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Coupling Coefficients For Imperfect Asymmetric Slab Waveguides

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This paper presents a collection of formulas that are necessary for the treatment of radiation and mode conversion phenomena of imperfect asymmetric slab waveguides. The coupled mode theory of dielectric waveguides is briefly reviewed, and general expressions for the coupling coefficients are given. The field expression of the guided and the radiation TE and TM modes of the asymmetric slab waveguide are stated, and are used to derive formulas for the coupling coefficient for slight core boundary irregularities.

I. INTRODUCTION

Mode coupling phenomena and radiation losses caused by core-cladding interface irregularities have been studied extensively for symmetric slab waveguides and for round optical fibers.[1–7] These results are not immediately applicable to the asymmetric slab waveguides used in integrated optics circuits. It is the purpose of this paper to collect the formulas for the normal modes of the asymmetric slab waveguides, and for the coupling coefficients between guided modes and guided and radiation modes caused by core boundary irregularities of these waveguides.

The coupling coefficients between guided modes are useful for the design of distributed feedback sections for lasers and for an evaluation of unintentional mode coupling caused by core boundary roughness. The results collected in this paper are further necessary for the evaluation of radiation losses caused by core boundary roughness.

Because of the many parameters that enter into the theory, it is impossible to evaluate the formulas in graphical form for all cases of practical interest. This paper is thus a collection of the required formulas which the reader can use to evaluate his particular problems.

II. SUMMARY OF THE COUPLED MODE THEORY

The coupling theory is based on an expansion of the solution of Maxwell's equations in terms of normal modes. The general theory of the mode expansion and the derivation of the coupling coefficients have been published by A. W. Snyder.[2,3] His theory is based on local normal modes. Local normal modes resemble the modes of the ideal asymmetric slab waveguide with perfect core boundary. However, the boundary of the perfect guide is allowed to change in such a way that it coincides with the actual deformed core boundary at the particular point z along the waveguide axis at which the coupling coefficients are to be evaluated. Since the waveguide width parameter is no longer a constant, the local normal modes are not themselves solutions of Maxwell's equations. They must be superimposed with z-dependent expansion coefficients to form such a solution. The fact that these modes form a complete orthogonal set and coincide with the modes of a fictitious waveguide, the width of which is locally (at the point z under consideration) the same as that of the deformed waveguide, explains the name "local normal modes." It is also possible to express the general field in terms of the modes of the ideal waveguide, the constant width of which differs slightly from that of the actual waveguide. This expansion suffers from convergence difficulties that are caused by the fact that the normal components of the electric field are discontinuous at the core boundary. The modes of the ideal guide are discontinuous at the dielectric interface of the ideal guide which does not coincide with that of the actual guide. The expansion in terms of ideal modes of the waveguide is thus discontinuous term by term at a point where the entire series must be continuous, and furthermore, it must describe a discontinuous field at the interface of the actual waveguide at a point where each individual term of the expansion remains continuous. The expansion in terms of local normal modes, on the other hand, describes the field discontinuity with a series, the individual terms of which are discontinuous in just the right way at the point of discontinuity of the entire series. The convergence behavior of this latter expansion can thus be expected to be superior to the expansion in terms of ideal modes.

The electric and magnetic fields of the imperfect asymmetric slab waveguide are expressed by the series expansions

$$\mathbf{E} = \sum_{\nu} \left(c_\nu^{(+)} \boldsymbol{\mathcal{E}}_\nu^{(+)} + c_\nu^{(-)} \boldsymbol{\mathcal{E}}_\nu^{(-)}\right) \tag{1}$$

$$\mathbf{H} = \sum_{\nu} \left(c_\nu^{(+)} \boldsymbol{\mathcal{H}}_\nu^{(+)} + c_\nu^{(-)} \boldsymbol{\mathcal{H}}_\nu^{(-)}\right). \tag{2}$$

The expansion coefficients $c_\nu^{(+)}$ and $c_\nu^{(-)}$ are functions of the length coordinate z. The superscripts $(+)$ and $(-)$ indicate waves traveling in positive and negative z-direction. The sums in (1) and (2) are symbolic representations of a summation over guided modes plus an integration over the radiation modes of the continuum.[4,5] In order to simplify the notation, both sum and integral are indicated by the same symbol. In the integral, the summation index ν is replaced by the continuous variable ν, and the sum must be understood as the integral

$$\sum_\nu \to \int_0^\infty d\nu. \tag{3}$$

The local normal modes E_ν and H_ν are solutions of the equations

$$\mp i\beta_\nu(\mathbf{e}_z \times \mathcal{H}_\nu^{(\pm)}) + \nabla_t \times \mathcal{H}_\nu^{(\pm)} = i\omega\epsilon_0 n^2 \mathcal{E}_\nu^{(\pm)} \tag{4}$$

$$\mp i\beta_\nu(\mathbf{e}_z \times \mathcal{E}_\nu^{(\pm)}) + \nabla_t \times \mathcal{E}_\nu^{(\pm)} = - i\omega\mu_0 \mathcal{H}_\nu^{(\pm)}. \tag{5}$$

The upper and lower signs and superscripts belong together. The symbols appearing in these equations have the following meaning.

β_ν = propagation constant of mode ν
$\mathbf{e}_z$ = unit vector in z-direction
∇_t = transverse part of the operator ∇
ω = radian frequency
ϵ_0 = dielectric permittivity of the vacuum
μ_0 = magnetic susceptibility of the vacuum
n = dielectric constant of the waveguide [$n = n(x, y, z)$].

