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Applications for Quantum Amplifiers in Simple Digital Optical Communication Systems

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Previously published results on the performance of optical direct detection digital receivers using avalanche detectors are extended to the case where incoherent noise due to quantum amplifiers in the transmission medium is present at the detector. These calculations are applied to determine the usefulness of quantum amplifiers in simple digital transmission systems where the optical source instability results in a required amplifier bandwidth which may be orders of magnitude greater than the modulation bandwidth. It is concluded that practical applications exist where quantum amplifiers can be used in analog repeaters between regenerating repeaters in a hybrid digital system; and also as front ends of regenerating repeaters to increase their sensitivities.

I. INTRODUCTION

Quantum amplifiers can be used in optical communication systems even if the optical sources are only partially coherent. They can serve as optical analog repeaters between regenerating repeaters in a hybrid digital system to compensate for transmission loss (see Fig. 1), and also as the front ends of regenerating repeaters which demodulate back to baseband.

This paper investigates the applications for quantum amplifiers in simple digital communication systems employing "on-off" intensity modulation. It will be assumed that due to source instability, the optical system bandwidth may be orders of magnitude greater than the bandwidth of the modulation, and that the quantum amplifiers have limited gain.

We shall calculate Chernov bounds on the required signal energy per pulse at the detector of a digital repeater (to be described in detail below) to achieve a 10^{-9} error rate as a function of the received sponta-

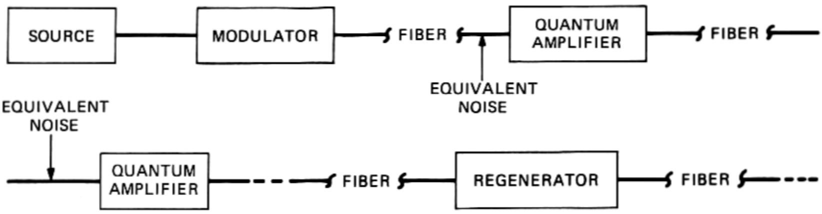

Fig. 1—Fiber communication system.

neous emission noise level from quantum amplifiers, or of the incoherent background noise level for some typical values of dark current of the detector, mean detector gain, and thermal noise introduced by circuitry following the detector. For these Chernov bound calculations, we shall assume a unilateral gain detector. Numerical results for other detectors and parameter values can be obtained by using the moment generating functions to be derived below with the results of two previous papers[1,2] concerning Chernov bounds for direct detection intensity modulation systems using avalanche gain.

We shall also derive some signal-to-noise ratio results which can be used to approximate the energy required per pulse to achieve a desired error rate. These signal-to-noise ratio results are consistent with previous published work of other authors.[3–5]

Throughout this paper it will be assumed that the modulation consists of varying the intensity of the transmitted signal in each baud interval to produce pulses at the regenerating repeater which are of one of two amplitudes, and such that the pulses are approximately constant in a baud interval of length T seconds. Generalization to other pulse shapes should be straightforward using the results below.

II. A MODEL FOR THE QUANTUM AMPLIFIER NOISE

Throughout this paper we shall model the source as follows. Its voltage in a single spatial mode will be given by

$$E_s(t) = \sqrt{2} re\{Ae^{i(\Omega+\omega)t}\} \tag{1}$$

where $|\omega| < 2\pi B/2$.

That is, the source will be nominally at optical frequency $\Omega/2\pi$ but due to source instability there will be an uncertainty of bandwidth B. (The conclusions and numerical results that follow also hold if the source is a randomly phase-modulated sinusoid having a bandwidth B of the form $E_s(t) = \sqrt{2} re\{Ae^{i(\theta(t)+\Omega t)}\}$.)

If the modulated signal is to be transmitted over a channel with quantum amplifiers and possibly with optical filters as well, then these devices must have a bandwidth of at least $B + 1/T$ to accommodate the modulated signal for all possible values of ω. (The term $1/T$ is due to the increase in bandwidth of the source due to the modulation.)

At the regenerating repeater input, the classical field will be modeled as follows (assuming only a single spatial mode)

$$E_r(t) = \sqrt{2}re\{m(t)e^{i(\Omega+\omega)t} + n(t)\} \tag{2}$$

where $m(t)$, the modulation, assumes one of the two possible pulse amplitudes (a pulse which is approximately constant in each baud interval T) and $n(t)$ is a complex Gaussian random process which represents the incoherent spontaneous emission noise introduced in the quantum amplifiers or represents incoherent background noise.[6] In each baud interval T, we can expand the field complex envelope in a Fourier series;† taking only enough terms to include the "system" bandwidth B'.‡

$$\epsilon_r(t) = m(t)e^{i\omega t} + n(t) = \sum_{-(L-1)/2}^{(L-1)/2} a_k \left[\frac{e^{i(2\pi kt/T)}}{\sqrt{T}}\right] \tag{3}$$

where L (the number of temporal modes) is given by $L = B'T \geqq 1 + BT$. Defining

$$m_k = \frac{1}{\sqrt{T}} \int_{\text{baud interval}} m(t)e^{i\omega t}e^{-i(2\pi kt/T)}dt\,,$$

$$n_k = \frac{1}{\sqrt{T}} \int_{\text{baud interval}} n(t)e^{-i(2\pi kt/T)}dt\,, \tag{4a}$$

we have for each value of k, $a_k = m_k + n_k$. Because we have a digital system, the signal components, m_k, take on one of two values for each k. The noise components n_k are complex Gaussian random variables.

