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Stability of a General Type of Pulse-Width-Modulated Feedback System

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Because of its theoretical and practical interest, the stability problem in pulse-width-modulated feedback systems has received an enormous amount of attention. Much of the reported literature deals with highly approximate methods, and the exact approaches, based on Lyapunov's direct method or functional analysis, are quite restrictive and do not easily lend themselves to systematic compensation or design.

In this paper, a quite general PWM is considered, and a frequency domain stability criterion is presented, yielding a geometric interpretation in the Popov plane.

I. INTRODUCTION

The stability of pulse-width-modulated control systems has been an active area of research since the early 1960's. A variety of graphical and analytical approaches to the problem have appeared in the literature.[1–4] Aside from the approximate methods, the main contribution of the early 1960's to exact stability criteria was in the application of Lyapunov's direct method.[5,6] As is often the case, this approach yields conservative results and does not easily lend itself to system compensation. Input-output stability via functional analytic techniques was reported in Skoog[7] and Skoog and Blankenship,[8] where conditions for the L_1 boundedness and continuity of the system operator are derived for PWM systems (considered there to belong to a larger class of pulse-modulated systems, i.e., that class of modulators for which the input is sampled). One drawback to the above type of criteria is the lack of a simple geometric interpretation; e.g., a Popov-type condition. In Skoog[7] a circle criterion is derived for PWM systems, operating in the "quasi-linear" mode; that is, where the modulator does not saturate. In its exact form, however, the above condition is rather difficult to apply (the radius of the circle is in the form of an infinite sum, involving an arbitrary parameter).

In all the previous cases, the pulse-width modulator considered is the periodic sampling type, where the input to the modulator is sampled and the polarity and width of the output pulse is determined from that sample. This paper will consider a similar PWM which is a *generalization* (GPWM) of the so-called natural sampling type.[9–11,*] In this scheme, the input is compared to a repetitive reference waveform, and pulses are emitted in accordance with some specified relation between the two signals.

It is the purpose of this paper to develop a geometric stability criterion for the GPWM system. The main result of the paper is a frequency domain condition for the stability of a feedback system containing a GPWM (described below) and a linear plant that may be either lumped or distributed. The condition, similar to a Popov type, is interesting in that it allows a tradeoff between the slope of the stability line and its intersection with the real axis of the Popov plane.

II. NOTATION

In this paper we are concerned with measurable functions of a real variable defined on $[0, \infty)$. We consider the function spaces $L_p(p \geqq 1)$, where

$$L_p(p \in [1, \infty)) = \left\{ x(t) : \int_0^\infty |x(t)|^p dt < \infty \right\}$$

and

$$L_\infty = \left\{ x(t) : \operatorname*{ess\,sup}_{t \geqq 0} |x(t)| < \infty \right\}.$$

The corresponding norms are defined by

$$\|x(t)\| L_p(p \in [1, \infty)) = \left(\int_0^\infty |x(t)|^p dt \right)^{1/p}$$

and

$$\|x(t)\| L_\infty = \operatorname*{ess\,sup}_{t \geqq 0} |x(t)|.$$

Also, we shall use the extensions[13] of these spaces, defined as:

$$L_{pe}(p \in [1, \infty)) = \left\{ x(t) : \int_0^\infty |x_T(t)| dt < \infty, \quad \forall\, T \in [0, \infty) \right\}$$

and

$$L_{\infty e} = \{ x(t) ; \operatorname*{ess\,sup}_{t \geqq 0} |x_T(t)|, \quad \forall\, T \in [0, \infty) \},$$

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* In a very recent paper, V. M. Kuntsevich[12] has treated this type of modulator by the discrete version of Lyapunov's direct method.

where

$$X_T(t) = \begin{cases} X(t), & t \leqq T \\ 0, & t > T \end{cases}.$$

And finally stability will be interpreted to mean that, for all inputs belonging to the spaces of interest, the composite system operator is a bounded mapping of those spaces into themselves.

