Nonorthogonal Optical Waveguides and Resonators

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The modes of propagation in optical systems which do not possess meridional planes of symmetry (nonorthogonal systems) are investigated in the case where the effect of apertures and losses can be neglected. The fundamental mode of propagation is obtained with the help of a complex ray pencil concept. An integral transformation of the field, based on a quasi-geometrical optics approximation and a first-order expansion of the point characteristic of the optical system, is given; it shows that the complex (three-dimensional) wavefront of the fundamental mode is transformed according to a generalized "ABCD law." A simple expression is also obtained for the phase-shift experienced by the beam. The higher order modes of propagation are obtained from a power series expansion of the fundamental mode. These higher order modes are expressed, in oblique coordinates, as the product of the fundamental solution and finite series of Hermite polynomials with real arguments. In the special case of systems with rotational symmetry, these series reduce to the well-known generalized Laguerre polynomials. The theory is applicable to media such as helical gas lenses and optical waveguides suffering from slowly varying deformations in three dimensions. Nonorthogonal resonant systems are also investigated. An expression for the resonant frequencies, applicable to any three-dimensional resonator, is derived. Numerical results are given for the resonant frequencies and the resonant field of a twisted path cavity which exhibits interesting properties: the usual polarization degeneracy is lifted and the intensity pattern of all of the modes possesses a rotational symmetry.

I. INTRODUCTION

An optical system, or a resonator, is called "nonorthogonal" when it is not possible to define two mutually orthogonal meridional planes of symmetry (Ref. 1, p. 240). The helical gas lens[2,3] is an example of a nonorthogonal lenslike medium. A conventional ring type cavity generally ceases to be orthogonal when its path is twisted, i.e., becomes nonplanar.[4]

Let us briefly review the major approaches in the theory of optical resonators. The field in a resonator can be expressed exactly in terms of known functions only for a few simple boundary surfaces. No exact solution is available for nonorthogonal systems. However, we are interested only in the high frequency operation of large resonators. In that limit, the waves have a tendency to follow closed curves in the resonator, either clinging to the concave parts of the boundary (whispering gallery modes[5]) or connecting opposite points of the boundary (bouncing ball modes). One defines the axial mode number as the number of wavelengths existing along such closed curves. The nodes of the field in the transverse planes define the transverse mode numbers. More insight concerning the mode structure and the resonant frequencies can be gained by using a geometrical optics approximation, or a paraxial form of the Huygens diffraction principle. The geometrical optics approach was developed by Keller and Rubinow.[6] It consists of setting up in the resonator a manifold of rays tangent to a caustic. The location of the caustic and the resonant frequencies are obtained from the condition that the variations of the eikonal along three independent closed curves are equal to an integral number of wavelengths (or an integer plus one-half or one-quarter). This theory, which is analogous to the Born approximation of quantum mechanics, gives the exact resonant frequencies of paraxial modes. The geometrical optics field, when extended in the shadow of the caustic by analytic continuation, provides an acceptable approximation to the exact field for large transverse mode numbers but, for the fundamental mode, it differs vastly from the exact field. The caustic line however, does coincide, in two dimensions, with the mode profile.[7,8] This geometrical optics method has been extended to nonorthogonal resonators incorporating homogeneous media by Popov,[9] who gave an expression for the resonant frequencies. Within the paraxial approximation, exact solutions for the field can be obtained from the Huygens principle; for that reason, the geometrical optics method, in spite of its general interest, will not be discussed further in this paper.

For the case of resonators incorporating inhomogeneous media, the Huygens principle must be supplemented by a quasi-geometrical optics approximation. This approximation consists of assuming that a point source at the input plane of the system creates at the output plane a field which can be adequately represented by the geometrical optics field. This approximation is generally applicable to optical waveguides and resonators if one disregards the effect of apertures and assumes that no diffraction gratings or other wavelength-dependent scatterers are present. This quasi-geometrical optics method provides an integral transformation for the field which is equivalent to a partial differential equation of the parabolic type (see Section II). The similarity between this parabolic equation and the Schroedinger equation has often been pointed out.[10–13] The matched modes of propagation in uniform lens-like media with hyperbolic secant refractive index laws, for instance, can be found in Landau and Lifshits' *Quantum Mechanics*[14] [whereas the ray trajectories are given in Ref. (1), p. 179]. The more general problem of unmatched beams in nonuniform lens-like media corresponds to the time-dependent Schroedinger equation with time-varying potentials. The adiabatic approximation usually applied to this problem, is based on conditions[12] which are too stringent for most optical systems. Generalized modes, where allowance is made for a wavefront curvature, were introduced by Goubau and Schwering[15] and Pierce[16] for the free-space case, in agreement with the theory of confocal resonators proposed by Boyd and Gordon.[17] These results were extended to orthogonal square law media.[18,19,20] The transformation of the complex curvature of beams through arbitrary optical systems with rotational symmetry and the resonant frequency of linear cavities was obtained by Kogelnik.[21,22] Vlasov and Talanov[23] have observed that, in two dimensions, the phase shift experienced by a matched beam in an optical system is equal to the phase of one of the two ray-matrix eigenvalues. This result is easily demonstrated and generalized to astigmatic orthogonal systems by using a complex ray pencil concept.[4,24]

The generalization to nonorthogonal systems is substantially more intricate. Arnaud and Kogelnik[24] have obtained a generalized gaussian mode of propagation in free space by giving complex values to the three parameters which define an astigmatic ray pencil, i.e., the position of the focal lines and the angular orientation of one of them. This solution can be used to obtain the beam transformation in a sequence of thin astigmatic lenses arbitrarily oriented, by matching the complex wavefronts at each lens. This method does not give, however, a general expression for the phase shift experienced by the beam, knowledge of which is essential in studying resonators. For that reason, a somewhat different approach is used here, where the ray pencil is defined by two of its rays. The field of the fundamental mode of propagation is obtained (Section III) by allowing these two rays to assume complex positions while remaining solutions of the ray equations.

The higher order modes of propagation are studied in Section IV. They are obtained by application of differential operators related to those used in quantum mechanics. An oblique coordinate system is introduced which diagonalizes the complex wavefront of the fundamental mode. In this oblique coordinate system, the higher order modes can be expressed as the product of the fundamental solution and finite series of Hermite polynomials with real arguments. An alternative procedure is also given which leads to Hermite polynomials in two complex variables. The simple formula for the resonant frequencies of linear resonators given by Popov[9,26] is shown to be applicable to ring type resonators incorporating inhomogeneous media (Section V). Finally these general results are applied to a new type of optical resonator called "cavity with image rotation" which presents interesting resonance and polarization properties (Section VI). Numerical results are presented.

The present theory is limited to paraxial first-order solutions in loss-less isotropic media. As indicated before, it is assumed that no apertures or diffraction gratings are present in the system, and the problem of mode selection is not discussed. The electromagnetic field is treated as a scalar quantity and the polarization effects are introduced only at a later stage; this is permissible within the paraxial approximation. Fresnel reflection at surfaces of discontinuity is also neglected.

II. PARABOLIC WAVE EQUATION AND INTEGRAL TRANSFORMATION OF THE FIELD

In this section an approximate form of the scalar Helmholtz equation is derived which is applicable to paraxial beams, i.e., to beams propagating at small angles with respect to the system axis. It is subsequently compared to an integral transformation derived from Huygens principle.

The scalar Helmholtz equation can be written in a x_1, x_2, z rectangular coordinate system

$$\frac{\partial^2 E}{\partial x_1^2} + \frac{\partial^2 E}{\partial x_2^2} + \frac{\partial^2 E}{\partial z^2} + k^2 n^2(x_1, x_2, z)E = 0, \tag{1}$$

where E is a component of the field and $n(x_1, x_2, z)$ the refractive index of the medium. Let us introduce a reduced field

$$\psi(x_1, x_2, z) = E(x_1, x_2, z) \exp\left[jk \int_0^z n(0, 0, z)\, dz\right], \tag{2}$$

and neglect the second derivative of ψ with respect to z. This approximation physically means that only waves propagating in a direction close to the z axis are considered. Denoting $n(0, 0, z)$ by n_0, for brevity, one obtains

$$\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} - 2jkn_0 \frac{\partial \psi}{\partial z} - jk\psi \frac{dn_0}{dz} + k^2(n^2 - n_0^2)\psi = 0. \tag{3}$$

This equation can be simplified if one introduces the following changes of function and variables[24]

$$\Psi = n_0^{\frac{1}{2}}\psi, \tag{4}$$

$$\zeta = \int_0^z dz/n_0 . \tag{5}$$

One obtains

$$\frac{\partial^2 \Psi}{\partial x_1^2} + \frac{\partial^2 \Psi}{\partial x_2^2} - 2jk \frac{\partial \Psi}{\partial \zeta} + k^2(n^2 - n_0^2)\Psi = 0. \tag{6}$$

Let us further assume that $n^2 - n_0^2$ is a quadratic form in x_1, x_2

$$n^2 = n_0^2 + n_{11}x_1^2 + 2n_{12}x_1x_2 + n_{22}x_2^2 . \tag{7a}$$

n_{11}, n_{12} and n_{22} are *real* functions of z since the losses in the medium are neglected. The quadratic form given in equation (7a) describes a nonorthogonal optical system when the directions of its axes change as z varies. In that case, the diagonal term $2n_{12}x_1x_2$ cannot be eliminated by rotating the coordinate system about z. We discuss this general case.

Let us rewrite equation (7a), for brevity, in matricial form

$$n^2 = n_0^2 + \tilde{r}\eta r \tag{7b}$$

where r denotes a column matrix with elements x_1, x_2 and η denotes a 2×2 real symmetrical matrix. The sign $\sim$ indicates a transposition. Inserting equation (7b) in equation (6), the wave equation assumes the form

$$\mathfrak{L}\Psi \equiv \left(\nabla^2 - 2jk \frac{\partial}{\partial \zeta} + k^2\tilde{r}\eta r\right)\Psi = 0 \tag{8}$$

where ∇^2 denotes the laplacian operator in the transverse x_1, x_2 plane.

It is henceforth assumed that $n^2 - n_0^2$ is small compared with unity. Within this (first-order) approximation, the refractive index law, equation (7b), becomes

$$n \simeq n_0 + \tilde{r}\eta r/(2n_0). \tag{9}$$

Let us now consider the ray trajectories. A ray $\mathcal{R}$ is defined at any transverse plane z by its position $q(z)$ and by the projection $p(z)$ on that plane of a vector directed along the ray, of length equal to the refractive index n. $q(z)$ and $p(z)$ are called respectively the position vector and the direction vector of the ray. It is convenient to represent these vectors by column matrices whose elements are the vector components on x_1, x_2. As long as only fixed coordinate systems are used, such matrices can be denoted without ambiguity $q(z)$ and $p(z)$, or simply q and p. The exact ray equations are (see, for instance, Ref. 1, p. 90)

$$\dot{p} = -n_0 \nabla H(r, p), \tag{10a}$$

$$\dot{q} = n_0 \nabla_p H(r, p), \tag{10b}$$

at $r = q$. In equation (10) the upper dots denote differentiations with respect to ζ, and $H(r, p)$ denotes the Hamiltonian of the system defined by

$$H(r, p) = -(n^2 - \tilde{p}p)^{\frac{1}{2}}; \tag{10c}$$

∇ denotes the gradient operator in the transverse x_1, x_2 plane, and ∇_p denotes a gradient operator relative to the p variables. Within the first order approximation [equation (9)], equations (10c), (10a) and (10b) reduce respectively to

$$H(r, p) = -n_0 - (\tilde{r}\eta r - \tilde{p}p)/(2n_0), \tag{11c}$$

$$\dot{p} = \eta q, \tag{11a}$$

and

$$\dot{q} = p. \tag{11b}$$

Equations (11a) and (11b) are called the paraxial ray equations.