$$\langle n_k n_j^* \rangle = N_o \delta_{kj}, \quad \langle n_k n_j \rangle = 0 \tag{4b}$$

where N_o is the classical incoherent noise spectral height§: $\langle n(t)n^*(\tau)\rangle = N_o\delta(t - \tau)$, and $\langle x \rangle$ stands for the expected value of x.

† A more rigorous and general approach taken in the Appendix is to expand the received field in a Karhunen-Loéve expansion[5] using the autocorrelation function of the noise $n(t)$ at the detector input as the kernel. The approach taken here is justified on grounds of simplicity and intuitiveness.

‡ The system bandwidth B' is the minimum of the quantum amplifier bandwidth, the detector optical bandwidth, and the bandwidths of any filters in the optical path preceding the detector. Of course $B' > B + (1/T)$, if we are to accommodate all possible signals with the unstable source described above.

§ That is, the number of watts of incoherent power falling on the detector in the bandwidth B' is N_oB'.

At the regenerating repeater it will be assumed that the field falls upon a detector with internal gain (e.g., an avalanche detector) and causes the detector to emit "primary" hole-electron pairs at rate (average pairs/second)

$$\lambda(t) = \frac{\eta}{\hbar\Omega}|\epsilon_r(t)|^2 \tag{5}$$

where $\hbar$ = Planck's constant/2π, Ω = optical frequency in radians/s, η = detector quantum efficiency, and $\epsilon_r(t)$ was defined in eq. (3) above.

Due to internal gain, each primary "count" (hole-electron pair) produces a random number of additional secondary counts. Because the modulation pulse is approximately constant throughout a baud interval, we will be interested in the total number of counts produced by the detector due to signal and incoherent noise in each baud interval. The moment generating function[7] of the random total number of counts, N, produced in each baud interval, T, is defined as

$$M_N(s) = \sum_{n=0}^{\infty} e^{sn}p(n) \tag{6}$$

where

$$p(n) = \text{probability that } N = n.$$

From previous work[1] we have

$$M_N(s) = M_C(\psi_G(s)) \tag{7}$$

where

$$\psi_G(s) \quad \text{is} \quad \ln[M_G(s)].$$

$M_G(s)$ is the moment generating function of the random internal gain G and $M_C(s)$ is the moment generating function of the total number of primary counts, C.

We can evaluate $M_C(s)$ as follows. Define the quantity Λ as

$$\Lambda = \frac{\eta}{\hbar\Omega}\int_{\text{baud interval}} |\epsilon_r(t)|^2 dt = \frac{\eta}{\hbar\Omega}\sum_{-(L-1)/2}^{(L-1)/2} |a_k|^2 \tag{8}$$

where Λ is the average number of received primary counts in a baud interval given $\epsilon_r(t)$. Λ is a random variable, since the $\{a_k\}$ are random variables having the following joint complex Gaussian probability density

$$p\{a_1, a_2\cdots a_L\} = \prod_{k=1}^{L} \frac{1}{\pi N_o} e^{-|a_k - m_k|^2/N_o}.$$

The probability distribution of the total number of primary counts

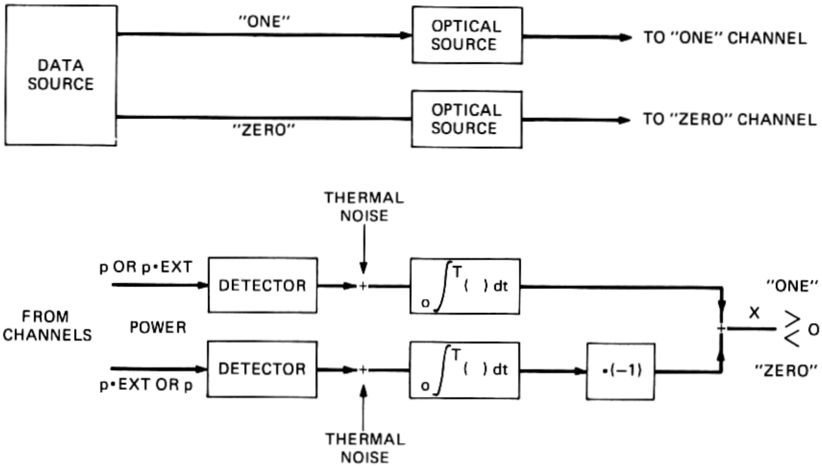

Fig. 2—Twin-channel system.