III. SYSTEM DESCRIPTION AND ASSUMPTIONS

Consider the feedback system of Fig. 1, where the output of the GPWM is:

$$m(t) = \sum_K M\epsilon_K[\mu(t - KT_d) - \mu(t - KT_d - \tau_K)]. \tag{1}$$

The constant M is the pulse height, $\mu(t)$ is the unit step function, and T_d is the period of the modulator. Also $\epsilon_K \triangleq \operatorname{sgn}[\sigma(KT_d)]$. Furthermore, if we define:

$$\omega_K(t) \triangleq [\sigma(t) - \epsilon_K A(t - KT_d)][\mu(t - KT_d) - \mu(t - (K+1)T_d)], \quad \forall K \in I^+, \tag{2}$$

where A (the slope) is a positive constant, then

$$\tau_K = \begin{cases} \min\{(t - KT_d) : \omega_K(t) = 0, & t \in [KT_d, (K+1)T_d)]\} \\ T_d, \quad \text{if} \quad \omega_k(t) \neq 0, & \forall t \in [KT_d, (K+1)T_d)] \end{cases}. \tag{3}$$

The above relations are illustrated in Fig. 2.

From eqs. (1) and (3) we see that the GPWM is a causal operator mapping L_{pe} into itself. Furthermore, it is interesting to note that the periodic PWM is derivable from the GPWM by inserting a sampler (operating every T_d seconds) and a zero order hold before the modulator, as shown in Fig. 3. Here the analog of eq. (3) would be

$$\tau_K = \begin{cases} \dfrac{1}{A}\dfrac{\sigma(KT_d)}{\epsilon_K} = \dfrac{1}{A}|\sigma(KT_d)|, & |\sigma(KT_d)| \leqq AT_d \\ T_d, & |\sigma(KT_d)| > AT_d \end{cases}$$

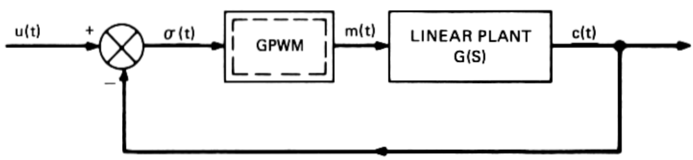

Fig. 1—GPWM feedback system.

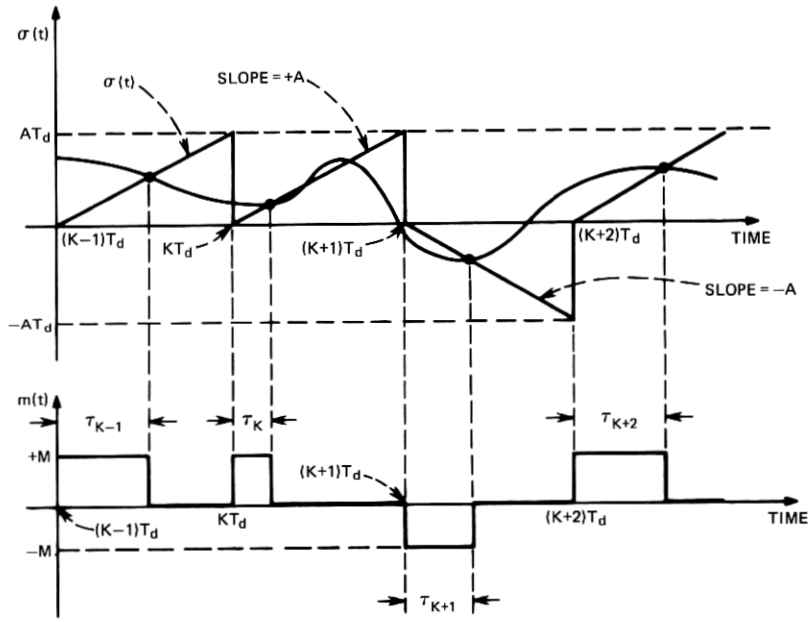

Fig. 2—Modulator definitions.

which is, indeed, the functional relation between the pulse width and the sampled input of a periodic PWM.

It is worth pointing out that various forms of the GPWM process could exist. The modulator may be one-sided ($\epsilon_K = +1, \forall K$) as, for example, in dc power conditioning; it may emit multiple pulses period; or the reference ramp may be replaced by a symmetrical triangle or other similar waveforms. The results of this paper may be extended to any of these variations.

