Optimum Phase Equalization of Filters for Digital Signals

By DAVID A. SHNIDMAN

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Digital information transmitted by means of a pulse amplitude modulation scheme depends critically on the pulse shape for reliable high speed communications. The pulse shape, in turn, depends in great measure on precise phase equalization. A new technique for the design of phase equalizers, based on digital mean square error, has been developed. This criterion is appropriate for a digital transmission system because it can be related to the system error rate.

A lower bound to the digital mean square error is first obtained by determining the theoretically optimum phase. A physical equalizer consists of a cascade of many (say N) constant resistance all-pass networks. For each of several different values of N, an optimization search over the parameters of the all-pass networks is then done. The smallest value of N which yields an error satisfactorily close to the lower bound is utilized for the optimum physical phase equalizer.

The major benefits derived from using this technique as opposed to the conventional one are:

(*i*) *A significant reduction in the number of all pass sections required.*
(*ii*) *More practical element values, facilitating network manufacture.*
(*iii*) *A substantial improvement in system performance.*

I. INTRODUCTION

Most information transmitted via telephone facilities today is sent over analog channels. It is anticipated, however, that future systems will be largely digital. New services such as *Picturephone*® will be transmitted over toll facilities in a digital format. The Bell System will need to provide a significant digital capacity by the late 1970s. In spite of this emphasis on digital transmission, analog demands will be rapidly increasing for some time to come. The L-4 system, originally conceived to provide increased analog capacity, is now being equipped to transmit in a digital mode as well. L-4 can therefore temporarily satisfy increasing analog and digital requirements. The eventual transition from analog to digital can be accomplished smoothly by apportioning analog and digital usage according to demand. Similarly, the L-5 system will be able to handle both analog and digital signals. Digital transmission on L-5 is expected to be available earlier than any purely digital system of comparable capacity. These systems with their dual capabilities promise to play an important role in the future of Bell System long-haul transmission.

The scheme (Fig. 1) employed for transmitting digital information over L-4 and L-5 is pulse amplitude modulation. The sequence of input digits, $\{a_k\}$, is used to scale translates (displaced by T seconds) of a given pulse shape. If these pulses do not overlap, then their amplitudes can be determined from samples of the received waveform (assuming noise free transmission) and the scale factors ascertained. Efficient use of channel capacity dictates simultaneous transmission of several signals, each confined to a specific bandwidth. The fulfillment of these bandwidth restrictions causes the transmitted pulses to overlap, and complicates the process of extracting the information digits. For bandwidths of greater than $1/2T$ (for single sideband), there exist a class of pulses, known as Nyquist pulses, which allow, at one point in each T second interval, for the extraction of an interference-free sample. Unfortunately, it is not possible to build filters to realize these pulse shapes exactly or to build sampling equipment with zero pulse widths. Consequently, one is left with an unavoidable amount of "intersymbol interference" such that the output sequence, $\{b_k\}$, will not be an exact duplicate of the input sequence, $\{a_k\}$.

The filters have a second function no less important than pulse shaping, namely, to prevent serious interference between signals in

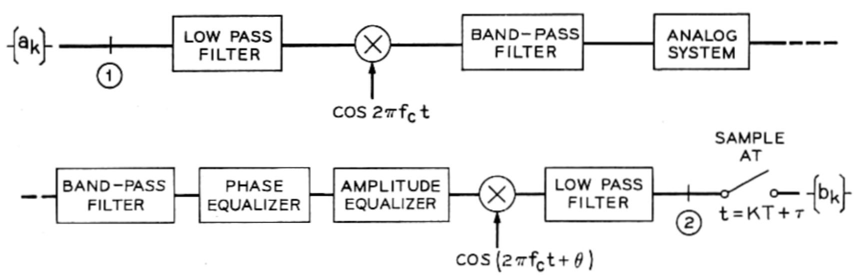

Fig. 1—Model for the digital communication system. [$V(f)$ is the transfer characteristic from 1 to 2.]

different bands. The filters must satisfy out-of-band rejection requirements as well as yield desirable pulse shapes.

With traditional filter design techniques, phase distortion is not considered initially.* An amplitude characteristic, fulfilling out-of-band rejection requirements, is realized as closely to a Nyquist shape as possible. Then the phase is modified by means of a cascade of all-pass sections so as to minimize the "curve fit" mean squared difference (over the pertinent bandwidth) between the resulting delay and a constant delay. This method (of reducing curve fit mean squared error) was initially used in L-4 for the design of filters for digital transmission. The result proved to be unacceptable. The major difficulty with this procedure is the fact that the resultant phase distortion, as determined above, cannot be readily related to an error between an input symbol, a_k, and the corresponding output, b_k. That is, not all errors of equal size as measured by this criterion affect the digital error to the same extent. Any criterion which does relate to differences between the $\{a_k\}$ and the $\{b_k\}$ would necessarily consider both amplitude and phase simultaneously. The digital mean square error, to be defined in the next section, is a criterion which is appropriate for our purposes and is related to the differences between digital input and output.