C in a baud interval *given* Λ is Poisson, i.e.,

$$p(c|\Lambda) = \frac{\Lambda^c e^{-\Lambda}}{c!} = \text{probability that } C = c \text{ given } \Lambda.$$

It follows that $M_C(s)$ is given by

$$\begin{aligned} M_C(s) &= \int_o^\infty \left[\sum_o^\infty p(c|\Lambda)e^{sc} \right] p(\Lambda)d\Lambda \\ &= \int_{-\infty}^\infty e^{\Lambda(e^s-1)} p(\Lambda)d\Lambda \\ &= \left[1 - \frac{\eta N_o}{\hbar\Omega}(e^s - 1) \right]^{-L} \\ &\quad \times e^{\{(\eta/\hbar\Omega)\Sigma|m_k|^2(e^s-1)/[1-(\eta/\hbar\Omega)N_o(e^s-1]\}}. \end{aligned} \tag{9}$$

III. SIGNAL-TO-NOISE RATIO RESULTS

From (7) and (9) we obtain the mean number of counts, $\langle N \rangle$, emitted by the avalanche detector in a baud interval as follows

$$\begin{aligned} \langle N \rangle &= \frac{\partial}{\partial s} M_N(s) = \frac{\partial}{\partial[\psi_G(s)]} M_C(\psi_G(s)) \frac{\partial}{\partial s}\psi_G(s) \bigg|_{s=0} \\ &= \bar{G}[m^2 + LN_o]\eta/\hbar\Omega \end{aligned} \tag{10}$$

where

$$m^2 = \int_{\text{baud interval}} |m(t)|^2 dt = \sum_{-(L-1)/2}^{(L-1)/2} |m_k|^2,$$

$N_o = \langle |n_k|^2 \rangle$ = classical spectral height of the incoherent noise at the detector input,

$L \geqq [B + 1/T]T = BT + 1,$

and $\bar{G}$ is the mean avalanche gain.

The variance of the total number of counts is

$$\langle N^2 \rangle - \langle N \rangle^2 = \frac{\partial^2}{\partial s^2} M_N(s)\bigg|_{s=0} - \left(\frac{\partial}{\partial s} M_N(s)\bigg|_{s=0}\right)^2$$

$$= \underbrace{(m^2 + LN_o)\frac{\eta}{\hbar\Omega}\overline{G^2}}_{\text{shot noises}} + \underbrace{(LN_o^2 + 2N_o m^2)(\bar{G})^2\left(\frac{\eta}{\hbar\Omega}\right)^2}_{\text{beat noises}\dagger} \tag{11}$$

where $\overline{G^2}$ is the mean square avalanche gain.

Consider a typical twin-channel digital system, shown in Fig. 2. There is light incident on each detector containing "on-off" modulated signal pulses of duration T and incoherent noise. A channel is in the "on" state when its signal pulse has optical power p. In the "off" state the signal pulse power is $p \cdot EXT$, where EXT is small compared to unity. During each baud interval, one or the other channel is "on." The detectors are assumed to have internal random gain (e.g., avalanche gain or photomultiplier gain) and there are assumed to be thermal noises added to the detector outputs due to the amplifiers following the detectors. It is assumed that the signaling rate is slow enough so that each signal pulse of light of duration T produces an output current from its detector of duration T that does not overlap with the currents from other pulses. The detector output current pulses plus the corresponding noises are integrated in each period T (or equivalently filtered). The output variable x is compared to the threshold after each integration to decide which channel is "on." An error is made if $x > 0$ when the "zero" channel is on, or vice-versa.

The baseband noise-to-signal ratio is defined as the variance of the output voltage x divided by the square of the mean of the output voltage x.

† The term "beat noise" has been used in literature[8] to describe those noise terms at the output of a square law detector which are due to fluctuations in the instantaneous power of a carrier which has a fluctuating amplitude.

$$\frac{\langle x^2\rangle - (\langle x\rangle)^2}{\langle x\rangle^2} = \underbrace{\frac{4k\theta T}{Re^2\lambda_s^2(1 - EXT)^2\bar{G}^2}}_{\text{thermal noise}} + \underbrace{\frac{2\lambda_d + 2L\overset{\swarrow}{\lambda_n} + \lambda_s(1 + EXT)}{\lambda_s^2(1 - EXT)^2}\,\frac{\overline{G^2}}{(\bar{G})^2}}_{\text{shot noises}} + \underbrace{\frac{2L\overset{\swarrow}{\lambda_n^2} + 2\lambda_s\overset{\swarrow}{\lambda_n}(1 + EXT)}{\lambda_s^2(1 - EXT)^2}}_{\text{beat noises}} \qquad (12)$$

where

$k\theta$ = Boltzman's constant·absolute noise temperature referred to the integrator input.