Substitution of the field expansions (1) and (2) into Maxwell's equations and use of the orthogonality relations [see (9)] lead to the set

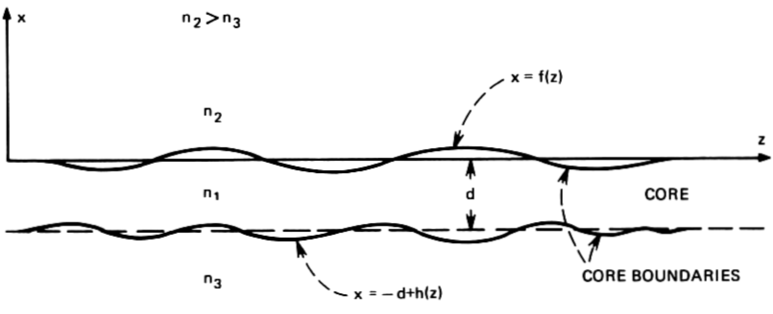

Fig. 1—Sketch of the asymmetric slab waveguide with distorted core boundaries.

of coupled wave equations[2,3,4]

$$\frac{dc_\mu^{(+)}}{dz} = -i\beta_\mu c_\mu^{(+)} + \sum_\nu \left(K_{\mu\nu}^{(+,+)} c_\nu^{(+)} + K_{\mu\nu}^{(+,-)} c_\nu^{(-)}\right) \tag{6}$$

$$\frac{dc_\mu^{(-)}}{dz} = i\beta_\mu c_\mu^{(-)} - \sum_\nu \left(K_{\mu\nu}^{(-,+)} c_\nu^{(+)} + K_{\mu\nu}^{(-,-)} c_\nu^{(-)}\right). \tag{7}$$

The coupling coefficients have the form[2,3]

$$K_{\mu\nu}^{(\pm,p)} = \frac{1}{4P}\int_{-\infty}^{\infty}\left\{\pm\left(\frac{\partial\,\mathcal{E}_\nu^{(+)}}{\partial z}\times\mathcal{H}_\mu^{(+)*}\right)\cdot\mathbf{e}_z - p\left(\mathcal{E}_\mu^{(+)*}\times\frac{\partial\mathcal{H}_\nu^{(+)}}{\partial z}\right)\cdot\mathbf{e}_z\right\}dx. \tag{8}$$

The asterisk indicates complex conjugation. The superscript p stands for $(+)$ or $(-)$ while the factor p assumes the values $+1$ and -1. P is a normalization parameter which is related to the power carried by the modes via the relation

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(\mathcal{E}_\nu^{(+)}\times\mathcal{H}_{\nu'}^{(+)*}\right)\cdot\mathbf{e}_z dx = P\delta_{\nu\nu'}. \tag{9}$$

The symbol $\delta_{\nu\nu'}$ indicates the Dirac delta function if both ν and ν' represent continuous variables, it represents the Kronecker delta symbol if both ν and ν' are discrete labels, and it is zero if one subscript belongs to discrete modes while the other indicates a mode of the continuum.

The coupling coefficients (8) are not very easy to evaluate since they are expressed in terms of derivatives of the mode functions. A. W. Snyder[3] has shown that the coupling coefficients can be transformed to the following more useful form. $(\beta_\nu^{(-)} = -\beta_\nu^{(+)})$

$$K_{\mu\nu}^{(\pm,p)} = -\frac{\omega\epsilon_0}{4P(\beta_\mu^{(\pm)} - \beta_\nu^{(p)})}\int_{-\infty}^{\infty}\frac{\partial n^2}{\partial z}\mathcal{E}_\mu^{(\pm)*}\cdot\mathcal{E}_\nu^{(p)}dx. \tag{10}$$

The coupled wave equations (6) and (7), with the coupling coefficients (10), provide an exact description of imperfect dielectric waveguides. The one-dimensional integral in (10) constitutes a specialization to a two-dimensional problem in view of our interest in the asymmetric slab waveguide. By extending the integration over the cross-sectional x, y plane, the general coupling coefficients are obtained.

For our purpose, it is advantageous to derive approximate coupling coefficients for asymmetric slab waveguides with discontinuous index distributions. We use the fact that the normal component E_x, of the electric field obeys the relation

$$n_1^2 E_{x'1} = n_2^2 E_{x'2}. \tag{11}$$

It is shown in Fig. 1 that n_1 and n_2 are the values of the refractive index at either side of the interface. In order to derive the desired expression for the coupling coefficient, we assume that the discontinuous index distribution is smoothed out (in an arbitrary way) into a continuous distribution. We assume that the wavelength of the radiation is very much larger than the region over which the refractive index varies continuously and write

$$E_{x'} = \frac{n_1^2}{n^2} E_{x'1}. \tag{12}$$

We show in the appendix how the integral in (10) can be evaluated and obtain the result

$$\begin{aligned} K_{\mu\nu}^{(\pm,p)} = & -\frac{\omega\epsilon_0}{4P(\beta_\mu^{(\pm)} - \beta_\nu^{(p)})} \Bigg\{ (n_1^2 - n_2^2)\frac{df}{dz}\left[\frac{n_1^2}{n_2^2}\mathcal{E}_{\mu x}^{(\pm)*}\mathcal{E}_{\nu x}^{(p)} + \mathcal{E}_{\mu y}^{(\pm)*}\mathcal{E}_{\nu y}^{(p)}\right. \\ & \left. + \mathcal{E}_{\mu z}^{(\pm)*}\mathcal{E}_{\nu z}^{(p)}\right]_{x=f(z)} - (n_1^2 - n_3^2)\frac{dh}{dz}\left[\frac{n_1^2}{n_3^2}\mathcal{E}_{\mu x}^{(\pm)*}\mathcal{E}_{\nu x}^{(p)}\right. \\ & \left. + \mathcal{E}_{\mu y}^{(\pm)*}\mathcal{E}_{\nu y}^{(p)} + \mathcal{E}_{\mu z}^{(\pm)*}\mathcal{E}_{\nu z}^{(p)}\right]_{x=-d+h(z)} \Bigg\}. \end{aligned} \tag{13}$$

The index distribution can now again be considered as discontinuous. The functions $f(z)$ and $h(z)$ describe the deformation of the upper and lower side of the core boundary (see Fig. 1). The field components are taken inside the core region at the core boundary. The refractive index of the core is n_1 while n_2 and n_3 are the indices above and below the core region. The electric field components are related in the following way.

$$\left.\begin{aligned} \mathcal{E}_{\nu x}^{(-)} &= \mathcal{E}_{\nu x}^{(+)} \\ \mathcal{E}_{\nu y}^{(-)} &= \mathcal{E}_{\nu y}^{(+)} \\ \mathcal{E}_{\nu z}^{(-)} &= -\mathcal{E}_{\nu z}^{(+)} \end{aligned}\right\}. \tag{14}$$