Let us now consider two arbitrary rays, $\mathcal{R}$ and $\hat{\mathcal{R}}$, defined by their position and direction vectors q, p and $\hat{q}$, $\hat{p}$, respectively, and let the "product" of these two rays be defined by the scalar expression

$$(\mathcal{R}; \hat{\mathcal{R}}) \equiv \tilde{q}\hat{p} - \tilde{\hat{q}}p. \tag{12}$$

$(\mathcal{R}; \hat{\mathcal{R}})$ is sometimes called the *Lagrange invariant* (see Ref. 1, p. 251).

It is easy to show that this quantity is independent of ζ (or z). Indeed, applying equations (11a) and (11b) to both $\mathcal{R}$ and $\hat{\mathcal{R}}$, and remembering that η is a symmetric matrix, one obtains†

$$\frac{d}{d\zeta}(\mathcal{R}; \hat{\mathcal{R}}) \equiv \frac{d}{d\zeta}(\tilde{q}\hat{p} - \tilde{\hat{q}}p) = \dot{\tilde{q}}\hat{p} + \tilde{q}\dot{\hat{p}} - \dot{\tilde{\hat{q}}}p - \tilde{\hat{q}}\dot{p} = 0. \tag{13}$$

The Lagrange invariant $(\mathcal{R}; \hat{\mathcal{R}})$ plays an important role in the present theory. Notice that n_0 does not appear explicitly in equations (8), (11a) and (11b). It can therefore be assumed, without loss of generality, that $n_0 \equiv 1$.

The properties of propagating beams are sometimes more easily understood by considering the transformation of the field between the input plane and the output plane of an optical system described by its point characteristic. Let us now choose as optical axis, for generality, an arbitrary ray $\mathcal{A}$ which need not be a straight line nor even a plane curve. Let us further define, at a distance z' from an origin 0, a rectangular coordinate system x_1', x_2', whose axes are oriented respectively along the principal normal and the binormal to $\mathcal{A}$ (see Fig. 1). At any given transverse plane, a ray is defined by its position vector q and its direction vector p. Let us assume that there is one ray, and only one ray which goes from a point r at $z = 0$ (input plane) to a point r' at $z = z'$ (output plane). This assumption implies, in particular, that the planes $z = 0$ and $z = z'$ are not conjugate. The optical length $\mathcal{V}(r, r')$ of such a ray is called the point characteristic of the optical system. As is well known, the direction vectors of a ray can be obtained from $\mathcal{V}$ by differentiation (Ref. 1, p. 97)

$$p = -\nabla\mathcal{V}(r, r'), \tag{14a}$$

$$p' = \nabla'\mathcal{V}(r, r'), \tag{14b}$$

at $r = q$, $r' = q'$. The primes always denote quantities at the output plane $z = z'$.

The law of transformation of the field can be obtained from the Huygens principle supplemented by a quasi-geometrical optics approximation.[28] The Huygens principle states that each point of an incident wavefront can be considered as the source of a secondary wave. The quasi-geometrical optics approximation consists of assuming that the field created at the output plane of the system by a point source at the input plane is adequately represented by the geometrical optics

† Recall also that, for any conformable matrices a and b, $(ab)^{\sim} = \tilde{b}\tilde{a}$ and that, for any scalar (one element matrix) c, we have $\tilde{c} \equiv c$.

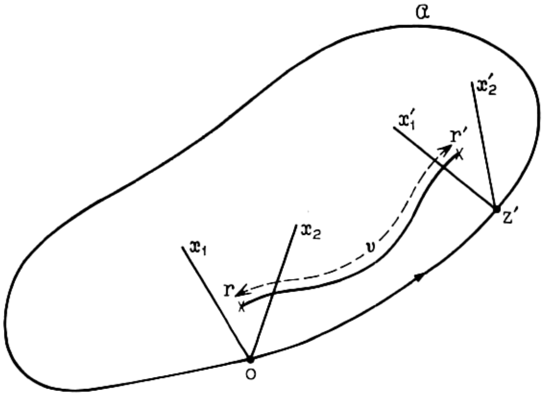

Fig. 1—Optical axis of a ring type resonator. $\mathcal{V}$ denotes the point characteristic of the system included between two transverse planes, $z = 0$ and $z = z'$.

field. These two assumptions allow us to express the field $E'(r')$ at the output plane as a function of the field $E(r)$ at the input plane. Within the paraxial approximation, we have

$$E'(r') = \pm\lambda^{-1} \iint_{-\infty}^{+\infty} E(r)K(r, r')\, d^2r, \tag{15a}$$

where

$$K(r, r') \equiv |\, \partial^2\mathcal{V}/\partial x_i\, \partial x_j' \,|^{\frac{1}{2}} \exp\left[-jk\mathcal{V}(r, r')\right]. \tag{15b}$$

The term $|\, \partial^2\mathcal{V}/\partial x_i \;\; \partial x_j' \,|^{\frac{1}{2}}$, where the bars denote a determinant, is obtained by recognizing that the power flowing through a small area at the output plane is equal to the power flowing in the corresponding cone of rays leaving the point source at the input plane, and using equation (14a).

To first order, the quantity $\mathcal{S} \equiv \mathcal{V} - z'$ is a quadratic form in x_1, x_2, x_1', x_2' which can be written, in matricial notation

$$\mathcal{S} = \tfrac{1}{2}(\tilde{r}Ur + \tilde{r}Vr' + \tilde{r}'\tilde{V}r + \tilde{r}'Wr'), \tag{16}$$

where U and W are 2×2 symmetric real matrices and V is a 2×2 real matrix. Equation (16) can be rewritten, more concisely

$$\mathcal{S} = \tfrac{1}{2}[\tilde{r}\;\tilde{r}']\begin{bmatrix} U & V \\ \tilde{V} & W \end{bmatrix}\begin{bmatrix} r \\ r' \end{bmatrix} \equiv \tfrac{1}{2}[\tilde{r}\;\tilde{r}'][\mathcal{S}]\begin{bmatrix} r \\ r' \end{bmatrix}. \tag{17}$$

Introducing equation (16) in equations (14a) and (14b), one obtains linear relations between p, p' and q, q' in the form

$$\begin{bmatrix} -p \\ p' \end{bmatrix} = \begin{bmatrix} U & V \\ \tilde{V} & W \end{bmatrix} \begin{bmatrix} q \\ q' \end{bmatrix} \equiv [\mathcal{S}] \begin{bmatrix} q \\ q' \end{bmatrix}. \tag{18}$$

It is sometimes convenient to introduce a *ray matrix* which relates q', p' to q, p. Simple relations exist between $[\mathcal{S}]$ and the ray matrix; they are given in Appendix A.

Let us now go back to the integral transformation and observe that, if $\mathcal{S}$ is a quadratic form [equation (16)], the determinant

$$| \partial^2 \mathcal{V}/\partial x_i \, \partial x'_j | = | \partial^2 \mathcal{S}/\partial x_i \, \partial x'_j | = | V | \tag{19}$$

is independent of r and r'. This term can consequently be taken out of the integral in equation (15). The integral transformation of the reduced field ψ [$\psi \equiv E \exp(jkz)$] becomes

$$\psi'(r') = \pm \lambda^{-1} | V |^{\frac{1}{2}} \iint_{-\infty}^{+\infty} \psi(r) \exp(-jk\mathcal{S}) \, d^2 r, \tag{20}$$

whose kernel is essentially

$$K_0 \equiv | V |^{\frac{1}{2}} \exp(-jk\mathcal{S}). \tag{21}$$

Let us show that, in a rectangular coordinate system, K_0 represents the Green function of the parabolic wave equation, equation (8), i.e., that

$$\mathcal{L}'K_0 \equiv \left(\nabla'^2 - 2jk \frac{\partial}{\partial z'} + k^2 \tilde{r}' \eta' r' \right) [| V |^{\frac{1}{2}} \exp(-jk\mathcal{S})] = 0. \tag{22}$$

The first term in equation (22) can be written, using equation (16)

$$\begin{aligned} \nabla'^2 [| V |^{\frac{1}{2}} \exp(-jk\mathcal{S})] \\ &= | V |^{\frac{1}{2}} (-jk\nabla'^2 \mathcal{S} - k^2 \nabla' \mathcal{S} \cdot \nabla' \mathcal{S}) \exp(-jk\mathcal{S}) \\ &= | V |^{\frac{1}{2}} \exp(-jk\mathcal{S})(-jk \, \mathrm{Spur}\, W - k^2 \nabla' \mathcal{S} \cdot \nabla' \mathcal{S}). \end{aligned} \tag{23}$$

To evaluate the second term in equation (22) one needs to know the derivative of $\mathcal{S}$ with respect to z'. We have (Ref. 1, p. 97)

$$\frac{\partial \mathcal{V}}{\partial z'} = -H(r', p') \cong 1 + (\tilde{r}' \eta' r' - \tilde{p}' p')/2, \tag{24}$$

where the paraxial approximation of H, equation (11c), has been used. Therefore, introducing the expression for p', equation (14b) in equation (24), one obtains

$$\frac{\partial \mathcal{S}}{\partial z'} = \frac{\partial \mathcal{V}}{\partial z'} - 1 = (\tilde{r}' \eta' r' - \nabla' \mathcal{S} \cdot \nabla' \mathcal{S})/2. \tag{25}$$

One also needs to know the derivative of V with respect to z'. It is obtained by introducing the quadratic form, equation (16), in both sides of equation (25)

$$\tilde{r}\frac{dU}{dz'}r + 2\tilde{r}\frac{dV}{dz'}r' + \tilde{r}'\frac{dW}{dz'}r'$$

$$= \tilde{r}'\eta' r' - (\tilde{r}V + \tilde{r}'W)(\tilde{V}r + Wr'). \qquad (26)$$

Equation (26) shows, upon identification, that

$$\frac{dV}{dz'} = -VW. \qquad (27)$$

Therefore (see Ref. 29)

$$\frac{d}{dz'}\,|\,V\,|^{\frac{1}{2}} = \tfrac{1}{2}\,|\,V\,|^{-\frac{1}{2}}\frac{d}{dz'}\,|\,V\,| = \tfrac{1}{2}\,|\,V\,|^{\frac{1}{2}}\,\mathrm{Spur}\left(V^{-1}\frac{dV}{dz'}\right)$$

$$= -\tfrac{1}{2}\,|\,V\,|^{\frac{1}{2}}\,\mathrm{Spur}\,W. \qquad (28)$$

Upon substitution of equations (23), (25) and (28), one finds that equation (22) is satisfied.

Consequently, within the first-order approximation, one may use indifferently the parabolic wave equation, equation (8), or the integral transformation, equation (20). Most of the demonstrations given in the following sections are based on both formulations.

III. FUNDAMENTAL MODE OF PROPAGATION

We know that in the high frequency limit, propagating beams closely resemble ray pencils. Let us therefore consider first the field of such ray pencils, and subsequently see how this solution can be generalized to take into account diffraction effects.