R = integrator equivalent thermal noise input resistance.

λ_d = mean dark current counts per detector per interval T before avalanche gain.

$\lambda_s = m^2\eta/\hbar\Omega$ = mean signal counts per interval T in "on" channel before avalanche gain.

$L\lambda_n$ = mean incoherent noise counts in either channel per baud interval T before avalanche gain.

$L \geqq BT + 1$, and equals the number of temporal modes detected.

EXT = Signal power in "off" channel/signal power in "on" channel.

In eq. (12), terms which are due to the incoherent spontaneous emission noises of the quantum amplifiers (or background noise) are marked with arrows.

We see that the optical incoherent noise, when detected to baseband, causes additional shot noise and also contributes two beat noise terms. One of these is proportional to the signal λ_s and one is proportional to L. One can use these signal-to-noise ratio results to approximate the error rate by assuming that the output variable x is roughly Gaussian in distribution.

In the next section we shall generate some curves that may give a clearer picture of the effects of L, λ_n, λ_s, etc., on performance.

IV. CHERNOV BOUNDS

The moment generating function defined in (9) was used with previously published results[1,2] on avalanche photo-diode gain statistics to obtain Chernov upper bounds on the energy per pulse required at the input of a digital twin-channel regenerating repeater of Fig. 2 to achieve a desired error rate as a function of the other parameters.

The general Chernov bound is given as follows.[7] Let χ be a random variable with moment generating function $M_\chi(s)$. Let $\Pr_\chi(x > \gamma)$ be the probability that an outcome x of χ exceeds γ. Then it follows that

$$\Pr_\chi(x > \gamma) \leqq e^{[\psi_\chi(s) - s\gamma]} \quad \text{for} \quad s > 0 \tag{13}$$

where

$$\psi_\chi(s) = \ln[M_\chi(s)].$$

The bound is optimized for s such that $(\partial\psi(s)/\partial s) = \gamma$ provided that value of s is greater than zero.

Similarly,

$$\Pr_\chi(x < \gamma) \leqq e^{[\psi_\chi(s) - s\gamma]} \quad \text{for} \quad s < 0 \tag{14}$$

where the optimal value of s is given by $(\partial\psi(s)/\partial s) = \gamma$ provided that value of s is less than zero.

To obtain Chernov bounds upon the probability of error for the twin-channel system of Fig. 2, one needs the moment generating

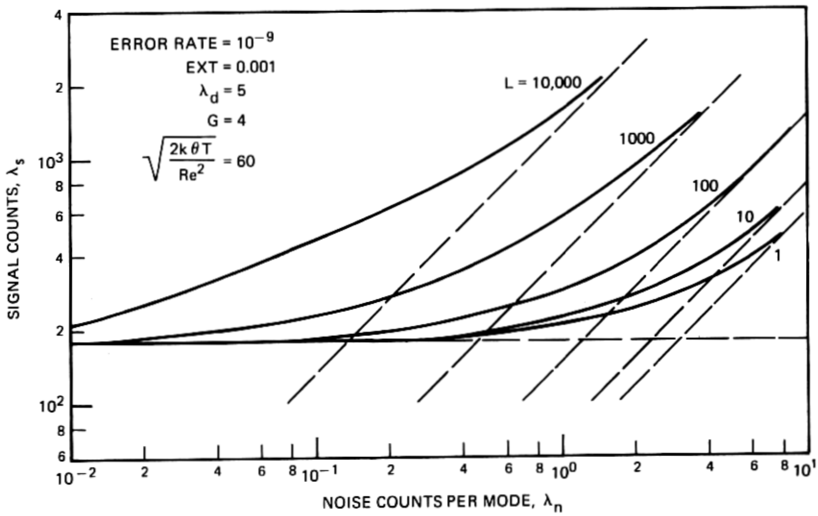

Fig. 3—Required energy per pulse normalized by $\eta/\hbar\Omega$ vs the incoherent noise level N_o at the detector, also normalized by $\eta/\hbar\Omega$.

