t) \equiv \sum_n p(t - nT) \cos \theta_n \tag{1d}$$

$$v_{\text{in}}(t) \equiv \sum_n p(t - nT) \sin \theta_n \tag{1e}$$

are respectively the in-phase and quadrature signal components. The envelope and phase of $s_i(t)$ are given by

$$r_{\text{in}}(t) = \sqrt{u_{\text{in}}^2(t) + v_{\text{in}}^2(t)} \tag{1f}$$

$$\Psi_{\text{in}}(t) = \tan^{-1} \frac{v_{\text{in}}(t)}{u_{\text{in}}(t)}. \tag{1g}$$

By letting

$$a_n = \cos \theta_n \tag{1h}$$

$$b_n = \sin \theta_n, \tag{1i}$$

we can write

$$s_i(t) = \left[\sum_n a_n p(t - nT)\right] \cos \omega_c t - \left[\sum_n b_n p(t - nT)\right] \sin \omega_c t. \tag{1j}$$

From eq. (1j), we can see that the bandlimited signal $s_i(t)$ is linear in both a_n and b_n.‡ As we have previously remarked, it is this linear

† We assume the usual Nyquist shaping in the sense that $p(kT) = 0,\ k \neq 0$.

‡ Note that any double-sideband signal, such as quadrature amplitude modulation (QAM), can be written in the form of (1j). What we have to say in the sequel is sufficiently general to apply to any such double-sideband signals.

representation that makes linear compensation first plausible then possible.

We note that the transmitter changes the value of θ_n once every T seconds, and that for differentially coded data, the customer information is the quantity $\theta_n - \theta_{n-1}$. Applying $s_i(t)$ to a bandpass channel with impulse response

$$2F_1(t) \cos \omega_c t - 2F_2(t) \sin \omega_c t,$$

produces an output signal

$$s_o(t) = u_o(t) \cos \omega_c t - v_o(t) \sin \omega_c t, \tag{2a}$$

where

$$\begin{aligned} u_o(t) &= F_1(t) \circledast u_{\text{in}}(t) - F_2(t) \circledast v_{\text{in}}(t) \\ v_o(t) &= F_2(t) \circledast u_{\text{in}}(t) + F_1(t) \circledast v_{\text{in}}(t), \end{aligned} \tag{2b}$$

with $\circledast$ denoting the convolution operation. To obtain a more compact representation, we define the in-phase channel pulse

$$x(t) = p(t) \circledast F_1(t), \tag{3a}$$

and the quadrature channel pulse

$$y(t) = p(t) \circledast F_2(t). \tag{3b}$$

In terms of the data components

$$\begin{aligned} a_n &= \cos \theta_n, \\ b_n &= \sin \theta_n, \end{aligned}$$

and the above notation, the channel output is given by

$$\begin{aligned} s_o(t) = \cos \omega_c t[\sum_n a_n x(t - nT) - b_n y(t - nT)] \\ - \sin \omega_c t[\sum_n a_n y(t - nT) + b_n x(t - nT)]. \end{aligned} \tag{4}$$

The above in-phase and quadrature representation clearly indicates the effect of the channel on the transmitted signal. A "good" channel would be one which has a small quadrature pulse† (ideally all its samples would be zero) and an in-phase pulse which is Nyquist. In any event, it is important to note that in passband the channel distortion appears in a linear fashion.

† If the channel has even amplitude symmetry and odd phase symmetry about the carrier, then $y(t)$ will be identically zero for all t.

Since the carrier phase reference is arbitrary, the output signal can also be written as

$$s_o(t) = \cos(\omega_c t + \phi)\Big[\sum_n a_n x^{(\phi)}(t - nT) - b_n y^{(\phi)}(t - nT)\Big]$$
$$- \sin(\omega_c t + \phi)\Big[\sum_n a_n y^{(\phi)}(t - nT) + b_n x^{(\phi)}(t - nT)\Big] \quad (5)$$

where $x^{(\phi)}(t)$ and $y^{(\phi)}(t)$ denote the in-phase and quadrature pulses with respect to the reference phase† ϕ. Comparing the right-hand sides (RHS) of (4) and (5) gives

$$\begin{bmatrix} x^{(\phi)}(t) \\ y^{(\phi)}(t) \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \quad (6)$$

Equation (6) indicates that changing the phase reference by ϕ radians corresponds to *rotating* the in-phase and quadrature signal vector $[x(t),\ y(t)]$ by this amount.‡ We wish to emphasize the above interpretation, since we will want to consider the effect of such a rotation on several quantities of interest. We indicate the incoherent nature of the system by including an arbitrary phase angle when describing the channel output, i.e., the absolute phase of the received signal is unknown to the receiver. For incoherent reception, it is common to use differential coding of the data and differentially coherent (comparison) detection. We will assume this mode of transmission and detection.