The approximation involved in the coupling coefficient (13) consists in using the x component $E_{\nu x}$ and z component $E_{\nu z}$ of the local normal modes instead of their normal and tangential components with respect to the interface. The approximation is valid provided that

$$\frac{df}{dz} \ll 1 \qquad \text{and} \qquad \frac{dh}{dz} \ll 1. \tag{15}$$

For many practical applications it is sufficient to use an approximate solution of the coupled wave eqs. (6) and (7). In particular, for the calculation of the radiation loss coefficient, we use the approximate solution of (6) ($\mu = \rho$)

$$c_\rho^{(\pm)}(z) = \pm c_\nu^{(p)} e^{-i\beta_\rho^{(\pm)} z} \int_0^z K_{\rho\nu}^{(\pm,p)}(u) \cdot \exp\left[-i\int_0^u (\beta_\nu^{(p)}(v) - \beta_\rho^{(\pm)})dv\right] du. \tag{16}$$

The coefficient $c_\nu^{(p)}$ is the amplitude of the guided mode, the losses of which we want to calculate, taken at $z = 0$. The propagation constant β_ν is a function of z since it belongs to a guided mode in a non-uniform waveguide, β_ρ is independent of z since it belongs to a radiation mode. The relative power loss $\Delta P_\nu / P_\nu$ that mode ν suffers in traveling from $z = 0$ to $z = L$ is given in Refs. 5 and 6.

$$\frac{\Delta P_\nu}{P_\nu} = \frac{1}{|c_\nu^{(p)}|^2} \sum \int_0^{n_2 k} |c_\rho^{(\pm)}(L)|^2 d\rho. \tag{17}$$

The sum in front of the integral sign indicates that we must add up the contributions of forward and backward traveling modes as well as the contributions from the various kinds of radiation modes that will be discussed in the next section. The integral extends over the range of ρ values that belongs to propagating (non-evanescent) radiation modes. We show below that the functional form of the radiation modes is not the same over the entire integration range.

III. MODES OF THE ASYMMETRIC SLAB WAVEGUIDE

We consider only the special case in which there is no field variation and no waveguide distortion in y direction. This fact is symbolically expressed by the equation

$$\frac{\partial}{\partial y} = 0. \tag{18}$$

With the restriction (18), the fields of the slab waveguide can be classified as either TE or TM fields.[8] The TE fields have only the following non-vanishing field components

$$E_y,\ H_x,\ H_z. \tag{19}$$

The TM fields have the non-vanishing field components

$$H_y,\ E_x,\ E_z. \tag{20}$$

It is assumed that the refractive indices of the waveguide are ordered in the following way

$$n_1 > n_2 \geqq n_3. \tag{21}$$

IV. GUIDED TE MODES

The x and z components of the magnetic field follow from the E_y component by differentiation

$$H_x = -\frac{i}{\omega\mu_0}\frac{\partial E_y}{\partial z} \tag{22}$$

$$H_z = \frac{i}{\omega\mu_0}\frac{\partial E_y}{\partial x}. \tag{23}$$

The guided TE modes of the asymmetric slab waveguide are obtained as follows (the factor $\exp[i(\omega t - \beta z)]$ is always suppressed):

$$\mathcal{E}_y = A_g e^{-\gamma x} \qquad \text{for} \qquad 0 \leqq x < \infty \tag{24}$$

$$\mathcal{E}_y = A_g\left(\cos\kappa x - \frac{\gamma}{\kappa}\sin\kappa x\right) \qquad \text{for} \qquad -d \leqq x \leqq 0 \tag{25}$$

$$\mathcal{E}_y = A_g\left(\cos\kappa d + \frac{\gamma}{\kappa}\sin\kappa d\right)e^{\theta(x+d)} \qquad \text{for} \qquad -\infty < x \leqq -d. \tag{26}$$

The parameters appearing in these field expressions are defined by the equations:

$$\gamma^2 = \beta^2 - n_2^2k^2 \tag{27}$$

$$\theta^2 = \beta^2 - n_3^2k^2 \tag{28}$$

$$\kappa^2 = n_1^2k^2 - \beta^2 \tag{29}$$

$$k^2 = \omega^2\epsilon_0\mu_0. \tag{30}$$

The propagation constant β is determined from the eigenvalue equation

$$\tan \kappa d = \frac{\kappa(\gamma + \theta)}{\kappa^2 - \gamma\theta}. \tag{31}$$

The normalization of the mode is obtained by expressing the amplitude coefficient A_g in terms of the power P carried by the mode.

$$A_g^2 = \frac{4\kappa^2 \omega \mu_0 P}{|\beta| \left(d + \frac{1}{\gamma} + \frac{1}{\theta} \right) (\kappa^2 + \gamma^2)}. \tag{32}$$

The mode labels ν, that were used in the coupled wave equations and the field expansions, are suppressed. The modes are labeled according to the solutions of the eigenvalue equation (31).

V. TE RADIATION MODES

The propagation constants of the radiation modes do not have a discrete set of values. However, the asymmetric slab waveguide has different types of radiation modes. In the range

$$n_3 k \leqq \beta \leqq n_2 k \tag{33}$$

we find only one type of radiation mode, the fields of which decay exponentially into the region three with refractive index n_3, but are standing waves in the space above the waveguide with refractive index n_2. We can visualize the physical mechanism for exciting these modes by assuming that a source at infinity sends a plane wave that impinges on the core of the slab waveguide under an angle that is given by

$$\cos \alpha = \frac{\beta}{n_2 k}. \tag{34}$$

The incident plane wave penetrates into the core region but is totally internally reflected if the angle α stays in the range given by (33). This total internal reflection occurs because we assumed that $n_3 < n_2$. In the space above the core we find a reflected wave added to the incident wave supplied by the external source. This explains the occurrence of standing waves in this region. It is not possible to find solutions of Maxwell's equations satisfying the boundary conditions which have only traveling waves outside of the core region.