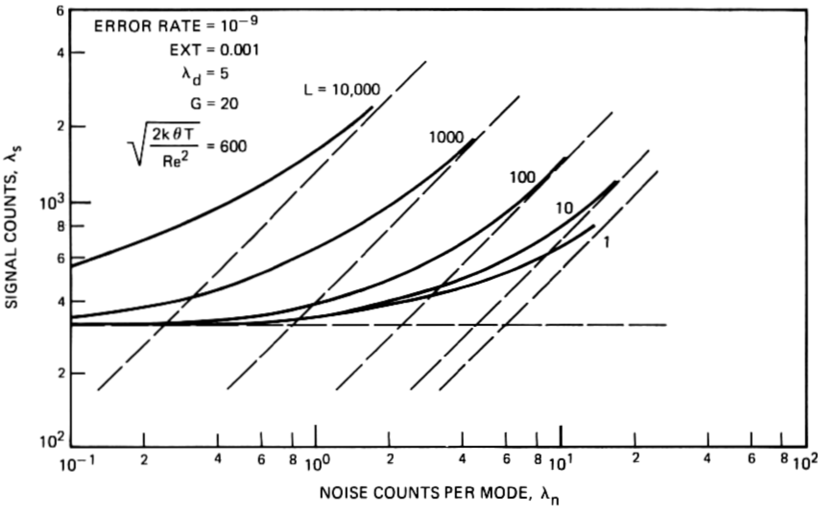

Fig. 4—Same as Fig. 3.

function of the output variable x. This can be obtained using (9) and the results of the Refs. 1 and 2, which are too detailed to duplicate here.

From simple cases where error rates can be calculated exactly, the differences in required power between those results and the bounds are typically a few dB or less. Experimental results also confirm the tightness of the bounds. Therefore, in this paper we shall take the liberty of comparing the effects of various parameters upon the required energy per pulse to achieve a desired error rate by comparing the bounds.

It was decided that the calculations should be presented graphically in two ways.

First, in Figs. 3 to 5, the required energy per pulse normalized by $\eta/\hbar\Omega$ (i.e., the mean number of detected signal photons per pulse) is plotted vs the incoherent noise level N_o at the detector also normalized by $\eta/\hbar\Omega$. This is done for various values shown of L, mean avalanche gain, thermal noise, dark current, error rate, and extinction ratio, for a low-noise unilateral gain avalanche detector (i.e., a detector in which only one type of carrier causes ionizing collisions, and where carrier injection is from one end of the high field region). The avalanche gains used in these calculations do not minimize the required energy per pulse for the given values of the other parameters, but were used for illustration.

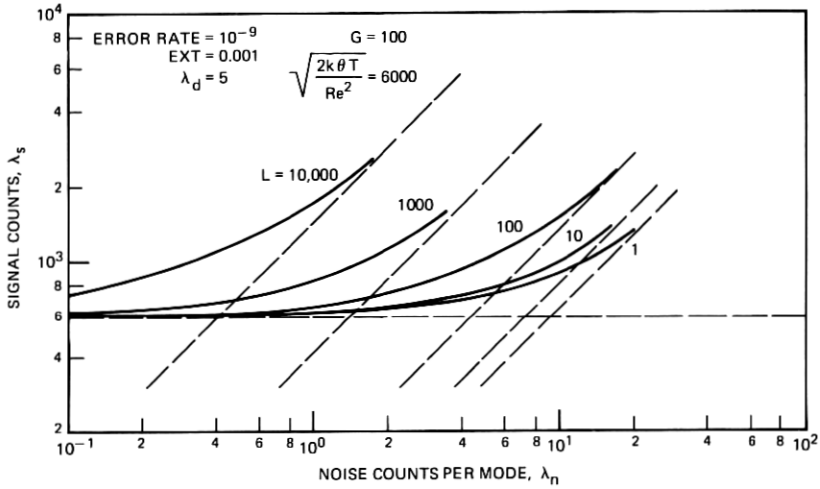

Fig. 5—Same as Fig. 3.

It was recognized that in a hybrid system, if the loss between the regenerating repeater and the analog repeater closest to it is increased, then the signal energy per pulse at the regenerating repeater

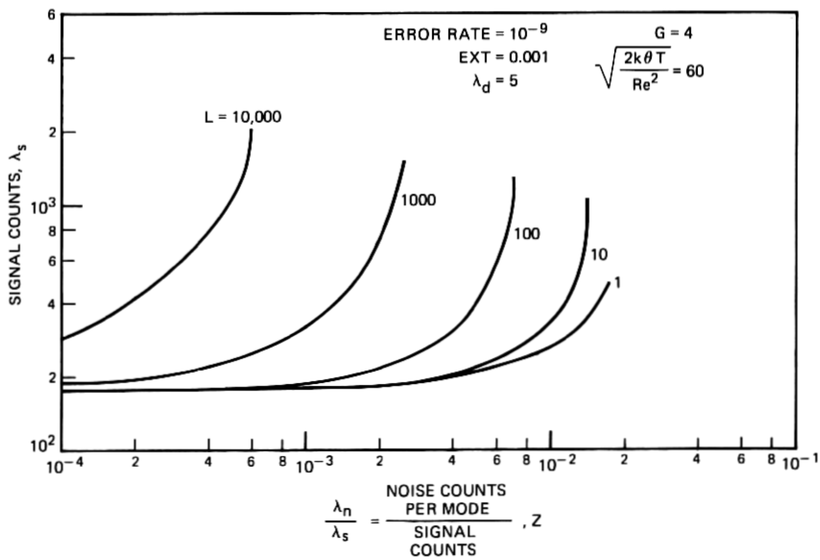

Fig. 6—Required energy per pulse normalized by $\eta/\hbar\Omega$ vs the ratio Z of spontaneous emission noise spectral height to signal energy.