A ray pencil is, in general, astigmatic; it can be defined, in free space, as the manifold of rays which intersect two mutually perpendicular focal lines. At any point, a surface exists, called the wavefront, which is perpendicular to all of these rays. The field of ray pencils propagating in inhomogeneous media can be written in a x_1, x_2, z rectangular coordinate system

$$E(x_1\,, x_2\,, z) = Ae^{-ikS}, \qquad (29)$$

where A and S are real functions of x_1, x_2 and z. A is an amplitude factor and S is called the eikonal of the geometrical optics field. The surfaces S = constant are the equations of the wavefronts associated with the manifold of rays. Let us assume that one of the rays coincide with the z-axis and that the refractive index of the medium is unity on that axis. Within the first-order approximation, $\Phi \equiv S - z$ is a quadratic form in the transverse variables x_1 and x_2, whose coefficients are slowly varying functions of z, and A is independent of x_1, x_2. Φ can be written, in matrix notation

$$\Phi(r, z) \equiv \tfrac{1}{2}\tilde{r}\mu(z)r, \tag{30}$$

where $\mu(z)$ is a 2×2 symmetrical matrix which generally depends on z. The law of conservation of power dictates that A and μ cannot be independent; a wavefront with a positive curvature, for instance, corresponds to a contraction of the ray pencil as z increases, which necessarily results in an increased intensity. To express this relation between A and μ (transport equation), let us choose any two rays of the ray pencil such as $\mathcal{R}$ and $\hat{\mathcal{R}}$. Since $\mathcal{R}$ and $\hat{\mathcal{R}}$ are both perpendicular to the wavefront, one has, from equation (30)

$$p = \nabla\Phi(q) = \mu q, \tag{31a}$$

$$\hat{p} = \nabla\Phi(\hat{q}) = \mu\hat{q}, \tag{31b}$$

where ∇ denotes as before the gradient operator in the x_1, x_2 plane. Equations (31a and b) can be written more concisely:

$$P = \mu Q, \tag{32a}$$

where we have defined

$$Q = [q\ \hat{q}], \tag{32b}$$

$$P = [p\ \hat{p}]. \tag{32c}$$

Equations (31) and (32) show that the product of $\mathcal{R}$ and $\hat{\mathcal{R}}$, defined in equation (12), is equal to zero at any plane

$$(\mathcal{R}; \hat{\mathcal{R}}) \equiv \tilde{q}\hat{p} - \tilde{\hat{q}}p = 0. \tag{33}$$

Any ray defined by a linear combination of $\mathcal{R}$ and $\hat{\mathcal{R}}$ also belongs to the ray pencil since its product with either $\mathcal{R}$ or $\hat{\mathcal{R}}$ is equal to zero. Therefore, the one-parameter manifold of rays $\epsilon\mathcal{R}$, $\epsilon\mathcal{R} + \hat{\mathcal{R}}$, $\epsilon\hat{\mathcal{R}}$, $\mathcal{R} + \epsilon\hat{\mathcal{R}}$, with $0 < \epsilon < 1$, defines a tube of rays in the ray pencil whose cross section is a parallelogram with sides ϵq, $\epsilon q + \hat{q}$, $\epsilon\hat{q}$ and $q + \epsilon\hat{q}$ (see Fig. 2). The area of this parallelogram is given by the length of the vector product of q and $\hat{q}$

$$h = q_1\hat{q}_2 - q_2\hat{q}_1 = |\, Q \,|. \tag{34}$$

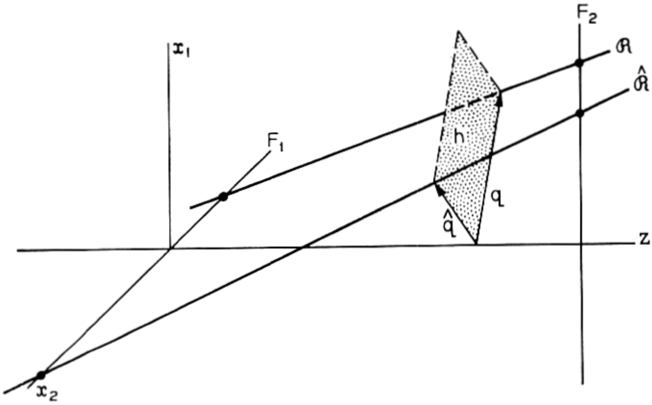

Fig. 2—An astigmatic ray pencil is defined in free space by the manifold of rays which intersect two mutually perpendicular focal lines such as F_1 and F_2. At any transverse plane the intensity of the field is inversely proportional to the square root of the area defined by q and $\hat{q}$, the position vectors of any two rays of the ray pencil ($\mathcal{R}$ and $\hat{\mathcal{R}}$).

Conservation of power requires that $A^2(z)h(z)$ be a constant. $A(z)$ can therefore be obtained from equation (34). Notice that, at a focal line, the sign of $h(z)$ changes from positive to negative. Therefore $A(z) \propto [h(z)]^{-\frac{1}{2}}$ becomes imaginary. If one insists on keeping $A(z)$ real, a $\pi/2$ phase shift must be subtracted from S at such points (anomalous phase shift).

The elements of the wavefront matrix μ can also be obtained from the components of two rays satisfying equation (33). One obtains, solving for μ equation (32a)

$$\mu_{11} = (\hat{q}_2 p_1 - q_2 \hat{p}_1)h^{-1}, \tag{35a}$$

$$\mu_{22} = (q_1 \hat{p}_2 - \hat{q}_1 p_2)h^{-1}, \tag{35b}$$

$$\mu_{12} = \mu_{21} = (q_1 \hat{p}_1 - \hat{q}_1 p_1)h^{-1},$$
$$= (\hat{q}_2 p_2 - q_2 \hat{p}_2)h^{-1}. \tag{35c}$$

The reduced field of the ray pencil is therefore

$$\psi(r, z; \mathcal{R}, \hat{\mathcal{R}}) = \pm h^{-\frac{1}{2}} \exp\left(-j\frac{k}{2}\tilde{r}\mu r\right), \tag{36}$$

where h and μ are given by equations (34) and (35a, b and c) respectively. The sign ambiguity in the expression of ψ can be resolved only by counting the number of focal lines along the ray pencil, from some reference plane.

Let us now show that the field of a ray pencil, as given by equation (36), is a solution of the parabolic wave equation, equation (8), i.e., that

$$\mathfrak{L}\psi(r, z; \mathfrak{R}, \hat{\mathfrak{R}}) \equiv \left(\nabla^2 - 2jk\frac{\partial}{\partial z} + k^2\tilde{r}\eta r\right) \cdot \left[h^{-\frac{1}{2}} \exp\left(-j\frac{k}{2}\tilde{r}\mu r\right)\right] = 0. \tag{37}$$

The first term on the right side of equation (37) is

$$\nabla^2\left[h^{-\frac{1}{2}} \exp\left(-j\frac{k}{2}\tilde{r}\mu r\right)\right] = h^{-\frac{1}{2}} \exp\left(-j\frac{k}{2}\tilde{r}\mu r\right) \times (-jk \operatorname{Spur} \mu - k^2\tilde{r}\mu^2 r). \tag{38}$$

The second term is

$$-2jk\frac{\partial}{\partial z}\left[h^{-\frac{1}{2}} \exp\left(-j\frac{k}{2}\tilde{r}\mu r\right)\right] = h^{-\frac{1}{2}} \exp\left(-j\frac{k}{2}\tilde{r}\mu r\right) \times (jk\dot{h}/h - k^2\tilde{r}\dot{\mu}r). \tag{39}$$

Using now equations (34), (32a) and (11b), one notices that

$$\dot{h}/h = \operatorname{Spur} \mu. \tag{40}$$

Differentiating both sides of equations (31a and b) with respect to z and using the paraxial ray equations [equations (11a) and (11b)] one obtains

$$(\dot{\mu} + \mu^2 - \eta)q = 0, \tag{41a}$$

$$(\dot{\mu} + \mu^2 - \eta)\hat{q} = 0. \tag{41b}$$

Since q and $\hat{q}$ are generally linearly independent, it results from equation (41) that

$$\dot{\mu} + \mu^2 = \eta. \tag{42}$$

Upon substitution of equations (38), (39), (40) and (42) in equation (37), one finds that the field of a paraxial ray pencil is, as expected, a solution of the parabolic wave equation.

It is important to remark that it has nowhere been specified that q, p, $\hat{q}$ and $\hat{p}$ are real quantities. The right side of equation (36) therefore remains a solution of the wave equation if $\mathfrak{R}$ and $\hat{\mathfrak{R}}$ are allowed to be complex valued while remaining solutions of the paraxial ray equations.

In that case, $\mu(z)$, whose elements are given by equations (35a, b and c), becomes a complex matrix and the exponential term in equation (36) describes the intensity pattern of the beam as well as its wavefront. As observed before[25] the axes of the constant intensity ellipse do not coincide, in general, with the axes of the wavefront surface. It is possible, however, to define at any plane an oblique coordinate system in which both the real part of μ, corresponding to the beam wavefront, and the imaginary part of μ, corresponding to the beam intensity, are diagonal. This coordinate transformation is given at the end of this section and used in Section IV to express in a convenient form the higher order modes of propagation.

$h(z)$, given by equation (34), and therefore the amplitude term $A(z)$, become complex quantities too. The $\pm j$ ambiguity pointed out for the case of ray pencils does not exist any more since the phase of $A(z)$ changes in a continuous manner along the z axis.

Let us now consider an optical system described by its point characteristic matrix [S] and calculate the transformation experienced by an incident gaussian beam whose reduced field has the form given in equation (36). Introducing this expression in equation (20), one obtains a reduced field at the output plane

$$\psi'(r') = \pm\lambda^{-1} \mid V \mid^{\frac{1}{2}} h^{-\frac{1}{2}}$$
$$\cdot \iint_{-\infty}^{+\infty} \exp\left\{-j\frac{k}{2}[\tilde{r}(U+\mu)r + 2\tilde{r}Vr' + \tilde{r}'Wr']\right\} d^2r. \qquad (43)$$

The integral in equation (43) is easily integrated if one notices that, for any nonsingular square matrix m and any comformable column matrices r and s one has

$$\tilde{r}mr + 2\tilde{r}s = (\tilde{r} + \tilde{s}m^{-1})m(r + m^{-1}s) - \tilde{s}m^{-1}s. \qquad (44)$$

Using equation (44) one finds that, if m is a symmetric matrix

$$\iint_{-\infty}^{+\infty} \exp\left[-(\tilde{r}mr + 2\tilde{r}s)\right] d^2r = \pi \mid m \mid^{-\frac{1}{2}} \exp(\tilde{s}m^{-1}s), \qquad (45)$$

provided the integral is defined, i.e., provided: $\tilde{r}$ (real part of m) r is a positive definite form. Substituting

$$m = j\frac{k}{2}(U + \mu) \qquad (46a)$$

and

$$s = j\frac{k}{2}Vr' \qquad (46b)$$

in equation (45), one obtains a reduced field at the output plane

$$\psi'(r') = \pm(-h \mid U + \mu \mid \mid V \mid^{-1})^{-\frac{1}{2}} \cdot \exp \left\{ -j \frac{k}{2} \tilde{r}' [W - \tilde{V}(U + \mu)^{-1} V] r' \right\}. \qquad (47)$$

This field has the same general form as the input field and describes a gaussian beam with a wavefront matrix

$$\mu' = W - \tilde{V}(U + \mu)^{-1} V, \qquad (48a)$$

or, in terms of the ray matrix (see Appendix A)

$$\mu' = (C + D\mu)(A + B\mu)^{-1}. \qquad (48b)$$

This interesting relation† generalizes the "ABCD law" which describes the transformation of the complex wavefront in two dimensions.[21,22] In some applications, it is also of interest to know the phase shift experienced by the beam through the optical system. It is given, from equation (47), to within Π, by the simple expression

$$\Theta = kz' - \tfrac{1}{2} \text{ Phase of } (\mid U + \mu \mid \mid V \mid^{-1}), \qquad (49)$$

where kz' is the geometrical optics phase shift. Equation (49) reduces to the expression given in Ref. 24 in the case of systems with rotational symmetry. One also verifies, after a few rearrangements, that the amplitude of the beam at the output plane assumes the form given in equation (34), i.e., that

$$h' \equiv -h \mid U + \mu \mid \mid V \mid^{-1} = q_1' \hat{q}_2' - q_2' \hat{q}_1' , \qquad (50)$$

where q' and $\hat{q}'$ denote the output (complex) ray position vectors. Equations (48), (49) and (50) completely define the transformation of fundamental gaussian beams propagating along the axis of nonorthogonal optical systems.