Although it was originally planned that the amplitude and phase equalization would be designed simultaneously, it was found to be advantageous not to do this initially. The problem becomes much more manageable by separating amplitude and phase designs into two interacting parts. In some cases, it was possible to achieve satisfactory amplitude characteristics by traditional methods. For these cases, the problem becomes one of designing a realizable phase characteristic approximating the optimum phase (which depends on the amplitude and which is not necessarily linear). Therefore, the first new effort is the development of a new technique for phase equalization using a criterion relevant to digital transmission.

II. DIGITAL MEAN SQUARE ERROR

The digital mean square error, D_k, of the kth digit is defined as the mean squared difference between the input, a_k, and its corresponding output, b_k [2,3]; that is,

$$D_k \equiv E\{(a_k - b_k)^2\} \tag{1}$$

* In his paper, "Synthesis of Pulse-Shaping Networks in the Time Domain," D. A. Spaulding does consider amplitude and phase simultaneously.[1]

where $E\{\cdot\}$ denotes expectation with respect to the random variables a_k and b_k. If the input sequence is wide-sense stationary and the channel is deterministic, conditions which we shall properly assume, then D_k is independent of k, and the subscript k will be dropped.

It can be shown (see, for example, Ref. 4) that equation (1) can be written as

$$D = m_0 + \int_{-\infty}^{\infty} M(f)V^*(f) \exp(j2\pi f\tau) \cdot \left\{ \frac{1}{T} \sum_{\alpha=-\infty}^{\infty} V\left(f - \frac{\alpha}{T}\right) \exp\left[-j2\pi\left(f - \frac{\alpha}{T}\right)\tau\right] - 2 \right\} df \qquad (2)$$

where

$V(f)$ is the transfer characteristic between points 1 and 2 of Fig. 1,
τ is a constant representing a delay in the sampling time,
T is the pulse repetition time interval,

and

$M(f)$ is a factor arising from the correlation between input digits $\{a_k\}$.

If we define

$$\rho_n = E\{a_k a_{k+n}\} = \rho_{-n}, \qquad (3)$$

then, assuming convergence, we have for $M(f)$,

$$M(f) = \sum_{n=-\infty}^{\infty} \rho_n \exp(-j2\pi f nT) = \sum_{n=0}^{\infty} m_n \cos 2\pi f nT \qquad (4)$$

where we have defined

$$m_0 = \rho_0$$

and

$$m_n = 2\rho_n, \qquad n \neq 0. \qquad (5)$$

The error, D, can be written in a more convenient form if we define an error term, $\epsilon(f)$, as:

$$\epsilon(f) = \frac{1}{T} \sum_{\alpha=-\infty}^{\infty} V\left(f - \frac{\alpha}{T}\right) \exp\left[-j2\pi\left(f - \frac{\alpha}{T}\right)\tau\right] - 1. \qquad (6)$$

Using equation (6) in equation (2) and noting the fact that the imaginary part of $\epsilon(f)$ is odd, we can write D as:

$$\begin{aligned} D &= m_0 + T \int_{-1/2T}^{1/2T} M(f)[\epsilon(f) - 1][\epsilon^*(f) + 1]\, df \\ &= 2T \int_0^{1/2T} M(f)\, |\, \epsilon(f)\, |^2\, df. \end{aligned} \qquad (7)$$

The function $\epsilon(f)$ is a measure of intersymbol interference. If the pulse shape were Nyquist, $\epsilon(f)$ would be zero. The term $M(f)$, due to the correlation of the input digits, acts as a weighting function in the error expression.