III. MEAN-SQUARE PASSBAND EQUALIZATION

We wish to develop an adaptive equalizer which is capable of compensating for the linear (in-phase and quadrature) channel distortion, while keeping the equalizer adaption (settling) time small. "Mean-square" adaptive equalization[1,2] has been successfully used in systems using linear modulation to meet the above objectives. Our approach will be to extend, with appropriate modification, this technique to the problem at hand. After specifying the equalizer structure, we select an appropriate cost function and indicate how the equalizer processes the received random signal to adapt to the optimum configuration. We begin by considering the output of the differential detector.

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† We reserve the right to suppress the phase reference when there is no possibility of confusion. Of course the phase reference can be chosen as convenience dictates.
‡ Clearly the same relation also holds for the sampled values of $x(t)$ and $y(t)$.

3.1 *The Detector Output*

The equalizer is to be compatible with the standard comparison detector[1] which is shown in Fig. 1. It is straightforward to show that the detector outputs at the nth sampling instant are†

$$I(nT + t_o) = \sum_i \sum_j \cos(\theta_i - \theta_j)[x_{n-i}x_{n-j-1} + y_{n-i}y_{n-j-1}] + \sum_i \sum_j \sin(\theta_i - \theta_j)[x_{n-i}y_{n-j-1} - y_{n-i}x_{n-j-1}] \quad (7)$$

$$Q(nT + t_o) = \sum_i \sum_j \cos(\theta_i - \theta_j)[y_{n-i}x_{n-j-1} - x_{n-i}y_{n-j-1}] + \sum_i \sum_j \sin(\theta_i - \theta_j)[x_{n-i}x_{n-j-1} + y_{n-i}y_{n-j-1}] \quad (8)$$

where

$$x(nT + t_o) = x_n$$

and

$$y(nT + t_o) = y_n.$$

We note that the intersymbol interference at the detector output is of a quadratic nature and thus is quite different than that encountered in linear modulation. It should also be noted that the channel-dependent terms in (7) are inner products, and hence are independent of the phase reference. The terms in (8) are components of cross products and hence also independent of phase. This is a manifestation of the differential nature of the receiver, i.e., only quantities which are invariant with respect to a change in phase reference can affect the detector output.

If we are fortunate enough to have a distortionless channel, i.e., for all integers n,

$$x_n = \delta_{no} \quad (9a)$$

$$y_n = 0, \quad (9b)$$

then detection can be simply accomplished by noting that

$$I(nT + t_o) = a_n a_{n-1} + b_n b_{n-1} = \cos(\theta_n - \theta_{n-1}) \quad (10a)$$

$$Q(nT + t_o) = b_n a_{n-1} - a_n b_{n-1} = \sin(\theta_n - \theta_{n-1}). \quad (10b)$$

Unfortunately, (9) will not be satisfied for an arbitrary channel. As mentioned above, the function of the proposed equalizer is to linearly

† We have, of course, neglected noise and have suppressed the phase reference. We also wish to emphasize that a desirable sampling epoch t_o should be determined. As usual, this is a problem in its own right, but is not considered further in this paper.

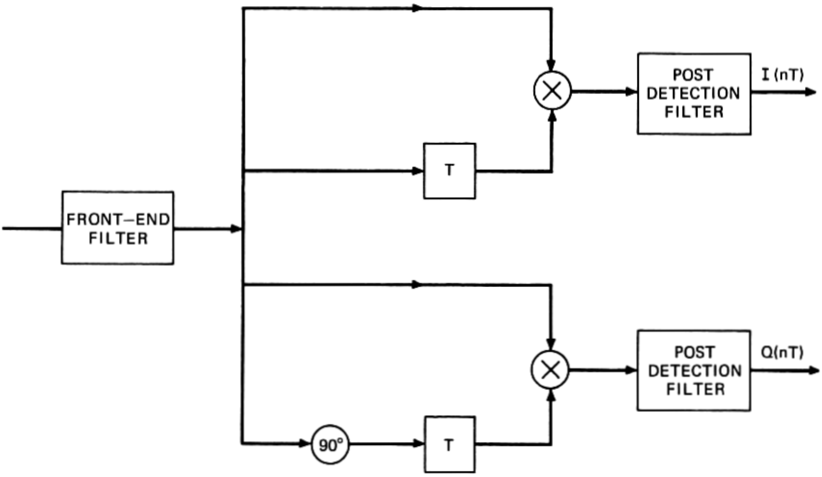

Fig. 1—Comparison (differential) detector.

operate on the in-phase and quadrature samples (the x_n's and y_n's) to produce an overall system response the samples of which (denoted by g_n's and h_n's) approximately satisfy eq. (9). The quality of this approximation is measured by the value of an appropriate cost function. We will have more to say about this later.