With the foregoing, the following assumptions are also made (see

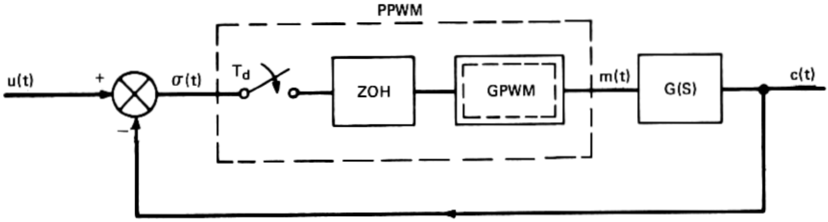

Fig. 3—Derivation of PPWM.

Fig. 1):

A1. $U(t)$ is absolutely continuous[14] on $[0, \infty)$, and $U(t)$, $\dot{U}(t) \in L_1 \cap L_2$, where $U(t)$ includes an external input and the zero input response of the linear plant.

A2. $g(t)$, $\dot{g}(t) \in L_1 \cap L_2$; $g(t) = 0$, $t < 0$, where $g(t)$ is the impulse response of the linear subsystem.

Normally, in input-output stability analysis the solution of the system is assumed to exist in the extended space under consideration. However, the constraints of the modulator make this unnecessary.

Lemma 1: Under assumptions A1 and A2, $\sigma(t) \in L_{pe}$ $(p = 1, 2)$.

Proof: From eq. (1), for any finite time $T \in [0, \infty)$ the modulator will produce a finite number of pulses. Thus $m(t) \in L_{pe}(p > 1)$, which implies by virtue of A2 that so does $c(t)$. Hence $\sigma(t) \in L_{pe}$ $(p = 1, 2)$ by A1 and the linearity of the L_p spaces.

IV. STABILITY

The objective of this section is to develop a geometric stability criterion for GPWM systems. Conditions for the system response to belong to $L_p(p \geqq 1)$ will be derived, yielding a geometric criterion in the Popov plane. The result will require that the linear subsystem have a measurable impulse response, satisfying A2, and thus may represent either a lumped or distributed plant. The following extension of a result due to Euler[15] will be useful in establishing the criterion.

Lemma 2: If $x(t)$ is absolutely continuous on $[0, T]$ for any $T \in [0, \infty)$, then:

$$\sum_{K=0}^{N} |x(KT_d)| \leqq \frac{1}{T_d} \int_0^T |x(t)|\,dt + \frac{1}{2} \int_0^T |\dot{x}(t)|\,dt + \tfrac{1}{2}[|x(0)| + |x(NT_d)|], \quad (4)$$

where $N = [T/T_d]$; i.e., the largest integer $\leqq T/T_d$, and the derivative $\dot{x}(t)$ exists almost everywhere.

Proof: For $K = 0, 1, 2, \cdots, N - 1$,

$$\int_{KT_d}^{(K+1)T_d} \left(\frac{t}{T_d} - K - \frac{1}{2}\right) \cdot \frac{d}{dt}|x(t)|\,dt = \tfrac{1}{2}[|x(KT_d)| + |x((K+1)T_d)|] - \frac{1}{T_d} \int_{KT_d}^{(K+1)T_d} |x(t)|\,dt$$

since both $x(t)$ and $(t/T_d - K - \frac{1}{2})$ are absolutely continuous on the interval.[16] In the integrand on the left, we can replace K by $[t/T_d]$

since, on the interval, $t/T_d - [t/T_d] - \frac{1}{2}$ differs from $t/T_d - K - \frac{1}{2}$ only on a set of measure zero. Now addition of the above for $K = 0, 1, 2, \cdots, N - 1$ and then the expression $\frac{1}{2}[|x(0)| + |x(NT_d)|]$ to both sides yields:

$$\sum_{K=0}^{N} |x(KT_d)| = \frac{1}{T_d}\int_0^{NT_d} |x(t)|dt + \int_0^{NT_d} (t/T_d - [t/T_d] - \tfrac{1}{2}) \cdot \frac{d}{dt}|x(t)|dt + \tfrac{1}{2}[|x(0)| + |x(NT_d)|].$$

Noting that:

$$\int_0^{NT_d} \left(\frac{t}{T_d} - \left[\frac{t}{T_d}\right] - \frac{1}{2}\right)\cdot\frac{d}{dt}|x(t)|dt \leqq \frac{1}{2}\int_0^{NT_d} |\dot{x}(t)|dt$$

and that $NT_d \leqq T$, we see:

$$\sum_{K=0}^{N} |x(KT_d)| \leqq \frac{1}{T_d}\int_0^{T} |x(t)|dt + \frac{1}{2}\int_0^{T} |\dot{x}(t)|dt + \tfrac{1}{2}[|x(0)| + |x(NT_d)|].$$

Q.E.D.