$\epsilon(f)$ is determined from the transfer functions of the block diagrams in Fig. 1. We can take advantage of the linearity of the vestigial sideband system and, by translating appropriately, lump several filters together. We define, therefore, the frequency transfer function for the combination of the two low-pass filters as $L(f)$, for the combination of two bandpass filters as $B(f)$, and for the combination of the all-pass networks (that is, the phase equalizer) as $A(f)$. The cable or analog system is represented by $C(f)$. With this notation, we can write the baseband equivalent to the passband portion of the system as:

$$H(f) = \tfrac{1}{2}[A(f + f_c)B(f + f_c)C(f + f_c) \exp(j\theta) + A(f - f_c)B(f - f_c)C(f - f_c) \exp(-j\theta)] \operatorname{rect}\left(\frac{fT}{2}\right) \tag{8}$$

where

$$\operatorname{rect}(x) \equiv \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

and where f_c is the carrier frequency and θ the phase offset (the difference in phase between the modulation and demodulation cosine functions). The overall transfer function becomes simply

$$V(f) = L(f)H(f). \tag{10}$$

In general $L(f)$, $B(f)$, and $C(f)$ are not measured separately. Instead $R(f)$ is measured where

$$R(f) = \begin{cases} \dfrac{1}{2T} L(f - f_c)B(f)C(f), & f > 0 \\[2ex] \dfrac{1}{2T} L(f + f_c)B(f)C(f), & f < 0 \end{cases} \tag{11}$$

with the result that

$$V(f) = T[R(f + f_c)A(f + f_c) \exp(j\theta) + R(f - f_c)A(f - f_c) \exp(-j\theta)] \operatorname{rect}\left(\frac{fT}{2}\right). \tag{12}$$

If we define $V_1(f)$ as

$$V_1(f) \equiv \frac{g}{T} V(f) \exp(-j2\pi f\tau), \tag{13}$$

where g is a scale factor, then $\epsilon(f)$ becomes, in terms of $V_1(f)$,

$$\epsilon(f) = \sum_{\alpha=-1}^{1} V_1\left(f - \frac{\alpha}{T}\right) - 1 \tag{14}$$

where the limits on the sum reflect the bandlimited nature of $V(f)$.

Since we are initially concerning ourselves with phase equalization only, we want to consider $R(f)$ as fixed and modify $\epsilon(f)$ by our choice of $A(f)$ where $|A(f)| = 1$. The optimum design for $A(f)$ is considered next.

III. OPTIMUM PHASE

In order to obtain a lower performance bound to the optimum design of the phase equilizer, it is necessary to determine, for a given amplitude shape, the phase associated with the minimum error, D. For this purpose, once again consider the error term,

$$\epsilon(f) = \sum_{\alpha=-\infty}^{\infty} \left| V_1\left(f - \frac{\alpha}{T}\right) \right| \exp\left(j\phi_1\left(f - \frac{\alpha}{T}\right)\right) - 1 \tag{15}$$

where $\phi_1(f)$ is the phase of $V_1(f)$.

Because $V_1(f)$ is zero, for $|f| > \frac{1}{T}$, then for $0 \leqq f \leqq \frac{1}{2T}$

we have

$$\begin{aligned} |\epsilon(f)|^2 &= \left| |V_1(f)| \exp(j\phi_1(f)) + \left| V_1\left(f - \frac{1}{T}\right) \right| \exp\left(j\phi_1\left(f - \frac{1}{T}\right)\right) - 1 \right|^2 \\ &= \left[|V_1(f)| \cos\phi_1(f) + \left| V_1\left(f - \frac{1}{T}\right) \right| \cos\phi_1\left(f - \frac{1}{T}\right) - 1 \right]^2 \\ &\quad + \left[|V_1(f)| \sin\phi_1(f) + \left| V_1\left(f - \frac{1}{T}\right) \right| \sin\phi_1\left(f - \frac{1}{T}\right) \right]^2. \end{aligned} \tag{16}$$

Clearly, minimizing $|\epsilon(f)|^2$ for each f minimizes D. Let us use the following shorthand notation

$$x = \cos \phi_1(f),$$

$$y = \cos \phi_1\left(f - \frac{1}{T}\right),$$

and

$$V = | V_1(f) |,$$

$$V_T = \left| V_1\left(f - \frac{1}{T}\right) \right|.$$

Then for f fixed we wish to minimize

$$J(x, y) = (Vx + V_Ty - 1)^2 + [V(1 - x^2)^{\frac{1}{2}} + V_T(1 - y^2)^{\frac{1}{2}}]^2 \qquad (17)$$

subject to the constraints

$$| x | \leqq 1 \quad \text{and} \quad | y | \leqq 1. \qquad (18)$$

Let $\bar{F}$ be the feasible region, that is, the set of points (x, y) which satisfy inequality (18). Let F be the interior of $\bar{F}$. Since $J(x, y)$ is differentiable in F, then the optimum point $(\hat{x}, \hat{y})$ is in F if and only if

$$\left.\frac{\partial J(x, y)}{\partial x}\right|_{(\hat{x}, \hat{y})} = \left.\frac{\partial J(x, y)}{\partial y}\right|_{(\hat{x}, \hat{y})} = 0. \qquad (19)$$

If $(\hat{x}, \hat{y})$ lies on the boundary of $\bar{F}$ then equation (18) is not necessarily satisfied.