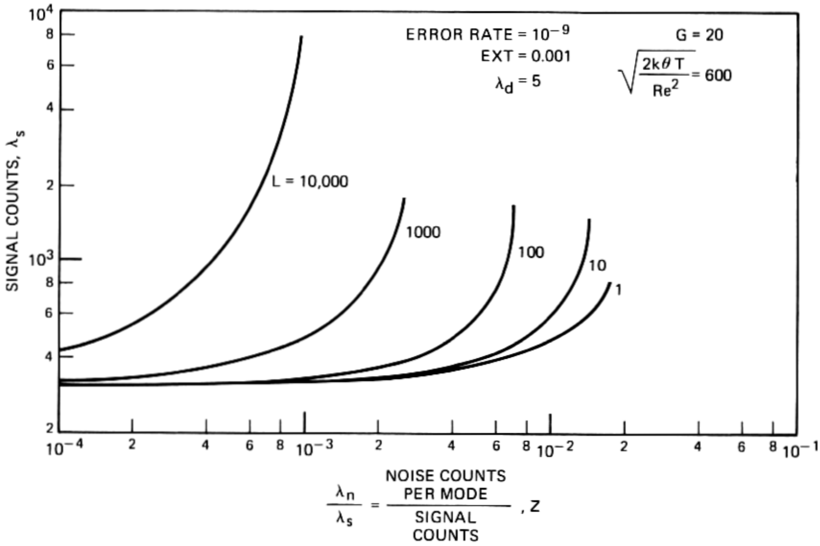

Fig. 7—Same as Fig. 6.

input will decrease while the ratio of signal energy per pulse to spontaneous emission noise spectral height at the regenerating repeater input will remain fixed. Thus in Figs. 6 to 8, the required energy per pulse normalized by $\eta/\hbar\Omega$ is plotted vs the ratio Z of spontaneous

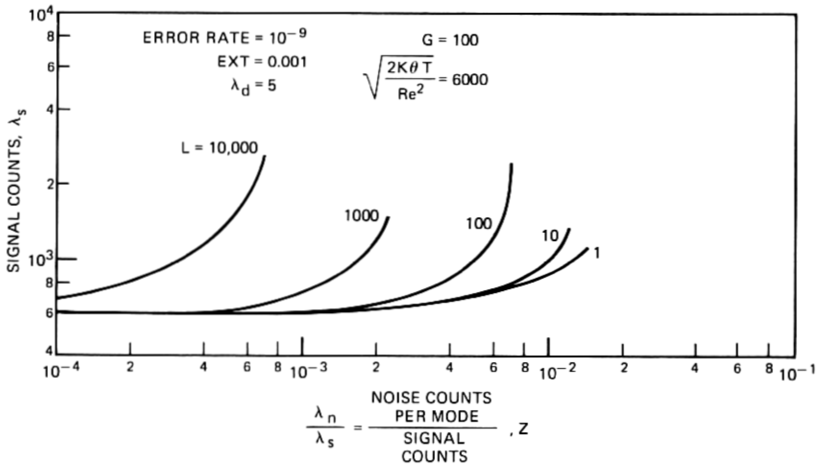

Fig. 8—Same as Fig. 6.

emission noise spectral height to signal energy, for the various values shown of other parameters.

V. APPLICATIONS AND EXAMPLES

5.1 *Analog Repeaters*

Suppose one used quantum amplifiers in analog repeaters placed between regenerating repeaters so as to increase the distance between regenerating repeaters. See Fig. 1. Each quantum amplifier introduces a spontaneous emission noise which has spectral height *referred to its input* given by[6]

$$N_{\text{input}} = F\hbar\Omega\left(\frac{G_q - 1}{G_q}\right) \tag{15}$$

where G_q is the quantum amplifier power gain and F is a noise figure which can be near unity for good quantum amplifiers and is typically less than 10.† If the input to analog repeater k is α_k nepers (in power) higher than the signal level at the input of the regenerating repeater, then the total spontaneous emission noise spectral height N_o at the input of the regenerating repeater is

$$N_o = \sum_1^R F_k\hbar\Omega\frac{(G_{qk} - 1)}{G_{qk}}e^{-\alpha_k} \tag{16}$$

where R = number of analog repeaters.

The ratio of N_o in (16) to the signal energy per pulse, $p \cdot T$ at the regenerating repeater input, see Fig. 2, is the parameter Z defined in Section IV above. Since the incoherent noise and the signal both experience equal loss per unit length from the fiber, the ratio Z is constant between the regenerating repeater and the analog repeater closest to it.