Along with the above result, the following observation concerning the modulator will be of interest in what follows.

Lemma 3: Consider Fig. 1. If $m(t) \in L_p$ for any $p \in [1, \infty)$, then it belongs to L_p for all $p \in [1, \infty]$.

Proof: Suppose $m(t) \in L_{\bar{p}}$, for some $\bar{p} \in [1, \infty)$. Then

$$\int_0^{\infty} |m(t)|^{\bar{p}} dt = M^{\bar{p}} \sum_{K=0}^{\infty} \tau_K < \infty$$

and, since M is a finite number, we see that, for any p,

$$\|m(t)\|_{L_p} = M^p \sum_{K=0}^{\infty} \tau_K < \infty$$

and thus $m(t) \in L_p$ for all $p \in [1, \infty]$ [of course, $m(t) \in L_\infty$ by virtue of (1)].

With the foregoing we are now in a position to state the main result of this paper.

Theorem: Consider the GPWM feedback system of Fig. 1. Suppose there exist two numbers, $q_1 \in R^+$, $q_2 \neq 0$, such that:

(i) $\dfrac{1}{q_1 q_2} > \dfrac{1}{T_d}\|g(t)\|_{L_1} + \tfrac{1}{2}[\|\dot{g}(t)\|_{L_1} + |g(0)|] - \dfrac{A}{M}$ *and*

(ii) $\text{Re}\,[(1 + j\omega q_1)G(j\omega)] \geqq \dfrac{1}{q_2}, \ \forall \omega \in R^+,$

where

$$G(j\omega) = \int_0^\infty g(t)e^{-j\omega t}dt.$$

Then

$$C(t) \in L_p(p \geqq 1).$$

Remark: If $U(t) \in L_p$ as well, then the system will be termed $L_1 \cap L_2 \cap L_p$ stable (bounded).

Proof: Consider Fig. 1. We note that condition (*ii*) implies (see, for example, Ref. 17) that:

$$\int_0^T \left[\sigma(t) + \frac{1}{q_2}m(t)\right]\cdot m(t)dt + q_1\int_0^T m(t)\dot{\sigma}(t)dt \leqq \int_0^T \bar{u}(t)\cdot m(t)dt, \quad \forall\, T \in [0, \infty), \quad (5)$$

where $\bar{u}(t) = u(t) + q_1\dot{u}(t)$. Now from the defining relations of the modulator (see also Fig. 2), the GPWM is an e-positive operator;[18] i.e.,

$$\int_0^T \sigma(t)\cdot m(t)dt \geqq 0^*, \quad \forall\, T \in [0, \infty).$$

Thus:

$$\frac{1}{q_2}\int_0^T m^2(t)dt + q_1\int_0^T m(t)\dot{\sigma}(t)dt \leqq \left[\int_0^T \bar{u}^2(t)dt\right]^{\frac{1}{2}}\left[\int_0^T m^2(t)dt\right]^{\frac{1}{2}}, \quad (6)$$

where Schwarz's inequality has been used on the rhs of (5). Using

$$m(t) = \sum_{K=0}^{N} M\epsilon_k[\mu(t - KT_d) - \mu(t - KT_d - \tau_K)] \quad \text{for } t \in [0, T):^\dagger$$

$$\frac{M^2}{q_2}\sum_{K=0}^{N}\tau_K + q_1M\sum_{K=0}^{N}[|\sigma(KT_d + \tau_K)| - |\sigma(KT_d)|] \leqq M\|\bar{u}(t)\|_{L_2}\left[\sum_{K=0}^{N}\tau_K\right]^{\frac{1}{2}}, \quad (7)$$

in which we have used

$$\epsilon_k\begin{Bmatrix}\sigma(KT_d + \tau_K)\\ \sigma(KT)\end{Bmatrix} = \begin{Bmatrix}|\sigma(KT_d + \tau_K)|\\ |\sigma(KT_d)|\end{Bmatrix}.$$

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* Note on L_2 this is not true for the periodic PWM.

† If the truncation time T should occur during the Nth pulse, then, of course, $\tau_N = T - NT_d$.