These solutions are easily generalized to the case where the axis of the incident beam is a ray $\bar{\mathcal{R}}(\bar{q}, \bar{p})$, distinct from the system axis. Let $\psi(r, z)$ denote the field of an arbitrary beam and $\bar{\mathcal{R}}$ denote an arbitrary ray; one can show[24] that

$$\psi(r, z; \bar{\mathcal{R}}) \equiv \psi(r - \bar{q}, z) \exp [-jk(\tilde{r}\bar{p} - \tfrac{1}{2}\tilde{\bar{q}}\bar{p})] \qquad (51)$$

is a solution of the parabolic wave equation, equation (8). An equivalent

† The transformation of the complex curvature of gaussian beams through nonorthogonal systems has been given before (in a very complicated form) by Y. Suematsu and H. Fukinuki.[30] Equation (48b) can alternatively be obtained without integration by writing down the laws of transformation of (real) astigmatic ray pencils, as suggested in Ref. 25.

result can alternatively be obtained from the integral transformation, equation (20), by introducing a change of variables. According to equation (51), a general form for the propagation of gaussian beams is obtained by introducing the expression for the field obtained before [equation (36)] into equation (51)

$$\psi(r, z; \mathcal{R}, \hat{\mathcal{R}}) = h^{-\frac{1}{2}} \exp\left\{-j\frac{k}{2}\left[(\tilde{r} - \tilde{\bar{q}})\mu(r - \bar{q}) + 2\tilde{r}\bar{p} - \tilde{\bar{q}}\bar{p}\right]\right\}. \quad (52)$$

Notice that $\bar{q}$ and $\bar{p}$ need not be real for the right side of equation (52) to satisfy the parabolic wave equation. It is merely required that they satisfy the paraxial ray equations. When $\bar{\mathcal{R}}$ assumes complex values, however, it cannot be interpreted any longer as a beam axis. Such solutions, with $\bar{\mathcal{R}}$ complex, are of interest to generate higher order modes of propagation, as shown in the next section.

Let us now show that the fundamental mode of propagation can be written in a form resembling the form obtained in the case of orthogonal systems. This can be done by introducing, at each transverse plane, a coordinate system in which μ is diagonal.

The reduced eikonal Φ can be written

$$\Phi \equiv \tfrac{1}{2}\tilde{r}\mu r \equiv \tfrac{1}{2}(\tilde{r}\mu^{r}r + j\tilde{r}\mu^{i}r), \quad (53)$$

where μ^r and μ^i denote the real and imaginary parts of μ, respectively. Two quadratic forms such as $\tilde{r}\mu^r r$ and $\tilde{r}\mu^i r$ can be simultaneously diagonalized if a proper (generally oblique) coordinate system is introduced.[31] The explicit expression for this transformation is not necessary here, because we are interested only in the general form of the field; it is given in Appendix C. To deal with oblique coordinates, it is convenient to introduce a tensorial notation. The expression for the scalar product of two real vectors† q and p in oblique coordinates assumes the form

$$q \cdot p = q^i p_i, \quad (54)$$

where q^1 and q^2 denote the (contravariant) components of q, obtained by drawing lines parallel to the axes from the tip of the vector q as shown in Fig. (3), and where p_1, p_2 denote the (covariant) components of p, obtained by drawing lines perpendicular to the axes. For brevity the summation sign over repeated indices is omitted.

† The following relations are also applicable to complex vectors since such vectors can be defined as linear combinations $V_r + jV_i$ of two real vectors V_r and V_i.

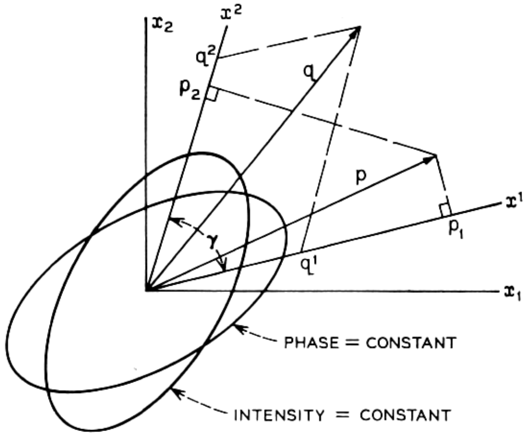

Fig. 3—This figure represents the oblique coordinate system, defined in the x_1x_2 transverse plane, which diagonalizes both the real and the imaginary parts of the wavefront (represented schematically by ellipses). The contravariant components of the position vector q, and the covariant components of the direction cosine vector p are also represented. It is assumed that the unit vectors of the coordinate system have a unit length. The index in the rectangular coordinate system is placed at a lower position only to distinguish it from the oblique coordinate system.

The reduced eikonal Φ is now written

$$\Phi \equiv \tfrac{1}{2}\mu_{ij}x^i x^j \tag{55}$$

where x^i, $i = 1, 2$ (or x^j, $j = 1, 2$) denote the contravariant components of a position vector r, and μ denotes a twice covariant tensor. With this notation, equations (31) and (32) are valid in any coordinate system. Therefore, in the coordinate system in which μ is diagonal ($\mu_{12} = 0$), we have

$$p_1 = \mu_{11}q^1, \tag{56a}$$

$$p_2 = \mu_{22}q^2, \tag{56b}$$

$$\hat{p}_1 = \mu_{11}\hat{q}^1, \tag{56c}$$

$$\hat{p}_2 = \mu_{22}\hat{q}^2. \tag{56d}$$

Let us set

$$\mu_{11} = \frac{p_1}{q^1} = \frac{\hat{p}_1}{\hat{q}^1} \equiv C_1 - 2jk^{-1}w_1^{-2}, \tag{57a}$$

$$\mu_{22} = \frac{p_2}{q^2} = \frac{\hat{p}_2}{\hat{q}^2} \equiv C_2 - 2jk^{-1}w_2^{-2}. \tag{57b}$$

In equation (57), C_i and $-2k^{-1}w_i^{-2}$ denote the real and imaginary parts of μ_{ii}, respectively. The reason for the notation $2k^{-1}w_i^{-2}$ is that w_i represent the beam radii along the coordinate axes, as shown in equation (59).

In the new coordinate system (with base vectors of unit lengths), the area of the parallelogram constructed on the vectors q and $\hat{q}$ is

$$\sin\gamma\ (q^1\hat{q}^2 - q^2\hat{q}^1) \tag{58}$$

where γ is the angle between the two coordinate axes. The field of the fundamental mode, equation (36), can consequently be rewritten

$$\begin{aligned}\psi_{00}(x^1, x^2, z; \mathcal{R}, \hat{\mathcal{R}}) = (\sin\gamma)^{-\frac{1}{2}}(q^1\hat{q}^2 - q^2\hat{q}^1)^{-\frac{1}{2}} \\ \cdot\exp\{-[(x^1/w_1)^2 + (x^2/w_2)^2]\} \\ \cdot\exp\left\{-j\frac{k}{2}[C_1(x^1)^2 + C_2(x^2)^2]\right\}.\end{aligned} \tag{59}$$

The first exponential term in equation (59) describes the beam intensity pattern and the second one describes the wavefront of the beam.

In the special case where the lens-like medium is orthogonal, one may choose $\mathcal{R}$ and $\hat{\mathcal{R}}$ in two mutually orthogonal planes. Assuming that these planes coincide with the x_1z and x_2z planes, respectively, we have

$$q_2 = p_2 = \hat{q}_1 = \hat{p}_1 = 0, \tag{60}$$

and equation (59) reduces to the known form (see, for example, Ref. 24)

$$\begin{aligned}\psi_{00}(x_1, x_2, z; \mathcal{R}, \hat{\mathcal{R}}) = q_1^{-\frac{1}{2}} \exp\left[-(x_1/w_1)^2 - j\frac{k}{2}C_1x_1^2\right] \\ \cdot\hat{q}_2^{-\frac{1}{2}} \exp\left[-(x_2/w_2)^2 - j\frac{k}{2}C_2x_2^2\right].\end{aligned} \tag{61}$$

IV. HIGHER ORDER MODES OF PROPAGATION

Two procedures are given in this section to obtain the higher order modes of propagation. One is based on the power series expansion of the field of off-set gaussian beams and the other is based on the application of differential operators on the fundamental mode. These two methods can be shown to be equivalent. They lead however to two different representations of the field, one in terms of Hermite polynomials in two complex variables and the other in terms of finite series of ordinary Hermite polynomials. Both representations are of interest.

It has been shown in the previous section that the field of a gaussian beam propagating along the axis of an optical system is fully described by two complex rays, denoted $\mathcal{R}$ and $\hat{\mathcal{R}}$, satisfying equation (33). This solution of the wave equation can be generalized to include the case where the beam axis is a ray $\bar{\mathcal{R}}$. It was pointed out also that $\bar{\mathcal{R}}$ may assume complex values provided its position and direction vectors $(\bar{q}, \bar{p})$ remain solutions of the ray equations. Let us define $\bar{\mathcal{R}}$ as a linear combination of the rays $\mathcal{R}^*$ and $\hat{\mathcal{R}}^*$, conjugate of $\mathcal{R}$ and $\hat{\mathcal{R}}$, respectively. We have

$$\bar{q} = \alpha_1 q^* + \alpha_2 \hat{q}^* \equiv Q^* \alpha, \tag{62}$$

$$\bar{p} = \alpha_1 p^* + \alpha_2 \hat{p}^* \equiv P^* \alpha, \tag{63}$$

where α_1 and α_2 are two arbitrary parameters. Introducing these expressions in equation (52) one obtains

$$\psi(r, z; \mathcal{R}, \hat{\mathcal{R}}; \alpha_1, \alpha_2) = \psi_{00}(r, z; \mathcal{R}, \hat{\mathcal{R}}) \times \exp(\tilde{\alpha} y - \tfrac{1}{2}\tilde{\alpha}\nu\alpha), \tag{64}$$

where $\psi_{00}(r, z; \mathcal{R}, \hat{\mathcal{R}})$ denotes the fundamental mode field and where we have defined

$$\alpha \equiv \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \tag{65a}$$

$$\nu \equiv \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{12} & \nu_{22} \end{bmatrix} \equiv -2k\tilde{Q}^* \mu^i Q^*, \tag{65b}$$

$$y \equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \equiv -2k\tilde{Q}^* \mu^i r. \tag{65c}$$

Notice that ν is a symmetric matrix as a result of equation (33).