We can now state the major result concisely as a theorem.

Theorem: If

$$V + V_T > 1 \quad \text{and} \quad |V - V_T| < 1 \qquad (20)$$

then $(\hat{x}, \hat{y}) \in F$ *and* $\hat{x}$ *and* $\hat{y}$ *are determined by*

$$\hat{x} = \frac{1 + (V + V_T)(V - V_T)}{2V}$$

and (21)

$$\hat{y} = \frac{1 + (V_T + V)(V_T - V)}{2V_T},$$

yielding an error

$$J(\hat{x}, \hat{y}) = 0:$$

however, if (20) *is not satisfied then*

$$\hat{x} = 1, \qquad \hat{y} = -1, \quad \text{if} \quad V \geqq V_T \qquad (22)$$

or

$$\hat{x} = 1, \qquad \hat{y} = 1, \quad \text{if} \quad V < V_T. \tag{23}$$

Proof: If $(\hat{x}, \hat{y}) \in F$ then equation (19) holds and solving for x and y we obtain equation (21). We must show that if inequality (20) holds, then equation (21) is consistent with $(\hat{x}, \hat{y}) \in F$. In order to do this, we define σ and Δ as:

$$\sigma = V + V_T - 1 \tag{24}$$

and

$$\Delta = 1 - (V - V_T). \tag{25}$$

Now inequality (20) implies that $\sigma > 0$ and $0 < \Delta < 2$. Therefore for $\sigma > 0$ and $0 < \Delta < 2$ we must obtain the values of $\hat{x}$ and $\hat{y}$ such that $| \hat{x} | < 1$ and $| \hat{y} | < 1$. Using equations (24) and (25) in equation (21), we obtain for $\hat{x}$

$$\hat{x} = \frac{1 + (1 + \sigma)(1 - \Delta)}{2 + \sigma - \Delta},$$

$$= 1 - \frac{\sigma\Delta}{2 + \sigma - \Delta}.$$

Since we have

$$0 < \frac{\sigma\Delta}{2 + \sigma - \Delta} < 2, \tag{26}$$

then $|\hat{x}|$ must be less than one. Similarly, we obtain

$$\hat{y} = \frac{1 - (1 + \sigma)(1 - \Delta)}{\sigma + \Delta},$$
$$= 1 - \frac{2 - \Delta}{1 + (\Delta/\sigma)}. \tag{27}$$

Since the quantity $1 + \Delta/\sigma$ is greater than 1 and $0 < 2 - \Delta < 2$, then $| \hat{y} | < 1$, proving that if inequality (20) is satisfied, then $(\hat{x}, \hat{y}) \in F$.

Substituting equation (21) into equation (17), we get

$$J(\hat{x}, \hat{y})$$
$$= [\tfrac{1}{2} + \tfrac{1}{2}(V + V_T)(V - V_T) + \tfrac{1}{2} - \tfrac{1}{2}(V + V_T)(V - V_T) - 1]^2$$
$$+ \left[\left(\frac{4V^2 - 1 - 2V^2 + 2V_T^2 - (V^2 - V_T^2)^2}{2}\right)^{\frac{1}{2}}\right.$$
$$\left. - \left(\frac{4V_T^2 - 1 + 2V^2 - 2V_T^2 - (V^2 - V_T^2)^2}{2}\right)^{\frac{1}{2}}\right] = 0. \tag{28}$$

If inequality (20) is not satisfied, then the point $(\hat{x}, \hat{y})$ is on the boundary. In order to minimize $J(x, y)$ we must eliminate the second term in J; that is, we must have $|\hat{x}| = |\hat{y}| = 1$. If $V \geqq V_T$ then $x = 1, y = -1$ minimizes J; otherwise $x = -1, y = 1$ minimizes J.

The optimum phases $\phi_1(f)$ and $\phi_1(f - 1/T)$ are obtained from the equations

$$\phi_1(f) = \cos^{-1} x,$$

and

$$\phi_1\left(f - \frac{1}{T}\right) = \begin{cases} -\cos^{-1} y, & y > 0; \\ -\pi + \cos^{-1} y, & y < 0. \end{cases} \tag{29}$$

In this manner, the optimum phase is determined for all f for the baseband signal $V_1(f)$. Except in the Nyquist roll-off region, $\phi(f)$ is zero. In the demodulation step, there is a similar overlapping in the vestigial roll-off region and an identical procedure can be done provided the two roll-off regions do not overlap. If they do, then the process is nearly the same except that one must first determine, for the overlapping part of the vestigial region, a phase difference between the terms that add together after demodulation and then determine the phase of the baseband waveform. The solutions are, in general, not unique.