Example: Suppose we make the following assumptions. A twin-channel system is used with a unilateral gain detector having mean gain 100 and with all the other parameter values necessary above so that the Chernov bound curves of Fig. 8 are applicable. The source is a Nd:YAlG laser having bandwidth 1 Å at wavelength 1 μm, i.e., $3 \cdot 10^{10}$ Hz. The modulation rate is 300 Mb/s so that $T \sim 3.33 \times 10^{-9}$ s.

† F is related to the population inversion in the amplifying medium which is assumed constant in this analysis.

We then have $L = 100$. There are 10 analog repeaters and they are spaced so that the signal level is the same at the input to each one.

From Fig. 8 (assuming that the upper bounds are tight enough so that we can comment upon the effects of various parameters on the required energy per pulse by observing their effects upon the bounds†) we see that when Z is less than 10^{-3}, the required signal energy at the regenerator input is 600 counts, i.e., $p \cdot T = \hbar\Omega/\eta \cdot 600$. This value of signal energy is the same as that which would be required if no spontaneous emission noise were present ($Z = 0$).

Thus for spontaneous emission noise to be negligible in this example, we must have the ratio of the signal energy per pulse at the regenerator input to N_o larger than 10^3.

This means [from (16)] that at the analog repeater inputs the signal level must exceed $10^3 \cdot \hbar\Omega RF[(G_q - 1)/G_q]$ where

R = number of analog repeaters = 10 in this example
G_q = gain of analog repeater (assumed the same for all repeaters)
F = noise figure of an analog repeater (assumed the same for all repeaters).

Looking again at Fig. 8, we see that for $L = 100$, Z can be as large as 5×10^{-3} before the required signal level at the regenerating repeater becomes large and enters the sensitive region. This means that the signal level at the inputs to the analog repeaters might be as low as $200 \cdot \hbar\Omega RF[(G_q - 1)/G_q]$ in which case the signal required at the regenerating repeater is somewhat larger, but still not extremely sensitive to small changes in Z. Suppose $F = 10$, $\eta = 1$, $G_q = 100$, and the maximum power output of any repeater is 1 mW. Suppose the loss of the medium is 10 dB/km. When spontaneous emission noise is negligible, we need $600\ \hbar\Omega = 1.2 \times 10^{-16}$ joules per pulse at the input to the regenerating repeater and we have 3.33×10^{-12} joules per pulse at the output. Without analog repeaters we can have about 44.5 dB of loss or 4.45 km between regenerating repeaters. Suppose on the other hand we use 10 analog repeaters starting where the signal level is 200 $\hbar\Omega RF[(G_q - 1)/G_q] = 4 \times 10^{-15}$ joules per pulse (i.e., $Z = 5 \times 10^{-3}$); or about 28.8 dB (2.88 km) from the regenerating repeater output. The string of 10 analog repeaters spaced at 20-dB intervals spans 200 dB or 20 km of distance; and we can have an additional 13 dB or 1.3 km of distance to the next regenerating repeater input resulting in the required 2×10^{-16} joules per pulse at that regenerating repeater input.

† See comment Section IV.

The total distance between regenerating repeaters is now about 24.2 km.†

It seems prudent that for a given value of L, one should avoid values of Z which are so large that small changes in Z result in large changes in the required signal energy at the regenerating repeater. Such small changes in Z might come about if the source power or quantum amplifier gains fluctuated slightly.

5.2 *Regenerator Repeater Front End*

Suppose that in the example above we had just one quantum amplifier (or equivalently, the spontaneous emission noise from any additional quantum amplifiers was negligible).

In the absence of spontaneous emission noise, the energy per pulse required at the regenerative repeater input is approximately 600 $\hbar\Omega/\eta$. Now suppose we place the quantum amplifier immediately before the regenerating repeater. If the gain is sufficiently large, then we can operate with Z as large as $7 \cdot 10^{-3}$. This means the energy per pulse at the input to the quantum amplifier need only be about $\hbar\Omega F/(7 \times 10^{-3}) \approx 140\hbar\Omega F$ (for large G_q). Thus, we see that if $140F < 600/\eta$, then the quantum amplifier increases the sensitivity of the regenerative repeater over that associated with an avalanche detector alone (in this example with $L = 100$).

For other values of L *in this example*, the condition for a quantum amplifier front end to increase the regenerative repeater sensitivity is

$$\frac{F}{Z_{\max}} < \frac{600}{\eta}$$

where $Z_{\max}$ is the maximum value of Z for reasonable required energy per pulse at the input to the regenerating repeater (following the quantum amplifier).

For other systems with different types of avalanche detectors and different parameters (avalanche gain, dark current, etc.) the number 600 in the above equation should be replaced by the required mean number of detected counts in the absence of a quantum amplifier.