Inequality (7) follows from the fact that $\sigma(t)$ is absolutely continuous, which will be shown below.

Now in view of (2) and (3), $\tau_K \leqq |\sigma(KT_d + \tau_K)|/A$, and thus:

$$M\left(q_1A + \frac{M}{q_2}\right)\sum_{K=0}^{N} \tau_K - M\|\bar{u}(t)\|_{L_2}\left[\sum_{K=0}^{N} \tau_K\right]^{\frac{1}{2}} \leqq q_1M\sum_{K=0}^{N}|\sigma(KT_d)|. \quad (8)$$

We observe that $C(t)$ [and hence $\sigma(t)$ by A1, Section III] is absolutely continuous, since it is the indefinite integral of a summable (Lebesque) function. Therefore, Lemma 2 is applicable to $\sigma(t)$ and:

$$\begin{aligned}\sum_{K=0}^{N}|\sigma(KT_d)| &\leqq \frac{1}{T_d}\int_0^T|\sigma(t)|dt \\ &\quad + \frac{1}{2}\int_0^T|\dot{\sigma}(t)|dt + \tfrac{1}{2}[|\sigma(0)| + |\sigma(NT_d)|] \\ &\leqq \int_0^T\left(\frac{1}{T_d}|u(t)| + \frac{1}{2}|\dot{u}(t)|\right)dt \\ &\quad + \int_0^T\left(\frac{1}{T_d}|C(t)| + \frac{1}{2}|\dot{C}(t)|\right)dt + \sup_{t\geqq 0}|\sigma(t)|. \end{aligned} \quad (9)$$

Furthermore,

$$\int_0^T|C(t)|dt \leqq M\|g(t)\|_{L_1}\sum_{K=0}^{N}\tau_K$$

and (10)

$$\int_0^T|\dot{C}(t)|dt \leqq M[\|\dot{g}(t)\|_{L_1} + |g(0)|]\sum_{K=0}^{N}\tau_k.$$

Using (9) and (10) in (8) then implies:

$$\begin{aligned}MZ&\left[\left(\sum_{K=0}^{N}\tau_K\right)^{\frac{1}{2}} - \frac{\|\bar{u}(t)\|_{L_2}}{2Z}\right]^2 \\ &\leqq q_1M\left(\frac{1}{T_d}\int_0^T|u(t)|dt + \frac{1}{2}\int_0^T|\dot{u}(t)|dt\right) \\ &\qquad + q_1M\sup_{t\geqq 0}|\sigma(t)| + \frac{M\|\bar{u}(t)\|_{L_2}^2}{4Z} \\ &\leqq q_1M\left(\frac{1}{T_d}\|u(t)\|_{L1} + \frac{1}{2}\|\dot{u}(t)\|_{L_1}\right) \\ &\qquad + q_1M\sup_{t\geqq 0}|\sigma(t)|^* + \frac{M\|\bar{u}(t)\|_{L_2}^2}{4Z} < \infty,\end{aligned}$$

* It is a simple matter to show that, under the hypotheses of the theorem, the system is L_∞ stable and, since $u(t) \in L_\infty$, $\sup_{t\geqq 0}|\sigma(t)| < \infty$.

where

$$Z = q_1 A - \frac{Mq_1}{T_d}\|g(t)\|_{L_1} - \frac{Mq_1}{2}[\|\dot{g}(t)\|_{L_1} + |g(0)|] + \frac{M}{q_2}.$$

For $Z > 0$ [condition (*i*) of the theorem], we have $\sum_{K=0}^{N} \tau_K \leqq Q$(independent of T) $< \infty$, and thus $m(t) \in L_p$, which implies by A2, Section III, that $C(t)$ does also, and the theorem is proved.

Comments: (*a*) Condition (*ii*) of the theorem is similar to a Popov condition for feedback systems with static, sector nonlinearities, although the GPWM does not strictly belong to that class.

(*b*) The condition allows a tradeoff between the slope of the stability line and its interaction with the real axis of the Popov plane.

(*c*) Because of the constraints of the modulator, the modified linear plant does not have to be a *strictly* positive operator, as is commonly the case.[18]

(*d*) Since the assumptions are sufficient to ensure that $U(t)$ [and $g(t)$] $\to 0$ as $t \to \infty$, the theorem also guarantees that $\sigma(t) \to 0$.

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