IV. REALIZABLE PHASE EQUALIZER

The optimum phase, as specified in the previous section, can not be realized. For physical systems we are restricted to those phase characteristics that can be achieved by cascading sections of constant resistance all-pass networks. A typical constant resistance all-pass network is shown in Fig. 2. Its transfer characteristic is

$$E_2/E_1 = \exp\,(-j\beta(f, f_n, b_n)) \tag{30}$$

where (Fig. 3)

$$\beta(f, f_n, b_n) = 2\tan^{-1} \frac{2}{b_n} \frac{f_n f}{(f_n^2 - f^2)}. \tag{31}$$

Using N such all-pass networks yields an overall response $A(f)$, such that

$$A(f) = \exp\left(-j \sum_{n=1}^{N} \beta(f, f_n, b_n)\right). \tag{32}$$

Substituting equation (32) into equation (12) and using measured characteristics for $R(f)$, we obtain $V_1(f)$, $\epsilon(f)$ and, in turn, D in terms

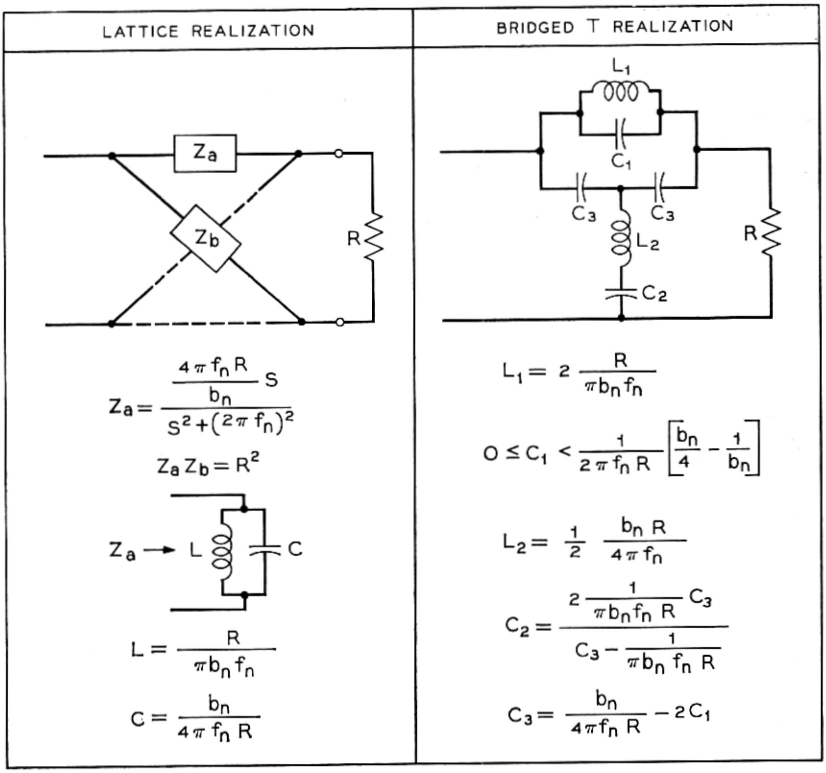

Fig. 2—Two realizations of a constant resistance all-pass network.

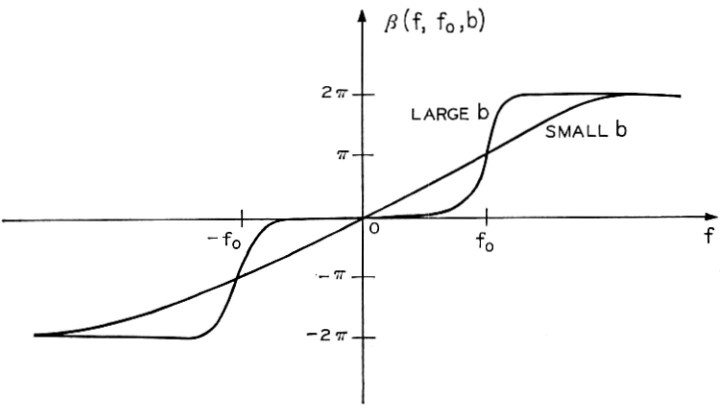

Fig. 3—Typical phase lag characteristics of an all-pass section.

of the b_n and f_n. The next step is the optimization search to choose the set of b_n's and f_n's to minimize D.