One now observes that the exponential term in equation (64) is the generating function for Hermite polynomials in two variables[32]

$$\exp(\tilde{\alpha} y - \tfrac{1}{2}\tilde{\alpha}\nu\alpha) = \sum_{m,n=0}^{\infty} \frac{\alpha_1^m}{m!}\frac{\alpha_2^n}{n!} H_{mn}(\nu^{-1} y; \nu), \tag{66}$$

where the polynomials H_{mn} have the form

$$H_{mn}(y_1, y_2; \nu) \equiv y_1^m y_2^n - \left[\frac{1}{2}\frac{m(m-1)}{1}\nu_{11} y_1^{m-2} y_2^n + \frac{mn}{1}\nu_{12} y_1^{m-1} y_2^{n-1} + \frac{1}{2}\frac{n(n-1)}{1}\nu_{22} y_1^m y_2^{n-2}\right] + \cdots \tag{67}$$

and, for $m + n \leqq 3$

$$\begin{aligned}
H_{00} &= 1, \\
H_{10} &= y_1 , \\
H_{01} &= y_2 , \\
H_{20} &= y_1^2 - \nu_{11} , \\
H_{11} &= y_1 y_2 - \nu_{12} , \\
H_{02} &= y_2^2 - \nu_{22} , \\
H_{30} &= y_1^3 - 3\nu_{11} y_1 , \\
H_{21} &= y_1^2 y_2 - 2\nu_{12} y_1 - \nu_{11} y_2 , \\
H_{12} &= y_1 y_2^2 - 2\nu_{12} y_2 - \nu_{22} y_1 , \\
H_{03} &= y_2^3 - 3\nu_{22} y_2 .
\end{aligned} \tag{68}$$

Each coefficient in the expansion of $\psi(r, z; \mathcal{R}, \hat{\mathcal{R}}; \alpha_1, \alpha_2)$ in power series of α_1, α_2 is necessarily a solution of the wave equation since α_1 and α_2 are arbitrary numbers. New solutions of the wave equation are therefore obtained in the form

$$\psi_{mn}(r, z; \mathcal{R}, \hat{\mathcal{R}}) = \psi_{00}(r, z; \mathcal{R}, \hat{\mathcal{R}}) H_{mn}(Q^{*-1} r; \nu). \tag{69}$$

It is demonstrated in Appendix D that this set of solutions forms an orthogonal system, provided the condition $(\mathcal{R}; \hat{\mathcal{R}}^*) = 0$ is satisfied [in addition to equation (33)]. The fact that y_1 and y_2 are complex does not raise any particular difficulty in calculating $H_{mn}(y_1, y_2; \nu)$ from equations (67) and (68). This prevents us, however, from identifying y_1 and y_2 with real coordinates. It is important to notice that multiplication of $\mathcal{R}$ by a factor λ (i.e., $q \to \lambda q$, $p \to \lambda p$) and $\hat{\mathcal{R}}$ by a factor $\hat{\lambda}$ ($\hat{q} \to \hat{\lambda}\hat{q}$, $\hat{p} \to \hat{\lambda}\hat{p}$) leaves essentially unchanged the field given in equation (69); it is merely multiplied by a constant. This property results from equation (65) and the general form of H_{mn} given in equation (67). Consequently $\mathcal{R}$ and $\hat{\mathcal{R}}$ need to be defined only to within constant factors.

In the special case where the optical system is orthogonal, one may choose $\mathcal{R}$ and $\hat{\mathcal{R}}$ in two mutually perpendicular meridional planes coincident with the $x_1 z$ and $x_2 z$ planes respectively ($q_2 = p_2 = 0$, $\hat{q}_1 = \hat{p}_1 = 0$). The matrix ν becomes diagonal and the Hermite polynomials in two variables reduce to a product of two Hermite polynomials in one variable. To within a constant one has, in that special case[32]

$$H_{mn}(y_1, y_2; \nu) = \nu_{11}^{m/2}\nu_{22}^{n/2}H_m(2^{-\frac{1}{2}}y_1\nu_{11}^{-\frac{1}{2}})H_n(2^{-\frac{1}{2}}y_2\nu_{22}^{-\frac{1}{2}}) \tag{70}$$

where $H_k(x)$ denotes a Hermite polynomial in one variable of order k (as defined in Ref. 33). Using equation (57), the right side of equation (70) can be written, to within a constant

$$(q_1^*/q_1)^{m/2}(\hat{q}_2^*/\hat{q}_2)^{n/2}H_m(2^{\frac{1}{2}}x_1/w_1)H_n(2^{\frac{1}{2}}x_2/w_2), \tag{71}$$

in agreement with previous results.[11]

The procedure just described for obtaining new solutions of the wave equation can be applied to an arbitrary field $\psi(r, z)$. The coefficients of the power series expansion are obtained in that case by repeated differentiation. If one calculates the coefficients for the few first orders, one finds that they assume the form

$$\psi_{mn}(r, z; \mathcal{R}, \hat{\mathcal{R}}) = \Lambda^m(\mathcal{R}^*)\Lambda^n(\hat{\mathcal{R}}^*)\psi(r, z), \tag{72}$$

where $\Lambda(\mathcal{R})$ and $\Lambda(\hat{\mathcal{R}})$ are differential operators defined by

$$\Lambda(\mathcal{R}) \equiv \tilde{p}r - jk^{-1}\tilde{q}\nabla, \tag{73a}$$

$$\Lambda(\hat{\mathcal{R}}) \equiv \tilde{\hat{p}}r - jk^{-1}\tilde{\hat{q}}\nabla. \tag{73b}$$

It is not difficult to show, using equation (33), that these two operators commute with one another. For generality, let us demonstrate equation (72) on the basis of the integral transformation, equation (20).†

Let $\psi(r)$ and $\psi'(r')$ denote fields at the input and output planes, respectively, of an optical system described by its reduced point characteristic $\mathcal{S}$. Let us prove that a field $\Lambda(\mathcal{R})\psi(r)$ is transformed into $\Lambda'(\mathcal{R}')\psi'(r')$ at the output plane, i.e., that

$$(\tilde{p}'r' - jk^{-1}\tilde{q}'\nabla')\left\{\iint_{-\infty}^{+\infty} \psi(r) \exp(-jk\mathcal{S})\, d^2r\right\}$$
$$= \iint_{-\infty}^{+\infty} [(\tilde{p}r - jk^{-1}\tilde{q}\nabla)\psi(r)] \exp(-jk\mathcal{S})\, d^2r. \tag{74}$$

Notice that the constant term $\pm\lambda^{-1} | V |^{\frac{1}{2}}$ in equation (20) can be dropped. The primes in equation (74) refer as before to quantities taken at the output plane.

Using equation (16), one finds that

$$\nabla' \exp(-jk\mathcal{S}) = -jk(\tilde{V}r + Wr') \exp(-jk\mathcal{S}). \tag{75}$$

† Alternatively one can show that the operator $\Lambda(\mathcal{R})$ [or $\Lambda(\hat{\mathcal{R}})$] commutes with the wave equation operator, equation (8). This result has been obtained before by Popov[26] for a special form of $\Lambda(\mathcal{R})$.

Therefore the left side of equation (74) can be written

$$\iint_{-\infty}^{+\infty} [\tilde{p}'r' - \tilde{q}'(\tilde{V}r + Wr')]\psi(r) \exp(-jk\mathcal{S})\, d^2r. \tag{76}$$

To evaluate the right side of equation (74), notice that, for any function $F(x_1, x_2)$ which tends exponentially to zero as $x_1, x_2 \to \pm\infty$, one has

$$\iint_{-\infty}^{+\infty} \nabla F(x_1, x_2)\, dx_1\, dx_2 = 0. \tag{77}$$

Therefore, setting

$$F(x_1, x_2) \equiv \psi(r) \times \exp(-jk\mathcal{S}) \tag{78}$$

in equation (77), one obtains

$$\iint_{-\infty}^{+\infty} \nabla\psi(r) \times \exp(-jk\mathcal{S})\, d^2r$$

$$= -\iint_{-\infty}^{+\infty} (-jk\nabla\mathcal{S})\psi(r) \times \exp(-jk\mathcal{S})\, d^2r. \tag{79}$$

Using again equation (16) to evaluate $\nabla\mathcal{S}$, the right side in equation (74) becomes

$$\iint_{-\infty}^{+\infty} [\tilde{p}r + \tilde{q}(Ur + Vr')]\psi(r) \exp(-jk\mathcal{S})\, d^2r. \tag{80}$$

The identity of the two terms in brackets in equations (76) and (80) results from the ray equations, equation (18).

The property established for $\Lambda(\mathcal{R})$ clearly holds true also for the operator $\Lambda^m(\mathcal{R})$ corresponding to m applications of $\Lambda(\mathcal{R})$, and for the operator $\Lambda^n(\hat{\mathcal{R}})$ associated with another ray $\hat{\mathcal{R}}$.

When applied to a gaussian beam (defined by $\mathcal{R}$, $\hat{\mathcal{R}}$), the operators $\Lambda(\mathcal{R})$ and $\Lambda(\hat{\mathcal{R}})$ give a result identically equal to zero. Higher order modes are obtained, however, if one considers the operators associated with the *conjugate* rays $\mathcal{R}^*$, $\hat{\mathcal{R}}^*$. One therefore calculates

$$\psi_{mn}(r, z; \mathcal{R}, \hat{\mathcal{R}}) = \Lambda^m(\mathcal{R}^*)\Lambda^n(\hat{\mathcal{R}}^*)\psi_{00}(r, z; \mathcal{R}, \hat{\mathcal{R}}). \tag{81}$$

To give a convenient form to the right side of equation (81), let us write down explicitely in tensorial notation (see Section III) the operator $\Lambda(\mathcal{R}^*)$, defined by equation (73a)

$$\Lambda(\mathcal{R}^*) = p_i^* x^i - jk^{-1} q^{i*} \nabla_i$$

$$\equiv \left(p_1^* x^1 - jk^{-1} q^{1*} \frac{\partial}{\partial x^1}\right) + \left(p_2^* x^2 - jk^{-1} q^{2*} \frac{\partial}{\partial x^2}\right). \tag{82}$$

Using equations (82) and (59) and the relation[33]

$$\frac{d}{dx} H_k(x) = 2xH_k(x) - H_{k+1}(x), \tag{83}$$

where $H_k(x)$ denotes a Hermite polynomial of order k, one finds that

$$\begin{aligned}\Lambda(\mathcal{R}^*)\{H_m(2^{\frac{1}{2}}x^1/w_1)H_n(2^{\frac{1}{2}}x^2/w_2)\psi_{00}(x^1, x^2, z; \mathcal{R}, \hat{\mathcal{R}})\} \\ = \psi_{00}(x^1, x^2, z; \mathcal{R}, \hat{\mathcal{R}})[q^{1*}/w_1 H_{m+1}(2^{\frac{1}{2}}x^1/w_1)H_n(2^{\frac{1}{2}}x^2/w_2) \\ + q^{2*}/w_2 H_m(2^{\frac{1}{2}}x^1/w_1)H_{n+1}(2^{\frac{1}{2}}x^2/w_2)].\end{aligned} \tag{84}$$

A similar relation holds for $\Lambda(\hat{\mathcal{R}}^*)$. These two relations show, by recurrence, that the field of the mode m, n can be written

$$\begin{aligned}\psi_{mn}(x^1, x^2, z; \mathcal{R}, \hat{\mathcal{R}}) &= \Lambda^m(\mathcal{R}^*)\Lambda^n(\hat{\mathcal{R}}^*)\psi_{00}(x^1, x^2, z; \mathcal{R}, \hat{\mathcal{R}}) \\ &= \psi_{00}(x^1, x^2, z; \mathcal{R}, \hat{\mathcal{R}}) \times [q^{1*}/w_1 H(2^{\frac{1}{2}}x^1/w_1) \\ &\quad + q^{2*}/w_2 H(2^{\frac{1}{2}}x^2/w_2)]^m \times [\hat{q}^{1*}/w_1 H(2^{\frac{1}{2}}x^1/w_1) \\ &\quad + \hat{q}^{2*}/w_2 H(2^{\frac{1}{2}}x^2/w_2)]^n,\end{aligned} \tag{85}$$

where the convention is made that, *after* multiplication of the two binomials, $H^k(x)$ actually represents a Hermite polynomial of order k: $H_k(x)$. This form of the field shows that the higher order modes of propagation can be obtained by multiplying the fundamental solution by a finite series of Hermite polynomials in one real variable.† Since q^2/q^1 and $\hat{q}^2/\hat{q}^1$ are generally complex, the wavefronts are different for each mode. It is shown in the next section that q^2/q^1 and $\hat{q}^2/\hat{q}^1$ happen to be real, however, at the end mirrors of linear resonators. From this observation, it results that the wavefronts of all of the resonating modes generally coincide with the end mirror surfaces.