3.2 *Equalizer Structure*

Since we would like to equalize both the in-phase and quadrature channel samples, a structure which combines two signals in quadrature is suggested. The equalizer, which is shown in Fig. 2, consists of a tapped delay line (TDL) where each of the $2N + 1$ delay outputs is fed into two branches. In each branch the delay outputs are multiplied by tap weights (c_i and d_i) and then summed. The equalizer output $q(t)$ is the sum of the upper output $q_1(t)$ and the Hilbert transform of the lower output $q_2(t)$. The 90-degree phase shift at the output of the lower branch provides the quadrature signal. Thus the taps are to be adjusted to minimize an appropriate (cost) function of the equalized in-phase and quadrature samples. We remark that a linear equalizer placed in the passband can compensate for linear channel distortion, while a linear equalizer placed after the comparison detector cannot hope to compensate for the quadratic distortion present in the detector output. Thus the equalizer precedes the detector and operates in the frequency passband. We now describe the equalizer operation in detail.

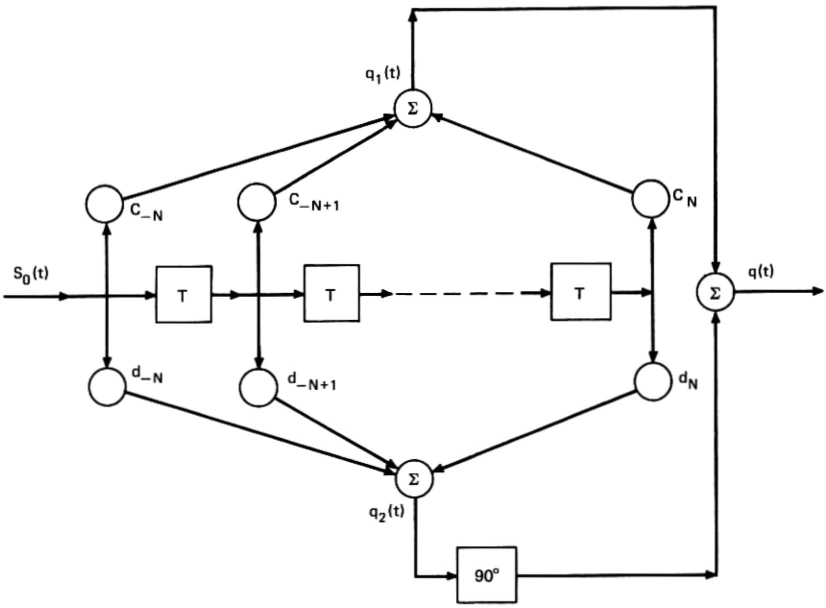

Fig. 2—The passband equalizer.

We can write the equalizer output, $q(t)$, as†

$$q(t) = \cos \omega_c t[\sum_n a_n g(t - nT) - \sum_n b_n h(t - nT)] - \sin \omega_c t[\sum_n a_n h(t - nT) + \sum_n b_n g(t - nT)] \quad (11)$$

where the in-phase and quadrature (equalized) signals, $g(t)$ and $h(t)$, are given by‡

$$g(t) = \sum_{n=-N}^{N} c_n x(t - nT) - \sum_{n=-N}^{N} d_n y(t - nT) \quad (12a)$$

$$h(t) = \sum_{n=-N}^{N} d_n x(t - nT) + \sum_{n=-N}^{N} c_n y(t - nT). \quad (12b)$$

We emphasize that $s_i(t)$ and $q(t)$ are defined with respect to the same

† Assuming that ω_c and T are chosen such that $\omega_c T$ is an integer multiple of 2π. If for some reason this is not convenient (perhaps a particular carrier frequency is desired), then $\omega_c T$ can be chosen to be any convenient angle. Some additional bookkeeping is then required at the receiver.

‡ The time reference has been taken at the center tap of the equalizer.

phase reference, but that the phase reference has not been indicated in the above equations. For a sampling epoch t_o, we denote the equalized samples by

$$g_j = g(jT + t_o), \quad h_j = h(jT + t_o). \tag{13}$$

Thus with the reference phase taken to be ϕ, the equalized samples are related to the channel samples and tap weights by

$$g_j^{(\phi)} = \sum_{-N}^{N} c_n x_{j-n}^{(\phi)} - \sum_{-N}^{N} d_n y_{j-n}^{(\phi)} \tag{14a}$$

$$h_j^{(\phi)} = \sum_{-N}^{N} d_n x_{j-n}^{(\phi)} + \sum_{-N}^{N} c_n y_{j-n}^{(\phi)}. \tag{14b}$$

Again we reserve the right to suppress the representation phase in writing (14). We now make some observations which indicate the effects of a change in reference phase and of a "tap-rotation" on the equalized samples.