In the range of propagation constants given by (33) we find the following expression for the field of the radiation modes

$$\mathcal{E}_y = A_r \cos \rho x + \frac{\sigma}{\rho} B_r \sin \rho x \qquad \text{for} \qquad 0 \leqq x < \infty \tag{35}$$

$$\mathcal{E}_y = A_r \cos \sigma x + B_r \sin \sigma x \qquad \text{for} \qquad -d \leqq x \leqq 0 \tag{36}$$

$$\mathcal{E}_y = (A_r \cos \sigma d - B_r \sin \sigma d) e^{-i\Delta(x+d)} \qquad \text{for} \qquad -\infty < x \leqq -d. \tag{37}$$

H_x and H_z are obtained from E_y with the help of (22) and (23). The constant B is related to the constant A by the equation

$$B_r = \frac{\Delta - i\sigma \tan \sigma d}{\Delta \tan \sigma d + i\sigma} A_r \tag{38}$$

and the parameters appearing in these equations are defined as follows

$$\sigma^2 = n_1^2 k^2 - \beta^2 \tag{39}$$

$$\rho^2 = n_2^2 k^2 - \beta^2 \tag{40}$$

$$\Delta^2 = n_3^2 k^2 - \beta^2. \tag{41}$$

Note that Δ is imaginary for β values in the range given by (33).

It is convenient to identify the parameter ρ with the mode label ν to label the radiation modes. We thus use the identity

$$\delta_{\nu\nu'} = \delta(\rho - \rho') \tag{42}$$

in (9) and find for the amplitude coefficient A the relation

$$A_r^2 = \frac{4\omega\mu_0 P}{\pi|\beta|} \frac{\rho^2\left(\sigma \cos \sigma d + \frac{\Delta}{i} \sin \sigma d\right)^2}{\rho^2\left(\sigma \cos \sigma d + \frac{\Delta}{i} \sin \sigma d\right)^2 + \sigma^2\left(\sigma \sin \sigma d - \frac{\Delta}{i} \cos \sigma d\right)^2}. \tag{43}$$

With β in the range (33) we find that ρ is confined to the region

$$0 \leqq \rho \leqq (n_2^2 - n_3^2)^{\frac{1}{2}} k. \tag{44}$$

Next we proceed to list the radiation modes that belong to propagation constants in the range

$$\left.\begin{array}{lll} & 0 \leqq \beta \leqq n_3 k & \beta \ \text{real} \\ \text{and} & 0 \leqq |\beta| < \infty & \beta \ \text{imaginary} \end{array}\right\}. \tag{45}$$

The corresponding range for ρ is given by

$$(n_2^2 - n_3^2)^{\frac{1}{2}} k < \rho < \infty. \tag{46}$$

For real values of β, these modes propagate along the z axis while they are evanescent waves in z direction for imaginary values of β. It is again possible to visualize the modes in the range (46) as being excited by a source outside of the waveguide core located at infinity. This source sends a plane wave toward the slab whose angle of propagation with respect to the z axis is given by (34). However, there is now no longer total internal reflection at the lower boundary of the core so that we obtain an incident and reflected (in x-direction) wave in the space above as well as inside the core. Below the core there is a transmitted propagating wave. However, we may now assume with equal justification that a second source sends a plane wave in the direction of the core from below. If both sources are turned on simultaneously, we obtain standing waves (in x-direction) below as well as above the waveguide core. The exact form of the radiation field depends on the relative phases between the two sources. It is thus not surprising that there should be an infinite number of ways in which orthogonal sets of radiation modes can be constructed.

We list only the E_y components of the modes and obtain the H_x and H_z components by differentiation from (22) and (23). ($i = 1, 2$)

$$\mathcal{E}_y = C_r\left(\cos \rho x + \frac{\sigma}{\rho} F_i \sin \rho x\right) \qquad 0 \leqq x < \infty \tag{47}$$

$$\mathcal{E}_y = C_r(\cos \sigma x + F_i \sin \sigma x) \qquad -d \leqq x \leqq 0 \tag{48}$$

$$\mathcal{E}_y = C_r\left\{(\cos \sigma d - F_i \sin \sigma d) \cos \Delta(x + d) + \frac{\sigma}{\Delta}(\sin \sigma d + F_i \cos \sigma d) \sin \Delta(x + d)\right\} \qquad -\infty < x \leqq -d. \tag{49}$$

The parameters σ, ρ and Δ are given by (39), (40) and (41), Δ is now a real constant. Whereas the amplitude coefficient B_r in (35) through (37) was related to A_r by the boundary conditions, we now face the situation where the amplitude coefficient F_i remains arbitrary. Equations (47) through (49) satisfy Maxwells' equations as well as the boundary conditions without any further restriction having to be imposed on the coefficient F_i. This freedom of choice is related to the arbitrary amplitude and phase of the two plane wave sources of the radiation mode mentioned above. We must choose F_i in such a way that two radiation modes with the same value of the propagation constant, but different values of F_i become mutually orthogonal. But even this requirement does not specify the possible values of F_i uniquely. We are thus free to choose F values according to our own convenience. Of the infinitely many possibilities, we choose the F_i coefficients such that in the limit $n_2 = n_3$, even and odd radiation modes result. In the asymmetric slab waveguide no even or odd modes exist. But the guided modes become either even or odd as n_2 approaches n_3. We obtain the same symmetries for the radiation modes by a suitable choice of the F_i coefficients.

$$F_{1,2} = \frac{1}{(\sigma^2 - \Delta^2)\sin 2\sigma d}\left\{(\sigma^2 - \Delta^2)\cos 2\sigma d + \frac{\Delta}{\rho}(\sigma^2 - \rho^2) \pm \left[(\sigma^2 - \Delta^2)^2 + 2\frac{\Delta}{\rho}(\sigma^2 - \Delta^2)(\sigma^2 - \rho^2)\cos 2\sigma d + \frac{\Delta^2}{\rho^2}(\sigma^2 - \rho^2)^2\right]^{\frac{1}{2}}\right\}. \quad (50)$$

The $+$ sign ($-$ sign) belongs to the odd (even) mode in the limit $n_2 = n_3$, $\Delta = \rho$. We identify the mode label ν again with the transverse propagation constant ρ, and obtain from the normalization condition (9) and (42) the relation between the amplitude coefficient C_r and the power P

$$C_r^2 = \frac{4\omega\mu_0 P}{\pi|\beta|}\left[\frac{\Delta}{\rho}(\cos\sigma d - F_i \sin\sigma d)^2 + \frac{\sigma^2}{\Delta\rho}(\sin\sigma d + F_i\cos\sigma d)^2 + 1 + \frac{\sigma^2}{\rho^2}F_i^2\right]^{-1}. \quad (51)$$

We have thus obtained two distinct sets of radiation modes, the propagation constants of which lie in the range (45). The two sets are distinguished by the labels $i = 1$ and $i = 2$. F_1 and F_2 follow from (50) if we use both signs of the square root expression. It is noteworthy that the following relation holds.

$$F_1 F_2 = -1. \quad (52)$$

This listing contains the complete set of TE guided and radiation modes consistent with the restriction (18). All modes are mutually orthogonal to each other. We have concentrated on the forward traveling modes. The backward traveling modes are obtained simply by changing the sign of β, $(\beta^{(-)} = -\beta^{(+)})$.