V. THE MINIMIZATION OF D

Since D is now seen to be a function of many parameters, it is more appropriately written as

$$D(\mathbf{f}, \mathbf{b}, \tau, g, \theta) = 2T \int_0^{1/2T} M(f) \mid \epsilon(f, \mathbf{f}, \mathbf{b}, \tau, g, \theta) \mid^2 df \tag{33}$$

where $V_1(f)$ in equation (14) is equal to $gV_2(f)$ and

$$V_2(f) = \Bigg[\mid R(f + f_c) \mid \exp \Bigg[j\Bigg(\phi(f + f_c) - \sum_{n=1}^{N} \beta(f + f_c, f_n, b_n) + \theta\Bigg)\Bigg]$$
$$+ R(-f + f_c) \exp \Bigg[-j\Bigg(\phi(f + f_c) - \sum_{n=1}^{N} \beta(-f + f_c, f_n, b_n) + \theta\Bigg)\Bigg]\Bigg]$$
$$\cdot \exp(j2\pi f\tau) \operatorname{rect}\left(\frac{fT}{2}\right) : \tag{34}$$

g is a scale factor for $V_1(f)$; $\mathbf{b}$ and $\mathbf{f}$ are parameter vectors

$$\mathbf{b} \equiv \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \qquad \mathbf{f} \equiv \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \tag{35}$$

and $\phi(f)$ is the phase of $R(f)$.

If we adopt the following inner product notation

$$[\alpha(f), \beta(f)]_M = 2T \int_0^{1/2T} M(f)\alpha(f)\beta^*(f)\, df, \tag{36}$$

then D can be concisely written as

$$D(\mathbf{f}, \mathbf{b}, \tau, g, \theta) = [\epsilon(f, \mathbf{f}, \mathbf{b}, \tau, g, \theta), \epsilon(f, \mathbf{f}, \mathbf{b}, \tau, g, \theta)]_M . \tag{37}$$

We must first minimize D with respect to the gain, g, the delay, τ, and phase offset, θ, before determining what changes to make in $\mathbf{b}$ and $\mathbf{f}$. After changing $\mathbf{b}$ and $\mathbf{f}$, we repeat the procedure by optimizing g, τ, and θ, and then obtaining new $\mathbf{b}$ and $\mathbf{f}$. After each iteration, we evaluate D and stop when D ceases to improve more than a predetermined amount.

To ascertain the optimum value for g, we solve $\partial D/\partial g = 0$ for g

and obtain

$$g = \frac{\mathrm{Re}\left\{\left[\sum_{\alpha=-1}^{1} V_2\left(f - \frac{\alpha}{T}\right), 1\right]_M\right\}}{\left[\sum_{\alpha=-1}^{1} V_2\left(f - \frac{\alpha}{T}\right), \sum_{\alpha=-1}^{1} V_2\left(f - \frac{\alpha}{T}\right)\right]_M}. \tag{38}$$

g does depend on the phase of $V_2(f)$; however, if the roll-off region is narrow, g is relatively insensitive to it, especially when **b** and **f** are near their optimum values. We can, therefore, eliminate repeating the calculation of g at each iteration by replacing $V_2(f)$ by $| V_2(f) |$ in equation (38) to obtain

$$g = \frac{\left[\sum_{\alpha=-1}^{1} \left| V_2\left(f - \frac{\alpha}{T}\right) \right|, 1\right]_M}{\left[\sum_{\alpha=-1}^{1} V_2\left(f - \frac{\alpha}{T}\right), \sum_{\alpha=-1}^{1} V_2\left(f - \frac{\alpha}{T}\right)\right]_M} \tag{39}$$

which, since it is independent of phase, need be calculated only once.

We shall take a similar approach for the evaluation of θ. Optimization with respect to θ would have to be done jointly with optimization with respect to τ. This would enormously complicate the minimization of D, since it would have to be repeated at each iteration. Instead we can choose θ as

$$\theta = -\phi(f_c) + \sum_{n=1}^{N} \beta(f_c, f_n, b_n) + \theta_0 \tag{40}$$

where θ_0 can be chosen so that θ is very close to its optimum value for the final set of **b** and **f**. Thus, as **b** and **f** improve, the value chosen for θ approaches its optimum. θ_0 depends on $R(f)$ and is determined from the optimum phase.