Another special case of interest is the case where q^2/q^1 and $\hat{q}^2/\hat{q}^1$ are both equal to j. This happens in the case of systems with rotational symmetry, such as the "cavities with image rotation" which are investigated in Section VI.

V. NONORTHOGONAL RESONATORS

We are concerned in this section with the resonant fields and the resonant frequencies of nonorthogonal resonators. Ring-type resonators

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† In the case where $m = 0$ (or $n = 0$) it is not difficult to show that the two expressions given for the mode mn in equations (69) and (85), respectively, coincide. This result can be obtained by writing equation (69) in the coordinate system in which μ (but not necessarily ν) is diagonal and using an expansion formula [equation (22), p. 371 of Ref. 32] for H_{mn} and a condensation formula [equation (31), p. 345 of Ref. 32] for the right side of equation (85). In the general case a direct comparison of the two expressions appears to be difficult.

being conceptually simpler than linear resonators, their properties are considered first. A ring-type resonator is essentially a section of waveguide closed on itself. An optical beam is a mode of the resonator, if, after a round trip, its field reproduces itself exactly.

The general form of the solutions obtained in the previous sections (Sections III and IV) is preserved as the beams propagate through an optical system. In general, however, the field distribution at the output plane of a section of waveguide does not coincide with the field distribution at the input plane [see, for instance, the transformation law, equation (48b) for the fundamental mode]. By a proper choice of the mode parameters it is possible, however, to achieve coincidence between the fields at the two planes (except, perhaps, for a constant phase factor). In that case, the beam is said to be *matched* to the section of waveguide considered. Clearly, such a beam would also be matched to a sequence of identical sections, forming a periodic waveguide. For the fundamental mode, the matching condition can be obtained by specifying that $\mu' = \mu$ in equation (48b) and solving for μ. However it is more convenient to look first for rays which reproduce themselves after a round trip in the system (except for a constant factor) and calculate the wavefront matrix μ associated with these rays. Such rays are called eigenrays; they are always complex in the case of stable resonators.

To obtain the eigenrays, let us replace q' and p' by λq and λp, respectively, in equation (18). One obtains the relations

$$-p = (U + \lambda V)q, \tag{86a}$$

$$p = (\lambda^{-1}\tilde{V} + W)q, \tag{86b}$$

and, by addition and subtraction

$$0 = (U + W + \lambda V + \lambda^{-1}\tilde{V})q, \tag{87a}$$

$$p = \tfrac{1}{2}(W - U + \lambda^{-1}\tilde{V} - \lambda V)q. \tag{87b}$$

Equation (87a) actually represents a system of two homogeneous linear equations which admit a solution only if

$$| U + W + \lambda V + \lambda^{-1}\tilde{V} | = 0, \tag{88}$$

where the bars denote a determinant. Equation (88) can be rewritten as a second-degree equation in $(\lambda + \lambda^{-1})$ as shown previously[27] for a special case. One obtains

$$| V | (\lambda + \lambda^{-1})^2 + [V_{11}K_{22} + K_{11}V_{22} - K_{12}(V_{12} + V_{21})](\lambda + \lambda^{-1}) + | K | - (V_{12} - V_{21})^2 = 0, \tag{89}$$

where we have defined

$$K \equiv U + W. \tag{90}$$

The resonator is stable when the solutions of equation (89) for

$$(\lambda + \lambda^{-1})/2 \equiv \cos\theta \tag{91}$$

are real and are in the range -1 to $+1$; this is assumed henceforth. In that case, two real characteristic angles, denoted θ and $\hat{\theta}$, are obtained, the two other characteristic angles being clearly $-\theta$ and $-\hat{\theta}$.

If one introduces one of the four eigenvalues $\lambda = \exp(j\theta)$, $\hat{\lambda} = \exp(j\hat{\theta})$, $\lambda^* = \exp(-j\theta)$ or $\hat{\lambda}^* = \exp(-j\hat{\theta})$ in equations (87a) and (87b), one obtains (to within arbitrary constants) the components of the four eigenrays denoted respectively $\mathcal{R}$, $\hat{\mathcal{R}}$, $\mathcal{R}^*$ and $\hat{\mathcal{R}}^*$. Let us show that the product of $\mathcal{R}$ and $\hat{\mathcal{R}}$ [defined in equation (12)] is equal to zero.

Since $(\mathcal{R}; \hat{\mathcal{R}})$ is invariant, one may choose a reference plane along the path where the matrix V is symmetric. At such a plane, equations (87a) and (87b) assume the form

$$0 = [U + W + (\lambda + \lambda^{-1})V]q, \tag{92a}$$

$$p = \tfrac{1}{2}[W - U + (\lambda^{-1} - \lambda)V]q. \tag{92b}$$

Since both $U + W$ and V are symmetric, one has[31]

$$\tilde{q}V\hat{q} = \tilde{\hat{q}}Vq = 0, \tag{93}$$

provided the absolute values of θ and $\hat{\theta}$ are distinct. Therefore

$$\begin{aligned}(\mathcal{R}; \hat{\mathcal{R}}) &\equiv \tilde{q}\hat{p} - \tilde{\hat{q}}p \\ &= \tfrac{1}{2}\tilde{q}[W - U + (\hat{\lambda}^{-1} - \hat{\lambda})V]\hat{q} - \tfrac{1}{2}\tilde{\hat{q}}[W - U + (\lambda^{-1} - \lambda)V]q \\ &= \tfrac{1}{2}(\hat{\lambda}^{-1} - \hat{\lambda})\tilde{q}V\hat{q} - \tfrac{1}{2}(\lambda^{-1} - \lambda)\tilde{\hat{q}}Vq = 0. \end{aligned} \tag{94a}$$

One also has, replacing $\hat{\lambda}$ by $\hat{\lambda}^{-1}$ and/or λ by λ^{-1}

$$(\mathcal{R}^*; \hat{\mathcal{R}}^*) = 0; \qquad (\mathcal{R}; \hat{\mathcal{R}}^*) = 0; \qquad (\mathcal{R}^*; \hat{\mathcal{R}}) = 0. \tag{94b}$$

Therefore, according to the results of Section III, each pair of eigenrays in equation (94) defines a gaussian beam. The choice between the four pairs of eigenrays can be made by giving either a positive or a negative sign to θ and $\hat{\theta}$. It is made in such a way that the imaginary part of μ is a negative definite form. This ensures that the power carried by the beam in finite. After traversing a period of the optical system, the position q and the direction p of $\mathcal{R}$ become $q \exp(j\theta)$ and $p \exp(j\theta)$ respectively. Similarly, $\hat{q}$ and $\hat{p}$ become $\hat{q} \exp(j\hat{\theta})$ and $\hat{p} \exp(j\hat{\theta})$. Equations (35a through c) and (34) show that μ assumes its original value after a period (round trip) in the optical system; h, however, is multiplied by exp $[j(\theta + \hat{\theta})]$. The field of the fundamental gaussian beam defined by $\mathcal{R}$ and $\hat{\mathcal{R}}$ consequently reproduces itself after a period except for an additional phase shift equal to $kL - (\theta + \hat{\theta})/2$, where L denotes the period (round trip) path length.

To clarify the above discussion let us observe that the modal matrix

$$\begin{bmatrix} jkQ^* & Q \\ jkP^* & P \end{bmatrix}, \tag{94c}$$

where Q and P were defined before in equations (32b and c), *is itself a ray matrix*, i.e., satisfies equation (112). As shown in Appendix D, the imaginary part of μ can be written $-(k^{-1}/2)(Q\tilde{Q}^*)^{-1}$; this is clearly a negative definite form, as required. It can also be shown, using equations (111), (16) and (21), that the mode generating function $\psi(r; \alpha)$ given in equation (64) is precisely the output field created by a point source located at the input plane of a (lossy) optical system whose ray matrix is the modal matrix, equation (94c).

Considering now the form, equation (85), obtained for the higher order modes, it appears that the operators Λ^m and Λ^n are responsible for an increase of the phase of ψ_{mn} equal to $-m\theta - n\hat{\theta}$. Therefore, the general expression for the resonant frequencies is

$$k_{\ell mn}L = (m + \tfrac{1}{2})\theta + (n + \tfrac{1}{2})\hat{\theta} + 2\ell\pi, \tag{95}$$

where ℓ is an integer defining the number of wavelengths along the system axis. This result was obtained by Popov[9,26] for the special case of linear nonorthogonal resonators incorporating homogeneous internal media. It is shown here to be applicable to the general case.

Let us now investigate the case of linear cavities (cavities with folded optical axis). It is convenient to replace the two curved end mirrors of such resonators by plane mirrors and thin lenses, and take the reference plane at one of the end mirrors. In a round trip along the folded optical axis two optical systems are encountered which are mirror images of one another. It is shown in Appendix B that the point characteristic matrix assumes in that case the simple form

$$[\mathcal{S}] = \begin{bmatrix} U & V \\ V & U \end{bmatrix}, \tag{96}$$

where both U and V are real and symmetric. The characteristic equations (92a) and (92b) become simply

$$(U + \cos\theta V)q = 0, \tag{97}$$

$$p = -j \sin \theta V q, \tag{98}$$

where θ is the characteristic angle.

Let q, p and $\hat{q}$, $\hat{p}$ denote two solutions of equations (97) and (98) (eigenrays). Equation (97) shows that the ratio of the two components of q and $\hat{q}$, q_2/q_1 and $\hat{q}_2/\hat{q}_1$, respectively, are real (in any coordinate system) since, for a stable resonator, the solutions θ and $\hat{\theta}$ of the characteristic equation

$$| U + \cos \theta V | = 0 \tag{99}$$

are real. One also observes that the wavefront matrix μ is imaginary. This result shows that, at the end mirrors of linear resonators, the wavefront of all of the modes coincide with the mirror surfaces, except perhaps in some cases of degeneracy. Since U and V are symmetrical, one further notices that[31]

$$\tilde{q} V \hat{q} = 0, \tag{100}$$

provided the absolute values of θ and $\hat{\theta}$ are distinct. Therefore, from equation (98),

$$\tilde{q} \hat{p} = \tilde{\hat{p}} q^* = 0. \tag{101}$$

This relation is useful in checking numerical calculations.