VI. GUIDED TM MODES

The TM modes are very similar to the TE modes except that the roles of the field components are interchanged. We now list only the H_y component and obtain the E_x and E_z components of the field by differentiation

$$E_x = \frac{i}{n^2\omega\epsilon_0}\frac{\partial H_y}{\partial z} \tag{53}$$

$$E_z = \frac{-i}{n^2\omega\epsilon_0}\frac{\partial H_y}{\partial x} \tag{54}$$

$$\mathcal{H}_y = D_g e^{-\gamma x} \qquad 0 \leqq x < \infty \tag{55}$$

$$\mathcal{H}_y = D_g\left(\cos\kappa x - \frac{n_1^2}{n_2^2}\frac{\gamma}{\kappa}\sin\kappa x\right) \qquad -d \leqq x \leqq 0 \tag{56}$$

$$\mathcal{H}_y = D_g\left(\cos\kappa d + \frac{n_1^2}{n_2^2}\frac{\gamma}{\kappa}\sin\kappa d\right)e^{\theta(x+d)} \qquad -\infty < x \leqq -d. \tag{57}$$

The parameters κ, γ and θ are defined by (27) through (29). The eigenvalue equation is

$$\tan\kappa d = \frac{n_1^2\kappa(n_3^2\gamma + n_2^2\theta)}{n_2^2n_3^2\kappa^2 - n_1^4\gamma\theta} \tag{58}$$

and the amplitude coefficient is given by

$$D_g^2 = \frac{4\omega\epsilon_0 P}{|\beta|} \times \frac{n_1^2 n_2^4\kappa^2}{(n_2^4\kappa^2 + n_1^4\gamma^2)\left[d + \dfrac{n_1^2n_2^2}{\gamma}\dfrac{\kappa^2+\gamma^2}{n_2^4\kappa^2 + n_1^4\gamma^2} + \dfrac{n_1^2n_3^2}{\theta}\dfrac{\kappa^2+\theta^2}{n_3^4\kappa^2+n_1^4\theta^2}\right]}. \tag{59}$$

VII. TM RADIATION MODES

The TM radiation modes must again be split up into two ranges. In the range $0 \leqq \rho \leqq (n_2^2 - n_3^2)^{\frac{1}{2}}k$, the fields have the form

$$\mathcal{H}_y = D_r \cos \rho x + \frac{n_2^2}{n_1^2}\frac{\sigma}{\rho} G_r \sin \rho x \qquad 0 \leqq x < \infty \tag{60}$$

$$\mathcal{H}_y = D_r \cos \sigma x + G_r \sin \sigma x \qquad -d \leqq x \leqq 0 \tag{61}$$

$$\mathcal{H}_y = (D_r \cos \sigma d - G_r \sin \sigma d) e^{-i\Delta(x+d)} \qquad -\infty \leqq x \leqq -d. \tag{62}$$

The boundary conditions require that G_r is related to D_r in the following way:

$$G_r = \frac{n_1^2 \Delta \cos \sigma d - i n_3^2 \sigma \sin \sigma d}{n_1^2 \Delta \sin \sigma d + i n_3^2 \sigma \cos \sigma d} D_r. \tag{63}$$

The parameters σ, ρ and Δ are defined by (39) through (41), $-i\Delta$ is a real positive quantity. The amplitude coefficient D_r is related to the power coefficient P

$$D_r^2 = \frac{4 n_2^2 \omega \epsilon_0 P}{\pi |\beta|} \frac{1}{1 + \dfrac{n_2^4}{n_1^4}\dfrac{\sigma^2}{\rho^2}\dfrac{G_r^2}{D_r^2}}. \tag{64}$$

In the range $(n_2^2 - n_3^2)^{\frac{1}{2}}k \leqq \rho \leqq \infty$ the radiation modes have the form ($i = 1, 2$)

$$\mathcal{H}_y = S_r\left(\cos \rho x + \frac{n_2^2}{n_1^2}\frac{\sigma}{\rho} R_i \sin \rho x\right) \qquad 0 \leqq x < \infty \tag{65}$$

$$\mathcal{H}_y = S_r(\cos \sigma x + R_i \sin \sigma x) \qquad -d \leqq x \leqq 0 \tag{66}$$

$$\mathcal{H}_y = S_r\left\{(\cos \sigma d - R_i \sin \sigma d) \cos \Delta(x + d) + \frac{n_3^2}{n_1^2}\frac{\sigma}{\Delta}(\sin \sigma d + R_i \cos \sigma d) \sin \Delta(x + d)\right\} \qquad -\infty < x \leqq -d. \tag{67}$$

The coefficients R_i are again arbitrary. Our choice

$$R_{1,2} = \frac{1}{(n_3^4\sigma^2 - n_1^4\Delta^2)\sin 2\sigma d}\Bigg\{(n_3^4\sigma^2 - n_1^4\Delta^2)\cos 2\sigma d + \frac{n_3^2}{n_2^2}\frac{\Delta}{\rho}(n_2^4\sigma^2 - n_1^4\rho^2) \pm \bigg[(n_3^4\sigma^2 - n_1^4\Delta^2)^2 + \frac{n_3^4}{n_2^4}\frac{\Delta^2}{\rho^2}(n_2^4\sigma^2 - n_1^4\rho^2)^2 + 2\frac{n_3^2}{n_2^2}\frac{\Delta}{\rho}(n_3^4\sigma^2 - n_1^4\Delta^2)(n_2^4\sigma^2 - n_1^4\rho^2)\cos 2\sigma d\bigg]^{\frac{1}{2}}\Bigg\} \quad (68)$$

causes the modes with index $i = 1$ to be orthogonal to the modes with index $i = 2$ and, in addition, assures that these modes become even and odd in the limit $n_2 = n_3$. [The $+$ sign ($-$ sign) belongs to the odd (even) mode.] The normalization is given by

$$S_r^2 = \frac{4\omega\epsilon_0 P}{\pi|\beta|}\left\{\frac{1}{n_2^2} + \frac{n_2^2}{n_1^4}\frac{\sigma^2}{\rho^2}R_i^2 + \frac{n_3^2}{n_1^4}\frac{\sigma^2}{\rho\Delta}(\sin\sigma d + R_i\cos\sigma d)^2 + \frac{1}{n_3^2}\frac{\Delta}{\rho}(\cos\sigma d - R_i\sin\sigma d)^2\right\}^{-1}. \quad (69)$$

All amplitude coefficients for the TE and TM modes were taken to be real quantities. This assumption does not lead to a loss of generality since the necessary phase factors are incorporated in the expansion coefficients c_ν.