With g and θ fixed, the optimization with respect to τ has been simplified. τ_{opt} is a solution of the equation

$$\begin{aligned} \frac{\partial D}{\partial \tau} &= 2\,\mathrm{Re}\left[\epsilon(f, \mathbf{f}, \mathbf{b}, \tau), \frac{\partial \epsilon(f, \mathbf{f}, \mathbf{b}, \tau)}{\partial \tau}\right]_M, \\ &= 2\,\mathrm{Re}\left[\epsilon(f, \mathbf{f}, \mathbf{b}, \tau), \sum_{\alpha=0}^{1} j2\pi\left(f - \frac{\alpha}{T}\right) V_1\left(f - \frac{\alpha}{T}\right)\right]_M = 0. \end{aligned} \tag{41}$$

An initial value of τ is chosen as described below and $\partial D/\partial\tau$ is evaluated. If $\partial D/\partial\tau > 0$, then τ is decremented, or if $\partial D/\partial\tau < 0$, then τ is incremented, until $\partial D/\partial\tau$ changes sign. Using the last two values of $\partial D/\partial\tau$, we can employ the method of "false position" (regula falsi)[5] to determine a value of $\partial D/\partial\tau < 10^{-5}$. The τ yielding this value of $\partial D/\partial\tau$ is accepted as τ_{opt}.

τ_i, the initial value of τ, is chosen to correspond to the best linear mean square error curve fit to the resulting baseband phase. Thus we have

$$\tau_i = -\mu/2\pi$$

where

$$\mu = \frac{\sum_{k=1}^{K} (f_k - f_c)\psi(f_k)}{\sum_{k=1}^{K} (f_k - f_c)^2}, \tag{42}$$

$$\psi(f) = \phi(f) - \sum_{n=1}^{N} \beta(f, f_n, b_n) + \theta,$$

and the f_k are the discrete values of frequency between $f = 0$ and $f = 1/2T$ used in evaluating D by a finite sum.

The minimization is done by a parameter search utilizing the Fletcher–Powell algorithm.[6,7] Since the effectiveness of the method depends on the accuracy with which the gradient is determined, it is particularly advantageous to use an analytic expression for its calculation. This expression is presented in Appendix A.

The computer program to achieve the above minimization was written in its entirety by Mrs. Barbara E. Forman for the CDC 3300 and the IBM 360/75.

Practical problems can arise in performing the parameter search. They are due mostly to the existance of many local minima. The Fletcher–Powell algorithm will approach a local minimum as readily as a global minimum. This difficulty is mitigated by three factors: (*i*) We have been able to determine, for a given amplitude characteristic, the theoretical phase which will yield the lowest value of D and thereby determine the practical utility of trying to find the global optimum—we have been able to find local minima which yield digital mean square errors insignificantly greater than the lower bound. (*ii*) We are able to specify the largest acceptable value of D. (*iii*) The parameter values at the global optimum may be impractical to realize by physical networks.

The system error rate objective for L-4 and L-5 is that the probability of error (P_E) be less than 10^{-8} per regenerative section. Walter E. Norris[8] has been able to relate a P_E of 10^{-8} to a value of D of approximately 10^{-3}. This relationship is not unique, but it is accurate to within a practically acceptable tolerance. We set our requirement for the theoretical filter at $D = 5 \times 10^{-4}$ since the physical realization will result in a larger error. We will accept a larger D for the implemented filter provided its value for D is reasonably close to this. If a minimization design yields a value of D near the lower bound and below the requirement, it is of no serious consequence that we do not achieve the global minimum. Since our design is based on a mathematical description of the network, it is important, however, that the parameter values be such that the physical realization approximates closely its mathematical representation. It is possible for a local optimum in a favorable parameter region to produce a better result than the global optimum in an unfavorable region due to attendant amplitude distortion and problems in its manufacture.

VI. COMPARISON OF DESIGNS

The data available from the L-4 system affords an opportunity to compare the conventional techniques of phase equalization with our method. The digital sequences on the system will be encoded in a partial response[9] format which yields an $M(f)$ of $M(f) = 4 \sin^2 4\pi fT$. The amplitude characteristic of the cascade of filters for digital transmission is shown in Fig. 4. By conventional methods of phase equalization, 20 all-pass sections were designed to reduce the phase deviation (deviation from linear phase, see Appendix B) to the "best achievable" level across the entire bandwidth. The theoretical performance of the 20-section equalizer (which does not consider the amplitude distortion of the phase equalizer) yields a phase deviation as shown in Fig. 5 and a digital mean square error of 1.45×10^{-4}. The "b" parameters (a measure of the steepness of the slope of the phase characteristics of an all-pass section, see Fig. 3) of several all-pass sections had values well above 25, the highest being 34.39. Values of b above 25 are undesirable for purposes of manufacture. The actual phase deviation as measured by J. S. Ronne[10] is shown in Fig. 6 and yields an error of 1.05×10^{-2}.