VI. CAVITIES WITH IMAGE ROTATION

As an example of application of the general theory discussed in the previous sections, let us calculate the resonant frequencies and the resonant field of a new type of optical cavity that one may call "cavity with image rotation."

Consider a nonplanar closed path (see Fig. 4) and let Ω be the rotation experienced after a round trip by rays parallel to the optical axis. (The value of Ω for a given orientation of the mirrors can be found in Ref. 4.) The case where the optical system has a rotational symmetry is of particular interest. Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be the 2×2 ray matrix of the optical system with rotational symmetry introduced along the path. The round trip point characteristic matrix of the resonator is, in rectangular coordinates

$$[\mathrm{S}] = b^{-1} \left[\begin{array}{cc|cc} a & 0 & -\cos \Omega & -\sin \Omega \\ 0 & a & \sin \Omega & -\cos \Omega \\ \hline -\cos \Omega & \sin \Omega & d & 0 \\ -\sin \Omega & -\cos \Omega & 0 & d \end{array}\right]. \tag{102}$$

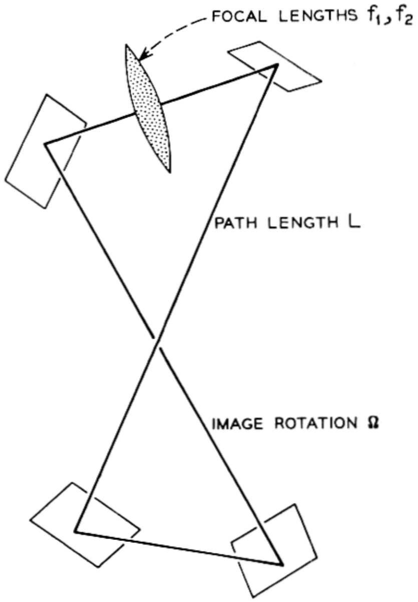

Fig. 4—A cavity with image rotation is represented. It incorporates a lens and four plane mirrors which define a nonplanar path. As a result of the twist of the path, this resonator is nonorthogonal. When the lens is astigmatic, the resonating modes do not exhibit the same patterns as in the case of more conventional cavities.

Equations (89) and (102) show that the characteristic angles are simply

$$\theta = \theta_0 + \Omega, \tag{103a}$$

$$\hat{\theta} = \theta_0 - \Omega, \tag{103b}$$

where we have defined

$$\cos \theta_0 \equiv (a + d)/2. \tag{104}$$

The resonant frequencies are therefore given, from the general relation equation (95), by

$$k_{\ell mn}L = (m + n + 1)\theta_0 + (m - n \pm 1)\Omega + 2\ell\pi. \tag{105}$$

The additional term $\pm\Omega$ in equation (105) is to be introduced when polarization effects are taken into account. It has been assumed that the mirrors are perfect conductors, even in number, and that the medium is isotropic. In that case the polarization vector experiences the same transformation as an image,[4] i.e. a rotation Ω. The + and − signs in equation (105) refer to the clockwise and counterclockwise polarization states, whose degeneracy is therefore lifted.

The eigenvectors q, p and $\hat{q}$, $\hat{p}$ have respectively clockwise and counterclockwise circular polarizations too, as one expects from the rotational symmetry of the system; they are independent of the image rotation Ω. The components of $\mathcal{R}(q,\ p)$ and $\hat{\mathcal{R}}(\hat{q},\ \hat{p})$ are respectively, to within arbitrary constants

$$\mathcal{R}\begin{cases} q(jb,\ b) \\ p(-\sin\,\theta_0\ ,\ j\sin\,\theta_0) \end{cases} \qquad \hat{\mathcal{R}}\begin{cases} \hat{q}(jb,\ -b) \\ \hat{p}(-\sin\,\theta_0\ ,\ -j\sin\,\theta_0). \end{cases} \tag{106}$$

Setting for brevity, $2^{\frac{1}{2}}x^1/w = x_1$ and $2^{\frac{1}{2}}x^2/w = x_2$, where w is the beam radius, the mode ψ_{mn} assumes in rectangular coordinates, from equation (85), the form

$$\psi_{mn} = [H(x_1) + jH(x_2)]^m[H(x_1) - jH(x_2)]^n\psi_{00}\ . \tag{107a}$$

This expression, being independent of Ω, should coincide with known forms (see Ref. 15 or 11) which can be written

$$(-1)^n n!\, 2^{m+n} Z^{m-n} L_n^{m-n}(ZZ^*)\psi_{00} \quad \text{if} \quad m \geqq n, \tag{107b}$$

$$(-1)^m m!\, 2^{m+n} Z^{*n-m} L_m^{n-m}(ZZ^*)\psi_{00} \quad \text{if} \quad m \leqq n, \tag{107c}$$

where L_p^l denotes a generalized Laguerre polynomial, and

$$Z \equiv x_1 + jx_2\ . \tag{108}$$

A relation between Hermite polynomials and generalized Laguerre polynomials was given before, in a different form, by J. R. Pierce and S. P. Morgan (private communication). The identity of the right side of equation (107a) and equations (107b) and (107c) is easily demonstrated for the special cases n (or m) $= 0$, and $m = n$, using well-known formulas,* and verified for the first values of m, n. The field consequently assumes the same form as in ordinary cavities. A rotation Ω about the z axis of the beam pattern can be expressed by a multiplication of Z by $\exp\ (j\Omega)$ and consequently, from equation (107), by a *phase shift* $(m - n)\Omega$, in agreement with equation (105). The distinctive feature of cavities with image rotation compared with ordinary cavities, in addition to the polarization properties mentioned before, is that the intensity pattern of the resonant field has necessarily a circular symmetry.

When the optical system introduced along the nonplanar path is

* See Ref. 33. Notice that a factor 2^{2n} is missing in equation (32), p. 195, of this book.

astigmatic, one must use the general expressions given in the previous sections. This case has been studied numerically, using equation (85), for the case of a resonator incorporating a single spherical mirror of radius $R = 6m$, operating at an incidence angle of 30° and an odd number of plane mirrors. The spherical mirror is equivalent to an astigmatic lens of focal lengths $f_1 = 2.6m$ and $f_2 = 3.47m$. Assuming a round trip path length $L = 1m$ and an image rotation $\Omega = 20°$, one obtains for the point characteristic matrices, from equation (115), with $d = 1, \nu = 0$

$$U = \begin{bmatrix} 0.615 & 0 \\ 0 & 0.712 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.94 & -0.34 \\ 0.34 & -0.94 \end{bmatrix},$$

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The characteristic angles are

$$\theta = -13°3,$$

$$\hat{\theta} = -54°,$$

from which the resonant frequencies can be obtained. The components of the eigenrays $\mathcal{R}$ and $\hat{\mathcal{R}}$ are respectively, in a rectangular coordinate system

$$\mathcal{R}\begin{cases} q(1, -j1.35) \\ p(0.19 - j0.66, -0.62 - j0.19), \end{cases}$$

$$\hat{\mathcal{R}}\begin{cases} \hat{q}(1, j0.91) \\ \hat{p}(0.19 - j0.57, 0.49 + j0.13). \end{cases}$$

These two eigenrays fulfill, as expected, the condition $\tilde{q}\hat{p} = \tilde{\hat{q}}p$; they define a wavefront matrix

$$\mu = \begin{bmatrix} 0.096 - j0.305 & 0.0206 \\ 0.0206 & 0.072 - j0.246 \end{bmatrix}$$

whose imaginary part is a negative definite form, as required. The intensity pattern for the mode ψ_{20} is shown in Fig. 5. It is intermediate between the circularly symmetric patterns observed when

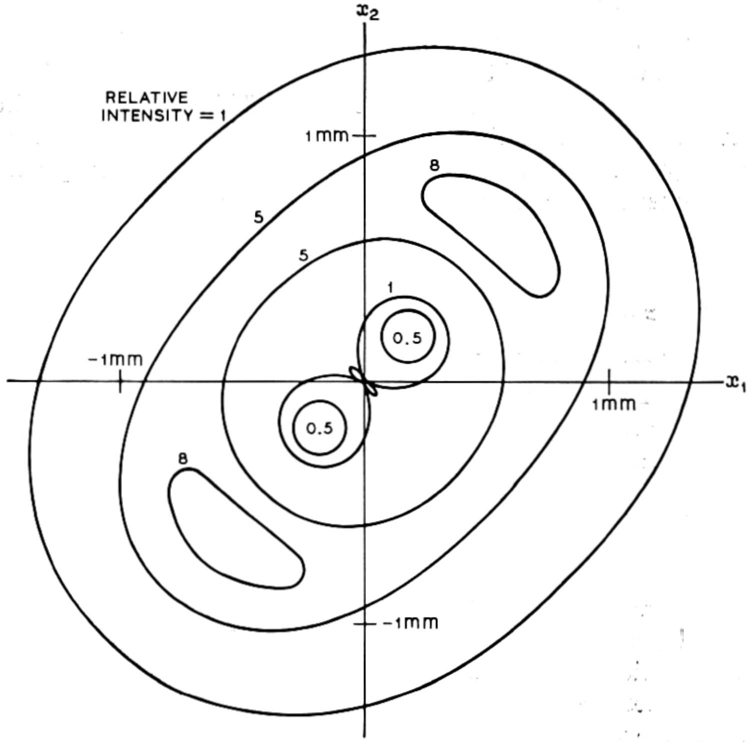

Fig. 5—This figure represents the constant intensity curves of the TEM_{20} mode in a nonorthogonal cavity incorporating a 6m radius mirror with an incidence angle of 30° and an image rotation of 20°. The optical axis path length is 1m, and the wavelength is 1μm.

$f_1 = f_2$, and the usual orthogonal patterns observed for $\Omega = 0$ (see, for instance, the TEM_{20} mode in Fig. 7 of Ref. 11).

VII. CONCLUSION

It has been shown that, within the first order approximation, the solutions of the scalar wave equation can be expressed in terms of the solutions of the (simpler) ray equations. The fundamental mode of propagation in nonorthogonal media was obtained by generalizing the expression for the field of astigmatic ray-pencils. An oblique coordinate system has been introduced which reduces this solution to the form assumed by ordinary gaussian beams. The higher order modes of propagation were also obtained; they can be expressed as the product of the fundamental solution and Hermite polynomials in one real variable.

The results of Popov[9,26] for the resonant frequencies of nonorthogonal resonators were extended to resonators incorporating arbitrary lens-like media and were applied to a new type of cavity which exhibits interesting resonance and polarization properties. This theory may also be useful for special optical waveguides such as the helical gas lenses, and for analysis of optical systems which are nominally orthogonal, but which suffer from small distortions in three dimensions.