VIII. COUPLING COEFFICIENTS

With the help of the expressions for the normal modes and the coupling coefficients (10), any problem of the asymmetric slab waveguide with arbitrary irregularities of its refractive index distribution can be solved, provided that the restriction (18) is imposed. Problems caused by core boundary irregularities or by gentle tapers can be solved with the help of the coupling coefficients (13). For convenience, a few coupling coefficients will be worked out explicitly.

As long as the restriction (18) applies, TE modes do not couple to TM modes. All coupling coefficients between TE and TM modes vanish. We restrict the discussion to listing the coupling coefficients between guided TE modes, guided TM modes, and to coupling from a guided TE or TM mode to its respective radiation modes for the case of core boundary irregularities.

The coupling coefficients between two guided TE modes can be obtained from (13) and (25). ($p = +$ or $-$)

$$K_{\mu\nu}^{(\pm,p)} = -\frac{\kappa_\mu\kappa_\nu\left[\dfrac{df}{dz} - \dfrac{\sin\kappa_\mu d\sin\kappa_\nu d}{|\sin\kappa_\mu d\sin\kappa_\nu d|}\dfrac{dh}{dz}\right]}{(\beta_\mu^{(\pm)} - \beta_\nu^{(p)})\left[|\beta_\mu\beta_\nu|\left(d + \dfrac{1}{\gamma_\mu} + \dfrac{1}{\theta_\mu}\right)\left(d + \dfrac{1}{\gamma_\nu} + \dfrac{1}{\theta_\nu}\right)\right]^{\frac{1}{2}}}. \quad (70)$$

The eigenvalue equation (31) was used to express (70) in this simple form. This coupling coefficient (and all others to be listed below) holds for the special case $f(z) \ll \pi/\kappa$, $h(z) \ll \pi/\kappa$ with κ of (29). Instead of using the values of the field at $x = f(z)$ and $x = -d + h(z)$, the field values at $x = 0$ and $x = -d$ were used. In order to see the agreement of this coupling coefficient with the coupling coefficient for the symmetric case [eq. (7) of Ref. 7], it is necessary to note that the core thickness d of this paper corresponds to $2d$ of Ref. 7. In addition, we need to keep in mind that only the Fourier components of $f(z)$ and $h(z)$ with spatial frequency $\beta_\mu^{(\pm)} - \beta_\nu^{(p)}$ contribute to coupling between modes μ and ν. The derivatives appearing in (70) are thus equivalent to the products $i(\beta_\mu^{(\pm)} - \beta_\nu^{(p)})\,f(z)$ and $i(\beta_\mu^{(\pm)} - \beta_\nu^{(p)})h(z)$. Keeping these remarks in mind, complete agreement of (70) with (7) of Ref. 7 is obtained for the special case $n_2 = n_3$, $\gamma = \theta$.

The coupling coefficient for coupling between a guided TE mode ν and a TE radiation mode ρ follows similarly from (13), (25), and (36) for radiation in the range $0 \leqq \rho \leqq (n_2^2 - n_3^2)^{\frac{1}{2}}k$

$$K_{\rho\nu}^{(\pm,p)} = -\frac{(n_1^2 - n_2^2)^{\frac{1}{2}}k\kappa_\nu\rho\left(\sigma\cos\sigma d + \dfrac{\Delta}{i}\sin\sigma d\right)}{(\beta_\rho^{(\pm)} - \beta_\nu^{(p)})}$$

$$\cdot\left\{\frac{df}{dz} - \frac{\sin\kappa_\nu d}{|\sin\kappa_\nu d|}\left(\frac{n_1^2 - n_3^2}{n_1^2 - n_2^2}\right)^{\frac{1}{2}}\frac{\sigma}{\sigma\cos\sigma d + \dfrac{\Delta}{i}\sin\sigma d}\frac{dh}{dz}\right\}$$

$$\cdot\left\{\pi|\beta_\rho\beta_\nu|\left(d + \frac{1}{\gamma_\nu} + \frac{1}{\theta_\nu}\right)\left[\rho^2\left(\sigma\cos\sigma d + \frac{\Delta}{i}\sin\sigma d\right)^2 + \sigma^2\left(\sigma\sin\sigma d - \frac{\Delta}{i}\cos\sigma d\right)^2\right]\right\}^{-\frac{1}{2}}. \quad (71)$$

The coupling coefficient of the guided TE mode ν to the TE radiation mode ρ in the range $(n_2^2 - n_3^2)^{\frac{1}{2}}k \leqq \rho < \infty$ is given by

$$K_{j\rho\nu}^{(\pm,p)} = -\frac{(n_1^2-n_2^2)^{\frac{1}{2}}k\kappa_\nu}{(\beta_\rho^{(\pm)} - \beta_\nu^{(p)})\left[\pi|\beta_\rho\beta_\nu|\left(d+\frac{1}{\gamma_\nu}+\frac{1}{\theta_\nu}\right)\right.^{\frac{1}{2}}} \cdot \frac{\frac{df}{dz} - \frac{\sin\kappa_\nu d}{|\sin\kappa_\nu d|}\left(\frac{n_1^2-n_3^2}{n_1^2-n_2^2}\right)^{\frac{1}{2}}(\cos\sigma d - F_j\sin\sigma d)\frac{dh}{dz}}{\left[\frac{\Delta}{\rho}(\cos\sigma d-F_j\sin\sigma d)^2+\frac{\sigma^2}{\rho\Delta}(\sin\sigma d+F_j\cos\sigma d)^2+1+\frac{\sigma^2}{\rho^2}F_j^2\right]^{\frac{1}{2}}}. \tag{72}$$

The factors F_1 and F_2 are obtained from (50). The radiation modes do not all propagate along the z-axis. Propagating modes are confined to the range $0 \leqq \rho \leqq n_2 k$.