Denoting the channel phase reference by ϕ, we can write the equalizer output as

$$q(t) = \cos(\omega_c t + \phi)[\sum_n a_n g^{(\phi)}(t - nT) - \sum_n b_n h^{(\phi)}(t - nT)]$$
$$- \sin(\omega_c t + \phi)[\sum_n a_n h^{(\phi)}(t - nT) + \sum_n b_n g^{(\phi)}(t - nT)], \tag{15}$$

and it is easy to see that

$$\begin{bmatrix} g^{(\phi)}(t) \\ h^{(\phi)}(t) \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}. \tag{16}$$

Thus we again observe that changing the reference phase by ϕ radians corresponds to rotating the in-phase and quadrature signals (and samples) by this amount, i.e., each two-tuple $(g_i^{(\phi)}, h_i^{(\phi)})$ is rotated by ϕ radians. Suppose that while keeping the phase reference fixed, we transform the taps (c_i and d_i) to new values ($\tilde{c}_i$ and $\tilde{d}_i$) via the rotation

$$\begin{bmatrix} \tilde{c}_i \\ \tilde{d}_i \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c_i \\ d_i \end{bmatrix} \qquad i = -N, \cdots, 0, \cdots N. \tag{17}$$

We can now express the equalized samples $\tilde{g}_i$ and $\tilde{h}_i$ (corresponding to $\tilde{c}_i$ and $\tilde{d}_i$), using (14) and (17), as

$$\begin{bmatrix} \tilde{g}_i \\ \tilde{h}_i \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} g_i \\ h_i \end{bmatrix} \qquad i = -N, \cdots, 0, \cdots N. \tag{18}$$

Thus rotating the equalizer taps by θ will rotate the equalized samples by the same amount.† It is important to note that the original and (tap) rotated equalized samples are defined with respect to the *same* phase reference (ϕ). This is in contrast to the rotation introduced when the reference phase is changed. We will use these notions in describing the equalizer adjustment algorithm, in fact, the key to the simplicity of the equalization system lies in recognizing that the possibility of tap-rotation allows us to fix the reference phase and still keep a rotational degree of freedom between all (g_i, h_i) pairs. The reference phase will be suppressed in the sequel.

It remains to choose an appropriate cost function (of the equalized samples) and to describe an iterative procedure for determining the tap weights that minimize this function.

3.3 *The Cost Function*

The cost function should be such that after minimization, the equalized channel is approximately ideal (i.e., the intersymbol interference has in some sense been suppressed). There are, of course, many such functions–thus our selection is influenced by additional considerations.

We begin by noting that one possible task to set for the equalizer would be the minimization of

$$D = \sum_{i \neq 0} g_i^2 + \sum_{i \neq 0} h_i^2, \tag{19}$$

subject to appropriate constraint on g_0 and h_0.‡ We observe that since D is independent of an equalizer tap rotation, the minimum value of D is the same for any constraint that satisfies

$$g_0^2 + h_0^2 = 1, \tag{20}$$

including the choices

$$g_0 = 1, \quad h_0 = 0. \tag{21}$$

Clearly a choice such as (21) fixes the reference phase used to describe the equalizer output, and a "rotation" of the equalizer taps can assure that it is met. Ultimately, the information needed to adjust the equalizer taps will indeed be obtained with respect to a particular

† It should be clear that if there were no d_0 tap, we could not perform this rotation.

‡ We note that this cost function would also be appropriate for a data signal which employs combined AM and PM. In fact, it is a simple matter to extend all of our results (i.e., the equalizer structure and algorithm) to this signaling format.

phase reference [in fact, we use the one implied by (21)]. Hence the tap adjustments will be made using coherently obtained information.

An additional interpretation of (21) is that the tap vectors are always required to lie on the hyperplanes $g_0 = 1$, $h_0 = 0$. Unfortunately, minimizing a cost function subject to this (hard) constraint cannot be done in real-time. This is due to the awkward requirement that the tap vectors be adjusted while constrained to lie on hyperplanes. For example, suppose we were using a steepest descent[3] algorithm to adjust the taps, then we would need the projected (onto the constraint hyperplanes) gradient of D at every iteration.† At present it is not known how to easily generate such a gradient from the available circuit voltages.

We circumvent this difficulty by imposing a quadratic penalty (a soft constraint) when the variables do not satisfy the constraints.[3] Thus the equalizer will try to minimize the unconstrained mean-square distortion

$$\mathcal{E} = \sum_{i \neq 0} g_i^2 + \sum_{i \neq 0} h_i^2 + (1 - g_0)^2 + h_0^2. \tag{22}$$

It should be noted that for small distortion, the optimum D [subject to (21)] and $\mathcal{E}$ will be essentially identical.‡

A useful consequence of the notions of phase-reference rotation and tap rotation will now be demonstrated. Suppose the phase reference has been chosen and is fixed at this value. Then $\mathcal{E}$, which can be rewritten as§

$$\mathcal{E} = \sum_{\text{all } i} (g_i^2 + h_i^2) + 1 - 2g_0, \tag{23}$$

clearly depends upon the chosen phase reference, since g_0 does. However, since we have shown that a simple *tap rotation* can be used to effect any desired phase, we are guaranteed that the minimization of $\mathcal{E}$ will be over *all* reference phase angles. That is, while (23) is in general not reference-phase invariant, the minimum of (23) over all tap settings is reference-phase invariant. A further manifestation of the tap-rotation property is the following necessary condition for the optimum tap setting: The optimum tap setting is such that $h_0 = 0$. To see that this must be the case, suppose that the taps have settled down and

† A more detailed discussion of this point is available in Ref. 4.
‡ See Ref. 4 for a discussion of this type of subject for baseband PAM.
§ Since the phase reference has been fixed, we do not indicate this quantity when writing the equalized samples.