APPENDIX A

Relations Between the Point Characteristic Matrix and the Ray Matrix

It has been shown in the main text [equation (18)] that the direction vectors p, p' of a ray at the input and output plane of an optical system are related to the position vectors q, q' by the following matricial relation

$$\begin{bmatrix} -p \\ p' \end{bmatrix} = \begin{bmatrix} U & V \\ \tilde{V} & W \end{bmatrix} \begin{bmatrix} q \\ q' \end{bmatrix} \equiv [\mathcal{S}] \begin{bmatrix} q \\ q' \end{bmatrix}, \tag{109}$$

where U and W are 2×2 real symmetric matrices, V is a 2×2 real matrix. $[\mathcal{S}]$ is a 4×4 symmetric matrix which has been called the point characteristic matrix. One also sometimes defines a *ray matrix*, $[\mathfrak{M}]$ which relates the position and direction vectors of a ray at the output plane to the values assumed at the input plane

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \equiv [\mathfrak{M}] \begin{bmatrix} q \\ p \end{bmatrix}, \tag{110}$$

where A, B, C, D are 2×2 real matrices. Since, from equation (109), only 10 numbers suffice to define the optical system, the elements of the 4×4 ray matrix $[\mathfrak{M}]$ must be related by $16 - 10 = 6$ relations. To obtain these relations, let us compare equations (109) and (110). One obtains readily

$$U = B^{-1}A, \tag{111a}$$

$$V = -B^{-1}, \tag{111b}$$

$$\tilde{V} = C - DB^{-1}A, \tag{111c}$$

$$W = DB^{-1}. \tag{111d}$$

Since U and W are symmetrical one has

$$A\tilde{B} - B\tilde{A} = 0, \tag{112a}$$

$$\tilde{B}D - \tilde{D}B = 0, \tag{112b}$$

and, by comparing the expressions obtained for V and $\tilde{V}$, equations (111b) and (111c), and using equation (112b), one finds that

$$\tilde{D}A - \tilde{B}C = 1. \tag{112c}$$

Equations (112a, b and c) are equivalent to those given by Luneburg.[1] They effectively correspond to six independent relations. The relations inverse of equations (111a through d) are

$$A = -V^{-1}U, \tag{113a}$$

$$B = -V^{-1}, \tag{113b}$$

$$C = \tilde{V} - WV^{-1}U, \tag{113c}$$

$$D = -WV^{-1}. \tag{113d}$$

APPENDIX B

Point Characteristic Matrix of a Sequence of Thin Lenses and Mirrors-Symmetrical Systems

The point characteristic matrix [S] of a sequence of thin astigmatic lenses and plane mirrors, arbitrarily oriented in space, can be obtained in closed form.

Let us first consider a thin astigmatic lens oriented at an angle ν with respect to the x_1 axis of a x_1x_2z rectangular coordinate system, with focal lengths f_1, f_2. This lens is followed by a section of free space of length d. For generality, one further assumes that the output coordinate system is rotated by an angle Ω about the z axis. This rotation has to be introduced in the case of non planar paths.[4,34] Using the expression for the optical thickness of a lens, and the paraxial approximation of the length of tilted rays in free space, one obtains

$$[S] \equiv \begin{bmatrix} U & V \\ \tilde{V} & W \end{bmatrix}, \tag{114}$$

with

$$U = \begin{bmatrix} \frac{1}{d} - \left(\frac{\cos^2 \nu}{f_1} + \frac{\sin^2 \nu}{f_2}\right) & \cos \nu \sin \nu \left(\frac{1}{f_1} - \frac{1}{f_2}\right) \\ \cos \nu \sin \nu \left(\frac{1}{f_1} - \frac{1}{f_2}\right) & \frac{1}{d} - \left(\frac{\cos^2 \nu}{f_2} + \frac{\sin^2 \nu}{f_1}\right) \end{bmatrix},$$

$$V = -d^{-1}\begin{bmatrix} \cos \Omega & \sin \Omega \\ -\sin \Omega & \cos \Omega \end{bmatrix},$$

$$W = d^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{115}$$

These expressions are applicable to curved mirrors under oblique incidence with little modification since a curved mirror is equivalent to a plane mirror and a lens, in the most general case.[34] It remains to calculate the point characteristic matrix of a sequence of optical systems such as the one described by equations (114) and (115).

The point characteristic matrix $[\mathcal{S}_t]$ of a sequence of two optical systems whose point characteristic matrices are respectively $[\mathcal{S}_1]$ and $[\mathcal{S}_2]$ is obtained by using equation (18) of the main text, and specifying that the rays are continuous at the junction between the two systems. One obtains

$$[\mathcal{S}_t] = \begin{bmatrix} U_1 - V_1(W_1 + U_2)^{-1}\tilde{V}_1 & -V_1(W_1 + U_2)^{-1}V_2 \\ -\tilde{V}_2(W_1 + U_2)^{-1}\tilde{V}_1 & W_2 - \tilde{V}_2(W_1 + U_2)^{-1}V_2 \end{bmatrix}. \tag{116}$$

In the special case where the second optical system is the mirror image of the first system with respect to their common plane, $[\mathcal{S}_t]$ reduces to

$$[\mathcal{S}_t] \equiv \begin{bmatrix} U_t & V_t \\ \tilde{V}_t & W_t \end{bmatrix} = \begin{bmatrix} U - \frac{1}{2}VW^{-1}\tilde{V} & -\frac{1}{2}VW^{-1}\tilde{V} \\ -\frac{1}{2}VW^{-1}\tilde{V} & U - \frac{1}{2}VW^{-1}\tilde{V} \end{bmatrix} \tag{117}$$

where the index 1 has been omitted. Equation (117) shows that, in a symmetric system, U_t is equal to W_t and V_t is a symmetric matrix.

Repeated applications of equation (116) and equations (114) and (115) give the point characteristic matrix of an arbitrary sequence of lenses or mirrors.

APPENDIX C

Diagonalization of a Complex Wavefront

The need for introducing an oblique coordinate system at each transverse plane has been outlined in the main text. Detailed transformation formulas are given in this appendix.

Let e_1, e_2 be the base vectors, of unit length, of the original rectangular coordinate system, and $\mathbf{e}_1$, $\mathbf{e}_2$ the base vectors, also of unit length, of a new coordinate system.† The $\mathbf{e}_i$, $i = 1, 2$ are linearly related to the e_j, $j = 1, 2$ by

$$\mathbf{e}_i = \delta_i^j e_j , \tag{118}$$

where δ_i^j is the mixed tensor which expresses the coordinate transformation. The reduced eikonal Φ is a complex quadratic form which was written in the original coordinate system [equation (53)]

$$\Phi \equiv \tfrac{1}{2}\tilde{r}\mu r \equiv \tfrac{1}{2}\tilde{r}(\mu^r + j\mu^i)r, \tag{119}$$

where μ^r and μ^i are real symmetric matrices.

By stipulating that, in the new coordinate system, the off-diagonal terms of μ^r and μ^i are both equal to zero, one obtains the transformation $[\delta]$ which diagonalizes Φ

$$[\delta] \equiv \begin{bmatrix} \delta_1^1 & \delta_1^2 \\ \delta_2^1 & \delta_2^2 \end{bmatrix} = \begin{bmatrix} (1 + v^2)^{-\frac{1}{2}} & v(1 + v^2)^{-\frac{1}{2}} \\ u(1 + u^2)^{-\frac{1}{2}} & (1 + u^2)^{-\frac{1}{2}} \end{bmatrix}, \tag{120}$$

where

$$u = (c/a)v = [-b + (b^2 - 4ac)^{\frac{1}{2}}]/2a, \tag{121}$$

$$a \equiv \mu_{11}^r\mu_{12}^i - \mu_{11}^i\mu_{12}^r , \tag{122a}$$

$$b \equiv \mu_{11}^r\mu_{22}^i - \mu_{11}^i\mu_{22}^r , \tag{122b}$$

$$c \equiv \mu_{22}^i\mu_{12}^r - \mu_{22}^r\mu_{12}^i . \tag{122c}$$

The law of transformation of the contravariant components of a vector q, denoted respectively q^i in the old system and $\mathbf{q}^i$ in the new system is (omitting the summation sign)

$$q^i = \delta_j^i \mathbf{q}^j. \tag{123a}$$

This relation is also applicable to the coordinate x^i

$$x^i = \delta_j^i \mathbf{x}^j. \tag{123b}$$

The covariant components of a vector p, denoted p_i in the old system and $\mathbf{p}_i$ in the new system, transform according to the inverse relation

$$\mathbf{p}_i = \delta_i^j p_j . \tag{124}$$

Expressions for the new components of μ are derived in the main text.

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† Quantities relative to the new system are denoted by bold face letters in this Appendix. Ordinary letters are used in the main text, where there is no risk of confusion.

APPENDIX D

Orthogonality of the Modes

Let q, p and $\hat{q}$, $\hat{p}$ be any two solutions of the paraxial ray equations, equations (11a and b), and assume that the matrix PQ^{-1}, where

$$Q \equiv [q\ \hat{q}], \tag{125a}$$

$$P \equiv [p\ \hat{p}], \tag{125b}$$

is symmetric.

An infinite set of solutions of the parabolic wave equation has been obtained in the main text in the form [equation (69)]

$$\psi_{mn}(r, z; Q, P) = |\ Q\ |^{-\frac{1}{2}} \exp\left(-j\frac{k}{2}\tilde{r}PQ^{-1}r\right)H_{mn}(\chi;\nu), \tag{126}$$

where H_{mn} denote the Hermite polynomial in two variables χ_1, χ_2

$$\chi \equiv Q^{*-1}r, \tag{127}$$

associated with the quadratic form $\tilde{\chi}\nu\chi$, where

$$\begin{aligned} \nu &\equiv jk\tilde{Q}^*(PQ^{-1} - P^*Q^{*-1})Q^* \\ &= JQ^{-1}Q^*, \end{aligned} \tag{128}$$

$$J \equiv -jk\begin{bmatrix} \tilde{q}p^* - \tilde{p}q^* & \tilde{\hat{q}}p^* - \tilde{\hat{p}}q^* \\ \tilde{q}\hat{p}^* - \tilde{p}\hat{q}^* & \tilde{\hat{q}}\hat{p}^* - \tilde{\hat{p}}\hat{q}^* \end{bmatrix}. \tag{129}$$

Let us now impose on the rays the additional condition

$$(\mathcal{R}; \hat{\mathcal{R}}^*) \equiv \tilde{q}\hat{p}^* - \tilde{p}\hat{q}^* = 0 \tag{130}$$

and assume that the diagonal terms of J are positive. Since, as pointed out in the main text, the two rays need be defined only to within constants, they can be normalized in such a way that J is the unit matrix. In that case we have, from equations (127) and (128)

$$\nu^{-1} = \nu^*, \tag{131}$$

$$\nu\chi = \chi^*. \tag{132}$$

Consequently

$$H^*_{m'n'}(\chi;\nu) = H_{m'n'}(\chi^*;\nu^*) = G_{m'n'}(\chi;\nu) \tag{133}$$

where we have introduced the adjoint polynomials G_{mn} defined by

$$G_{mn}(\chi;\nu) \equiv H_{mn}(\nu\chi;\nu^{-1}). \tag{134}$$

The orthogonality condition for two solutions ψ_{mn} and $\psi_{m'n'}$ [equation (126)] can now be written in the form

$$\iint_{-\infty}^{+\infty} \psi_{mn}(r)\psi^*_{m'n'}(r)\, d^2r$$

$$= |\, Q^{-1}Q^* \,|^{\frac{1}{2}} \iint_{-\infty}^{+\infty} \exp\left(-\tfrac{1}{2}\tilde{\chi}\nu\chi\right) H_{mn}(\chi;\nu) G_{m'n'}(\chi;\nu)\, d^2\chi,$$

$$= 2\pi m!\, n! \quad \text{if} \quad m' = m \quad \text{and} \quad n' = n,$$

$$= 0 \qquad\qquad \text{if} \quad m' \neq m \quad \text{or} \quad n' \neq n. \tag{135}$$

The biorthogonality property[32] of the polynomials H_{mn} and G_{mn} has been used in equation (135).

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