The reader should not be startled by the fact that the coupling coefficients (70) have the dimension m^{-1} while the coupling coefficients (71) and (72) have the dimension $m^{-\frac{1}{2}}$. The different dimensions are attributable to the fact that the coupling coefficients between guided modes occur under a summation sign while the coupling coefficients that describe coupling to radiation modes occur under an integral sign. The integration is performed with respect to ρ, the dimension of which is m^{-1}. The product $c_\rho K_{\rho\nu} d\rho$ has the dimension m^{-1} in agreement with the dimension of the coupling coefficients for guided modes.†

Finally, we list the coupling coefficients for the TM modes. Coupling between guided TM modes is described by the coupling coefficient ($p = +$ or $-$)

$$K_{\mu\nu}^{(\pm,p)} = -\frac{(n_1^2 - n_2^2)D_{g\nu}D_{g\mu}}{4P(\beta_\mu^{(\pm)} - \beta_\nu^{(p)})\omega\epsilon_0 n_1^2 n_2^4}\left\{(n_2^2\beta_\mu^{(\pm)}\beta_\nu^{(p)} + n_1^2\gamma_\nu\gamma_\mu)\frac{df}{dz}\right.$$
$$- \frac{\sin\kappa_\nu d\sin\kappa_\mu d}{|\sin\kappa_\nu d\sin\kappa_\mu d|}\frac{n_1^2 - n_3^2}{n_1^2 - n_2^2}\left[\frac{(n_2^4\kappa_\nu^2 + n_1^4\gamma_\nu^2)(n_2^4\kappa_\mu^2 + n_1^4\gamma_\mu^2)}{(n_3^4\kappa_\nu^2 + n_1^4\theta_\nu^2)(n_3^4\kappa_\mu^2 + n_1^4\theta_\mu^2)}\right]^{\frac{1}{2}}$$
$$\left.\cdot(n_3^2\beta_\mu^{(\pm)}\beta_\nu^{(p)} + n_1^2\theta_\nu\theta_\mu)\frac{dh}{dz}\right\}. \tag{73}$$

The coupling coefficients for the TM modes are far more complicated

† Note that $\int|c_\rho|^2 d_\rho$ is dimensionless so that the dimension of c_ρ is $m^{\frac{1}{2}}$.

than the corresponding coefficients for the TE modes. To simplify the notation, we did not substitute expression (59) for the mode amplitude into (73).

The coefficient for coupling from a guided TM mode to a TM radiation mode in the range $0 \leqq \rho \leqq (n_2^2 - n_3^2)^{\frac{1}{2}}k$ is

$$K_{\rho\nu}^{(\pm,p)} = -\frac{(n_1^2 - n_2^2)D_{g\nu}D_{r\rho}}{4P(\beta_\rho^{(\pm)} - \beta_\nu^{(p)})\omega\epsilon_0 n_1^2 n_2^2}\left\{\left(\beta_\rho^{(\pm)}\beta_\nu^{(p)} - \sigma\gamma_\nu\frac{G_{r\rho}}{D_{r\rho}}\right)\frac{df}{dz}\right.$$

$$-\frac{\sin\kappa_\nu d}{|\sin\kappa_\nu d|}\frac{n_1^2 - n_3^2}{n_1^2 - n_2^2}\left[\frac{n_2^4\kappa_\nu^2 + n_1^4\gamma_\nu^2}{n_3^4\kappa_\nu^2 + n_1^4\theta_\nu^2}\right]^{\frac{1}{2}}\left[\beta_\rho^{(\pm)}\beta_\nu^{(p)}\left(\cos\sigma d - \frac{G_{r\rho}}{D_{r\rho}}\sin\sigma d\right)\right.$$

$$\left.\left. + \sigma\theta_\nu\left(\sin\sigma d + \frac{G_{r\rho}}{D_{r\rho}}\cos\sigma d\right)\right]\frac{dh}{dz}\right\}. \quad (74)$$

The amplitude coefficients $D_{g\nu}$, $D_{r\rho}$ and $G_{r\rho}$ are obtained from (59), (63) and (64).

The coefficient for coupling from a guided TM mode to a TM radiation mode in the range $(n_2^2 - n_3^2)^{\frac{1}{2}}k \leqq \rho \leqq \infty$ is given by

$$K_{i\rho\nu}^{(\pm,p)} = -\frac{(n_1^2 - n_2^2)D_{g\nu}S_{r\rho}}{4P(\beta_\rho^{(\pm)} - \beta_\nu^{(p)})\omega\epsilon_0 n_1^2 n_2^2}\left\{(\beta_\rho^{(\pm)}\beta_\nu^{(p)} - \sigma\gamma_\nu R_i)\frac{df}{dz}\right.$$

$$-\frac{\sin\kappa_\nu d}{|\sin\kappa_\nu d|}\frac{n_1^2 - n_3^2}{n_1^2 - n_2^2}\left[\frac{n_2^4\kappa_\nu^2 + n_1^4\gamma_\nu^2}{n_3^4\kappa_\nu^2 + n_1^4\theta_\nu^2}\right]^{\frac{1}{2}}\left[\beta_\rho^{(\pm)}\beta_\nu^{(p)}(\cos\sigma d - R_i\sin\sigma d)\right.$$

$$\left.\left. + \sigma\theta_\nu(\sin\sigma d + R_i\cos\sigma d)\right]\frac{dh}{dz}\right\}. \quad (75)$$

The amplitude coefficients $D_{g\nu}$, $S_{r\rho}$, and R_i are obtained from (59), (68), and (69). The index i assumes the values 1 and 2, and corresponds to the two types of radiation modes that are distinguished by the + and − signs in (68). The superscripts, + and −, attached to the coupling coefficients are supposed to indicate whether the modes travel in + or − z-direction.

It can be shown that the coupling coefficients derived in this paper specialize to the correct expressions[4,5] of the symmetric slab waveguide in the limit $n_2 = n_3$, $\gamma = \Delta$, $\rho = \theta$.†

† Equations (9.5–26) and (9.5–27) of Ref. 5 must be divided by n_2^2, eq. (9.5–31) must, correspondingly, be divided by n_2^4.