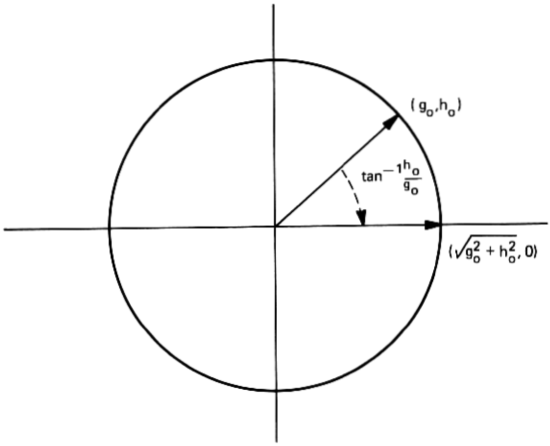

Fig. 3—Illustration of the tap-rotation property.

that $h_0 \neq 0$. Since $\sum_i g_i^2 + h_i^2$ is unchanged under a tap rotation, the equalizer taps could now be rotated so as to further minimize $\mathcal{E}$ by maximizing g_0 (i.e., making $h_0 = 0$). The rotation of the vector (g_0, h_0) into the vector $(\sqrt{g_0^2 + h_0^2}, 0)$ is shown in Fig. 3; thus the required rotation is $\tan^{-1}(h_0/g_0)$.

Recalling that a differential detector is only affected by phase-invariant quantities, we observe that if it were not possible to perform the above sort of minimization, then the utility of the structure and cost function would be suspect.

IV. EQUALIZER TAP ADJUSTMENT

Now that we have selected a cost function, we wish to demonstrate how the equalizer taps should be adjusted to minimize this function. We proceed in a manner reminiscent of mean-square equalization for baseband PAM–in fact it is shown that the tap increments are obtained by cross correlating error signals with delayed equalizer inputs.

4.1 *Deterministic Aspects*

We first wish to show that our criteria, $\mathcal{E}$, is an easily minimized function of the tap weights. We denote the tap vectors by

$$\begin{aligned} \mathbf{c}^T &= (c_{-N}, \cdots, c_0, \cdots, c_N) \\ \mathbf{d}^T &= (d_{-N}, \cdots, d_0, \cdots, d_N) \end{aligned} \tag{24}$$

where the superscript denotes transpose. It is convenient to introduce the channel correlation matrices[†]

$$[X]_{ij} = \sum_n x_{n-i}x_{n-j} \tag{25a}$$

$$[Y]_{ij} = \sum_n y_{n-i}y_{n-j} \tag{25b}$$

$$[K]_{ij} = \sum_n (y_{n-i}x_{n-j} - x_{n-i}y_{n-j}), \tag{25c}$$

and the truncated channel vectors

$$\begin{aligned} \mathbf{x}^T &= (x_N, \cdots x_0, \cdots x_{-N}) \\ \mathbf{y}^T &= (y_{-N}, \cdots y_0, \cdots y_N). \end{aligned} \tag{26}$$

We can now write the mean-square distortion, in matrix notation as

$$\mathcal{E} = (\mathbf{c}^T, \mathbf{d}^T) \begin{bmatrix} X + Y & K \\ -K & X + Y \end{bmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} - 2(\mathbf{x}^T, -\mathbf{y}^T) \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} + 1. \tag{27}$$

A few comments are in order on the above representation. We first note that X and Y are the in-phase and quadrature channel correlation matrices. Thus if the quadrature component is zero, we need only the $\mathbf{c}$ taps. The cross-correlation between the in-phase and quadrature channel is measured by the skew-symmetric matrix K. By introducing the bandpass channel correlation matrix

$$A = \begin{bmatrix} X + Y & K \\ -K & X + Y \end{bmatrix}, \tag{28}$$

the augmented channel vector

$$\mathbf{z}^T = (x^T, -y^T), \tag{29}$$

and the composite tap vector

$$\mathbf{b}^T = (\mathbf{c}^T, \mathbf{d}^T), \tag{30}$$

the mean-square distortion can be expressed as

$$\mathcal{E} = \mathbf{b}^T A \mathbf{b} - 2\mathbf{b}^T \mathbf{z} + 1. \tag{31}$$

Conceptually at least, the optimum tap settings are given by setting the gradient of $\mathcal{E}$, with respect to $\mathbf{b}$, to zero. This gives[‡]

$$A\mathbf{b} - \mathbf{z} = 0 \tag{32a}$$

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† $[X]_{ij}$ denotes the ijth element of the matrix X.