† A slightly larger total distance between regenerating repeaters can be obtained by starting the chain of analog repeaters 20 dB (rather than 28.8 dB) from the regenerating repeater output. In that case $Z \cong 5 \cdot 10^{-4}$ and the next regenerating repeater can be about 45 dB from the last analog repeater for a total span of 24.5 km between regenerating repeaters. Placing the analog repeaters as described in the above example allows some margin for overload.

5.3 *Background Noise*

As a final comment, it is clear from eq. (12) that if the incoherent noise spectral height, N_o, at the regenerating repeater input is small enough so that $(\eta/\hbar\Omega)N_o \ll \overline{G^2}/(\bar{G})^2$ then only the additional shot noise term is important amongst the three noise terms associated with the incoherent noise.

This inequality always holds for the case where the incoherent noise is background (thermal) radiation in equilibrium at temperatures below 10^4 °K, since for thermal background radiation we have

$$N_o(\text{Thermal}) = \frac{\hbar\Omega}{e^{\hbar\Omega/k\theta} - 1},$$

$k\theta$ = Boltzman's constant · absolute temperature.

At room temperature and at a wavelength of 1 μm, $\hbar\Omega/k\theta \approx 50$.

Therefore, in analyses where incoherent background radiation is included, one usually only includes the additional shot noise term $LN_o(\eta/\hbar\Omega) = L\lambda_n$ in the signal-to-noise ratio formulae.

VI. CONCLUSIONS

We have shown that quantum amplifiers can have applications in both analog repeaters to extend the distance between regenerating repeaters and as front ends of regenerating repeaters. Their usefulness is a function of the ratio of the optical bandwidth of the system to the modulation bandwidth; but is not limited to small values of this ratio. To choose system parameters, for example, the required signal levels at the analog and regenerating repeater inputs, various component parameters such as the mean avalanche gain, avalanche detector type, source bandwidth, baseband thermal noise, etc., must be given. Computations in addition to those presented, upper bounding the error rates, can be carried out with previously published Chernov bound results;[1,2] or approximate error-rate calculations can be made using the signal-to-noise ratio results of Section III above.

APPENDIX

Use of the Karhunen-Loéve Expression

Starting with eq. (2) of the text, we could expand the received complex envelope $\epsilon_r(t) = m(t)e^{i\omega t} + n(t)$ in a baud interval in terms of the Karhunen-Loéve eigenfunctions of the band limited incoherent noise $n(t)$, i.e., define

$$R_n(t,u), \quad \{\psi_k(u)\}, \quad \text{and} \quad \{\gamma_k\}$$

as follows

$$R_n(t,u) = \langle n(t)n^*(u)\rangle$$

$$\int_{\text{baud interval}} \psi_k(u)R_n(t,u)du = \gamma_k\psi_k(t) \qquad \text{for } t\epsilon \text{ baud interval,}$$

$$k = 1, 2, 3 \cdots.$$

Then

$$\epsilon_r(t) = \sum_1^\infty a_k\psi_k(t) \qquad \text{for} \quad t\epsilon(0,T)$$

where

$$a_k = m_k + n_k$$

$$m_k = \int_{\text{baud interval}} m(t)e^{i\omega t}\overset{*}{\psi}_k(t)dt$$

$$n_k = \int_{\text{baud interval}} n(t)\overset{*}{\psi}_k(t)dt$$

$$\langle n_k\overset{*}{n}_j\rangle = \gamma_k\delta_{kj}, \qquad \langle n_k n_j\rangle = 0$$

and

$$\int_{\text{baud interval}} \psi_k(t)\overset{*}{\psi}_j(t)dt = \delta_{kj}.$$

Then we would find that $M_C(s)$ of eq. (9) could be more rigorously given by

$$M_C(s) = \left[\prod_{k=1}^\infty \left[1 - \frac{\eta\gamma_k}{\hbar\Omega}(e^s - 1)\right]\right] \exp\left\{(\eta/\hbar\Omega)\sum_{k=1}^\infty\left\{|m_k|^2(e^s - 1)\Big/\left[1 - \frac{\eta}{\hbar\Omega}\gamma_k(e^s - 1)\right]\right\}\right\}.$$

Thus in eq. (9) N_o has been rigorously replaced by γ_k for each k and the finite number of terms L has been replaced by an infinite number of terms.

If we make the reasonable assumption that the incoherent noise is flat with spectral height N_o in a band of width $B' + 1/T$ then

$$\begin{aligned} \gamma_k &\approx N_o \quad \text{for} \quad 1 \geqq k \geqq L \\ &\approx 0 \quad \text{otherwise} \end{aligned} \tag{17}$$

where

$$L = B'T + 1.$$

Thus the form for $M_C(s)$ derived in the main text is identical to the more rigorous result under this approximation.

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