IX. CONCLUSIONS

We have collected the formulas for the modes of the asymmetric slab waveguide and have used this information to derive the coupling coefficients between guided modes as well as between guided and radiation modes for the case of very slight core boundary imperfections. Also presented is the general theory of coupled modes of dielectric waveguides and the general formulas for the coupling coefficients. The theory collected in this paper is useful for the description of mode conversion and radiation phenomena. Phenomena such as the grating coupler and the interaction of acoustic waves and guided light waves can readily be treated with the theory presented here. For an application to statistical irregularities of the core boundary, the reader is advised to consult Refs. 5, 6, and 7.

APPENDIX

Evaluation of the Integral (12)

We consider the index distribution of the slab waveguide as being smoothed out to avoid the discontinuity at the core boundary. It is assumed that the index varies only in a direction perpendicular to the core boundary. We use a coordinate system x', z', the axes of which are perpendicular, and parallel to the tangent at a particular point of the core boundary as shown in Fig. 2. In this coordinate system, we assume that the refractive index is of the form

$$n^2 = F(x'). \tag{76}$$

For values of $x' \leqq 0$ we have $F = n_1^2$; for values $x' > \xi$ we have $F = n_2^2$. At the end of our discussion, we let $\xi \to 0$. The scalar product

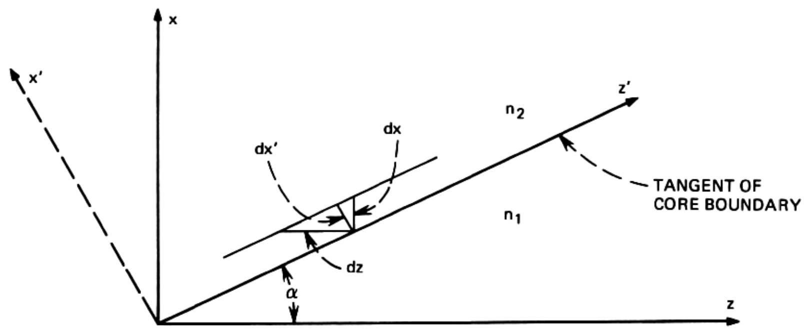

Fig. 2—Sketch of the coordinate systems used for the evaluation of integral (10).

of the electric field vectors can be expressed as

$$\boldsymbol{\mathcal{E}}_\mu^* \cdot \boldsymbol{\mathcal{E}}_\nu = \mathcal{E}_{\mu x'}^* \mathcal{E}_{\nu x'} + \mathcal{E}_{\mu t}^* \mathcal{E}_{\nu t}. \tag{77}$$

The coordinate t indicates a direction parallel to the core boundary. $\mathcal{E}_{\nu t}$ is continuous at the core boundary and can be taken out of the integral. $\mathcal{E}_{\nu x'}$, on the other hand, is discontinuous. We express it in terms of the field just inside of the core and, using (12), write

$$\mathcal{E}_{\nu x'} = \frac{n_1^2}{F(x')} (\mathcal{E}_{\nu x'})_{x'=0}. \tag{78}$$

The scalar product can thus be written in the form

$$\boldsymbol{\mathcal{E}}_\mu^* \cdot \boldsymbol{\mathcal{E}}_\nu = \frac{n_1^4}{F^2(x')} (\mathcal{E}_{\mu x'}^* \mathcal{E}_{\nu x'})_{x'=0} + \mathcal{E}_{\mu t}^* \mathcal{E}_{\nu t}. \tag{79}$$

We thus have to deal with two different integrals. We first consider the integral

$$\int_0^\infty \frac{\partial F(x')}{\partial z} dx = \frac{\partial x'}{dz} \int_0^\infty \frac{\partial F(x')}{\partial x'} dx. \tag{80}$$

We obtain from Fig. 2 the relations

$$\frac{\partial x'}{\partial z} = \sin \alpha \tag{81}$$

and

$$dx = \frac{dx'}{\cos \alpha}. \tag{82}$$

The integral can thus be evaluated

$$\int_0^\infty \frac{\partial F(x')}{\partial z} dx = [F(\xi) - F(0)] \tan \alpha. \tag{83}$$

At the upper core coundary, we have $\tan\alpha = df(z)/dz$, and at the lower core boundary we have $\tan\alpha = dh(z)/dz$. Taking both core boundaries into account, we find with the help of (76),

$$\int_{-\infty}^\infty \frac{\partial n^2}{\partial z} \mathcal{E}_{\mu t}^* \mathcal{E}_{\nu t} dx = -(n_1^2 - n_2^2) \frac{df}{dz} (\mathcal{E}_{\mu t}^* \mathcal{E}_{\nu t})_{x=f}$$

$$+ (n_1^2 - n_3^2) \frac{dh}{dz} (\mathcal{E}_{\mu t}^* \mathcal{E}_{\nu t})_{x=-d+h}. \tag{84}$$

The integral associated with the normal field components is essentially of the form

$$\int_0^\infty \frac{1}{F^2(x')} \frac{\partial F(x')}{\partial z} dx = (\tan \alpha) \int_0^\infty \frac{1}{F^2} \frac{\partial F(x')}{\partial x'} dx'$$
$$= (\tan \alpha) \frac{F(\xi) - F(0)}{F(0)F(\xi)}. \quad (85)$$

The integral containing the normal field components results in

$$\int_{-\infty}^\infty \frac{\partial n^2}{\partial z} \mathcal{E}^*_{\mu x'} \mathcal{E}_{\nu x'} dx = -(n_1^2 - n_2^2) \frac{n_1^2}{n_2^2} (\mathcal{E}^*_{\mu x'} \mathcal{E}_{\nu x'})_{x=f} \frac{df}{dz}$$
$$+ (n_1^2 - n_3^2) \frac{n_1^2}{n_3^2} (\mathcal{E}^*_{\mu x'} \mathcal{E}_{\nu x'})_{x=-d+h} \frac{dh}{dz}. \quad (86)$$

In (13) we replaced x' with x and t with z. These approximations are valid provided that the inequalities (15) apply. The error is of second order in the derivatives of $f(z)$ or $h(z)$.

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