‡ Positive definiteness of A is assumed. For the optimum tap setting it is easy to see that $\mathcal{E} = 1 - \mathbf{z}^T A^{-1} \mathbf{z}$.

or

$$\mathbf{b}^* = A^{-1}\mathbf{z} \tag{32b}$$

as the desired tap settings.

If a gradient (steepest-descent) algorithm is used to minimize† $\mathcal{E}$ by iteratively adjusting $\mathbf{b}$, then the speed of convergence of the algorithm (also referred to as the equalizer settling time) is determined by the matrix A. A simple bound on the speed of convergence is given in Ref. 4.

$$\mathcal{E}_n \leqq \frac{1}{\lambda_{\min}} \left(1 - \frac{\lambda_{\min}}{\lambda_{\max}}\right)^n \mathcal{E}_0,$$

where $\lambda_{\max}$ and $\lambda_{\min}$ denote, respectively, the maximum and minimum eigenvalues of A, and $\mathcal{E}_n$ is the mean-square error after the nth iteration. The ratio $\rho = (\lambda_{\max}/\lambda_{\min})$ was computed for several voice-grade telephone channels (note that the closer ρ is to unity, the smaller the bound on settling time). Typically, the effect of the quadrature channel was small and the A matrix was diagonally dominant with ρ in the range from 2 to 8. Chang[5] has computed ρ for baseband PAM transmission, and comparing our numerical values with his indicates that for most channels the passband equalizer will settle rapidly. In other words, there are no special phenomena arising in the equalization of a bandpass channel which might lead one to expect a larger settling time than that observed for baseband channels. Though (32b) tells us what the optimum settings should be, we have not as yet indicated how these settings can be generated from available circuit voltages.

4.2 *Real-Time Tap Adjustments Using an Ideal Reference*

Ultimately, the cost function actually minimized is determined by the ease with which the optimum tap settings can be obtained in real time. We first note that if, as assumed, A is positive definite, then $\mathcal{E}$ is a convex function of the tap vector $\mathbf{b}$. Since, as is well known, a convex function has a single minimum, a gradient (steepest-descent) algorithm can be used to adjust the taps. If $\mathbf{b}^{(k)}$ denotes the tap vector after the kth adjustment, then the next tap setting is given by

$$\mathbf{b}^{(k+1)} = \mathbf{b}^{(k)} - \alpha^{(k)} \nabla \mathcal{E}(\mathbf{b}^{(k)}), \tag{33}$$

where $\alpha^{(k)}$ is a gain (or step size) and $\nabla \mathcal{E}$ is the gradient of $\mathcal{E}$. Note

† The details of how one uses such an algorithm in this context are described in the next section. The reader not familiar with this type of algorithm should read Section 4.2 before reading the present discussion.

that the algorithm "turns itself off" only when the gradient is zero. For a careful choice of the sequence $\{\alpha^{(k)}\}$, the above algorithm will converge to the optimum tap setting. Clearly the major problem associated with a real-time implementation of (33) is generating the gradient–or a good (statistical) approximation. To obtain an estimate of the gradient, we parallel the approach taken in baseband PAM.

The gradient of $\mathcal{E}$ can be rewritten in component form, as

$$\frac{\partial \mathcal{E}}{\partial c_i} = \sum_n (g_n x_{n-i} + h_n y_{n-i}) - x_{-i} \tag{34a}$$

$$\frac{\partial \mathcal{E}}{\partial d_i} = \sum_n (h_n x_{n-i} - g_n y_{n-i}) + y_{-i} \qquad i = -N, \cdots 0, \cdots N. \tag{34b}$$

We remark that a reasonable statistical approximation (i.e., an estimate) of the true gradient would be one whose average value is the right-hand side of (34). To do this, we first use the sampled equalizer output and an ideal data reference to generate a quantity, the average value of which is precisely our cost function. Once we have done this, we interchange the (linear) operations of ensemble averaging and differentiation to obtain the desired estimate.

The equalizer output samples are given by†

$$q(jT + t_0) \equiv q_j = \cos(\omega_c t_0 + \phi)\left[\sum_n (a_n g_{j-n} - b_n h_{j-n})\right] - \sin(\omega_c t_0 + \phi)\left[\sum_n (a_n h_{j-n} + b_n g_{j-n})\right], \tag{35}$$

and for perfect equalization these samples would be

$$\overset{*}{q}_j = a_j \cos(\omega_c t_0 + \phi) - b_j \sin(\omega_c t_0 + \phi) = \cos(\omega_c t_0 + \phi_j + \phi). \tag{36a}$$

If we have an ideal reference (i.e., a_j and b_j are known at the receiver), then since the tap-rotation property permits the equalizer to obtain any phase, we have the liberty to choose ϕ. In other words, we establish a phase reference at the equalizer output, and we force (via the cost function penalty) the equalizer to find the phase angle of the vector $(g_0^{(\phi)}, h_0^{(\phi)})$ which minimizes the cost function. The equalizer does this by rotating the taps until $h_0^{(\phi)} = 0$. For convenience we choose the reference phase, ϕ, so that

$$\overset{*}{q}_j = (a_j - b_j), \tag{36b}$$

† Again, since the reference phase is arbitrary, we are free to choose ϕ as convenience dictates.

i.e., $\omega_c t_0 + \phi$ has been taken to be $\pi/4$. Note that we do not need an oscillator to generate the ideal reference. By analogy with baseband PAM, we next consider the mean-squared difference between the sampled equalizer output (q_j) and the ideal (or desired) sequence (q_j^*), i.e., $E[(q_j - q_j^*)^2]$. With

$$a^2 \equiv E[a_n^2] = E[b_n^2], \tag{37a}$$

the independence of the data sequences is used to show that

$$E[(q_j - q_j^*)^2] = a^2[\sum_n (g_n^2 + h_n^2) + 1 - 2g_0] = a^2\mathcal{E}. \tag{37b}$$

This is an extremely useful observation, since we now have available a quantity $(q_j - q_j^*)^2$, the average value of which is proportional to the cost function. Interchanging the linear operations of expectation and differentiation allows us to write

$$\frac{\partial}{\partial c_i} E[(q_j - q_j^*)^2] = 2E\left[(q_j - q_j^*)\frac{\partial q_j}{\partial c_i}\right] \tag{38a}$$

and

$$\frac{\partial}{\partial d_i} E[(q_j - q_j^*)^2] = 2E\left[(q_j - q_j^*)\frac{\partial q_j}{\partial d_i}\right]. \tag{38b}$$

Note that $q_j - q_j^*$ is the error (at the equalizer output) at the jth sampling instant. It is easy to see that the differentiated terms can be written as

$$\frac{\partial q_j}{\partial c_i} = \cos(\omega_c t_0 + \phi)[\sum_n a_n x_{j-i-n} - \sum_n b_n y_{j-i-n}] - \sin(\omega_c t_0 + \phi)[\sum_n a_n y_{j-i-n} + \sum_n b_n x_{j-i-n}], \tag{39a}$$

and

$$\frac{\partial q_j}{\partial d_i} = -\cos(\omega_c t_0 + \phi)[\sum_n a_n y_{j-i-n} + \sum_n b_n x_{j-i-n}] - \sin(\omega_c t_0 + \phi)[\sum_n a_n x_{j-i-n} - \sum_n b_n y_{j-i-n}]. \tag{39b}$$

We recognize that $\partial q_j/\partial c_i$ is the jth channel output sample delayed by iT seconds, and thus is available at the ith delay element of the equalizer. We also note that $\partial q_j/\partial d_i$ is precisely $\partial q_j/\partial c_i$, but with the carrier phase delayed by $\pi/2$ radians, and thus is available at the output of the ith delay followed by a $\pi/2$ phase shifter. The estimated gradient, which is given by the bracketed terms on the RHS of (38), can thus be obtained by correlating (muliplying and perhaps then averaging) the error signal $(q_j - q_j^*)$ and the tap outputs.† As the equalization becomes better, the ideal reference can be replaced by decisions, i.e., the receiver operates in a decision-directed mode. This then gives a complete description of the passband equalization of the PM data signal when an ideal reference is available.

At this juncture we wish to make the following remark. Suppose the channel is initially perfect [in the sense of eq. (9)], but the phase reference is such that h_0 is not zero:‡ What does the equalizer do? We first observe that the equalizer will rotate the taps until h_0 is zero, and thus lock in on the established reference phase. However, the differential detector output, which is insensitive to such a rotation, will have been error free from the outset.

4.3 *A Decision-Directed Tap Adjustment Procedure*

Suppose an ideal reference has been used to start-up the equalizer. It is desired that the equalizer then be capable of using decisions in place of an ideal reference. This mode of operation has been dubbed adaptive or decision-directed equalization.[1] The correction term needed for the tap adjustment algorithm is the product of the equalizer error signal $(q_j - q_j^*)$ and a delayed equalizer input. For decision-directed operation, q_j^* (which is $a_j - b_j$) would be replaced by the decided value of q_j. For four-phase operation, a_j and b_j are binary, thus $a_j - b_j = \cos(\phi_j + \pi/4)$ takes on the values -0.707 and 0.707. Thus $\hat{q}_j$ can be obtained as the output of a threshold device (with a threshold at zero).

V. FREQUENCY OFFSET AND PHASE JITTER

In this section we consider the effect of two common transmission impairments, frequency offset and phase jitter, on the operation of the equalizer.

5.1 *Frequency Offset*

Frequently, the transmission media perturbs the received carrier frequency ω_c so that it differs from the transmitted carrier frequency

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† Remembering that to obtain $\partial q_j/\partial d_i$ the ith tap signal has been shifted by $\pi/2$ radians.

‡ This could come about, for example, by a phase-hit impinging on an otherwise ideal channel.

$2\pi/T$ by Δ Hz.† As a result of this frequency offset, each tap signal has a different carrier phase. To obtain a more detailed understanding of this effect on the equalizer, we introduce the following rotated equalized samples

$$\begin{bmatrix} g_n(\Delta) \\ h_n(\Delta) \end{bmatrix} = \begin{bmatrix} \cos n\Delta & \sin n\Delta \\ -\sin n\Delta & \cos n\Delta \end{bmatrix} \begin{bmatrix} g_n \\ h_n \end{bmatrix} \tag{40}$$

and

$$\begin{bmatrix} \tilde{g}_{m,n}(\Delta) \\ \tilde{h}_{m,n}(\Delta) \end{bmatrix} = \begin{bmatrix} \cos m\Delta & \sin m\Delta \\ -\sin m\Delta & \cos m\Delta \end{bmatrix} \begin{bmatrix} g_n(\Delta) \\ h_n(\Delta) \end{bmatrix}, \tag{41}$$

where g_n and h_n are the equalized samples in the absence of frequency offset.

It is straightforward to show that the equalizer output, at the mth sampling instant, can be written in terms of the above quantities as

$$q(mT) = \sum_n a_{m-n}\tilde{g}_{m,n}(\Delta) - \sum_n b_{m-n}\tilde{h}_{m,n}(\Delta). \tag{42}$$

The equalized samples [which are precisely the $\tilde{g}_{m,n}(\Delta)$'s and the $\tilde{h}_{m,n}(\Delta)$'s] are thus a cascade of these two rotations. The rotation described by (40) is time-invariant and amounts to presenting the equalizer with a slightly different channel which is obtained by rotating the vector (g_n, h_n) by $n\Delta$ degrees. By observing that

$$g_n^2(\Delta) + h_n^2(\Delta) = g_n^2 + h_n^2$$

and

$$g_0(\Delta) = g_0, \tag{43}$$

it is clear that this effect can be neglected since the cost function is invariant under such a transformation.

The rotation described by (41) is *time-varying* and indicates that at the mth sampling instant, each channel pair $(g_n(\Delta), h_n(\Delta))$ is rotated by $m\Delta$ radians. Since each channel pair is rotated by the same amount, an equalizer tap-rotation can compensate for this rotation, the only requirement being that the equalizer settling time be much smaller than the period of rotation $1/\Delta$. Thus if the settling time of the equalizer is small, the equalizer automatically tracks the frequency offset by rotating each tap pair (c_i, d_i) at the offset frequency Δ.

5.2 *Phase Jitter*

Phase jitter is an additive random component $\theta(t)$ often present in the phase angle of the channel output, and is characterized as a low-

† Typically Δ is less than 0.5 Hz.

pass process with energy up to 150 Hz. In this section we give only a suggestive description of the effect of phase jitter on the equalizer operation, since a detailed study appears to be difficult and could well be the subject of a separate investigation. If one were to assume the jitter to be constant along the length of the equalizer, i.e.,

$$\theta(mT - iT) = \theta(mT) \quad i = -N, \cdots, 0, \cdots, N,$$

then at the mth sampling instant, each equalized sample pair (g_n, h_n) is rotated by $\theta(mT)$ radians. Such an assumption could only be valid for the extreme low-frequency components of the phase jitter, of course, and these would be successfully "tracked" by slow to-and-fro motions of the tap settings. The higher-frequency components could not be followed by the equalizer, but might be modeled by noise sources at each tap causing fluctuations of the individual settings about the optimum values. In such a model, it would be realistic to include correlations among these fluctuations at each spectral component and one would expect the correlations to decrease with increasing frequency.

VI. SUMMARY AND CONCLUSIONS

An approach to the mean-square equalization of a differentially phase-modulated data signal has been presented. An equalizer, of the transversal filter type, has been proposed which operates in the frequency passband and contains two sets of taps–in-phase and quadrature branches. By exploiting the tap-rotation property of the equalizer, a phase reference is established at the output of the equalizer. Among the manifestations of the tap-rotation property are the ability to control the equalizer by using coherently obtained output samples and the ability to track small amounts of frequency offset and phase jitter. The equalized output is used to automatically (as well as adaptively) adjust the equalizer taps so as to minimize a mean-square distortion function. The required operations (correlating an error signal with tap voltages) are those performed when equalizing baseband PAM. Thus much of the existing knowledge concerning the technology of baseband PAM equalization can be applied to the equalization of phase-modulated data signals.

We make two final remarks. The equalizer structure and tap adjustment algorithm can be applied, with minor modification, to the equalization of any double-sideband modulated data signals, e.g., combined amplitude and phase modulation. As is the case with all equalization systems, the precise dynamic behavior of the proposed equalizer can only be studied, at present, by experiment.

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