An optimum phase deviation for the system, as determined by the method described in an earlier section, is shown in Fig. 7 and yields a digital mean square error of 5.69×10^{-5}. If the phase had been exactly linear, the error would be 7.52×10^{-5}. Our program will attempt to achieve the optimum phase.

We must first determine the number of sections necessary to yield an acceptable error. In Fig. 8, the error, D, for the optimum design is plotted as a function of the number of all-pass sections. The curve shows

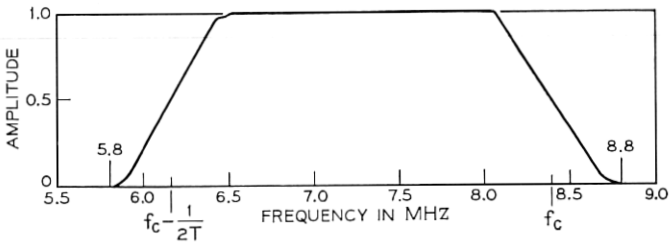

Fig. 4—Amplitude characteristics of cascade of filters.

that one obtains diminishing returns after 10 sections and that these 10 sections have a theoretical error (1.41×10^{-4}) less than that of the original 20-section design. Taking into account the reduced amplitude distortion in realizing 10 sections as opposed to 20 sections, and a minimal theoretical improvement in going from 10 to 20, we would, in all likelihood, achieve better performance from the optimum 10-section design than from the optimum 20-section design. A comparison of calculated amplitude distortions is presented in Fig. 9. It indicates that it is very desirable to use as few sections as is possible.

The optimum 10-section phase equalizer (all of whose b parameters

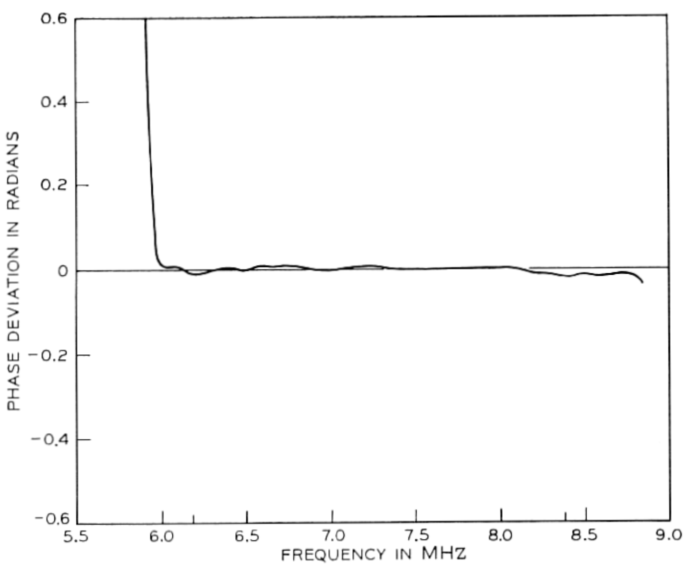

Fig. 5—Theoretical phase deviation of original 20-section design.

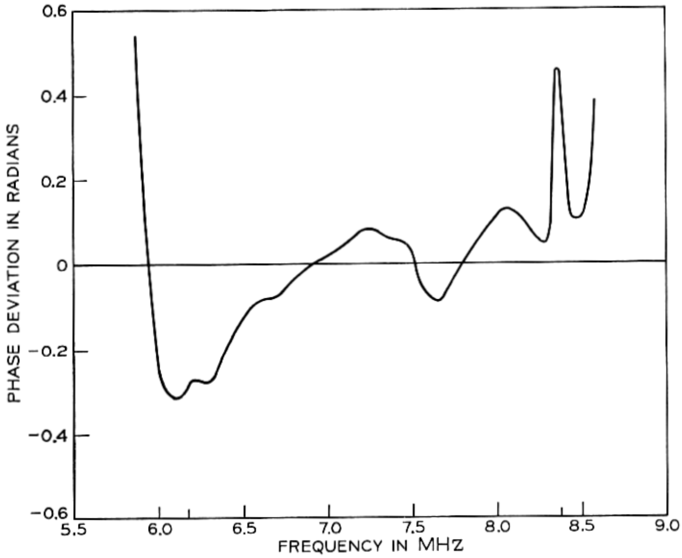

Fig. 6—Phase deviation of original 20-section design as determined from measured characteristics.

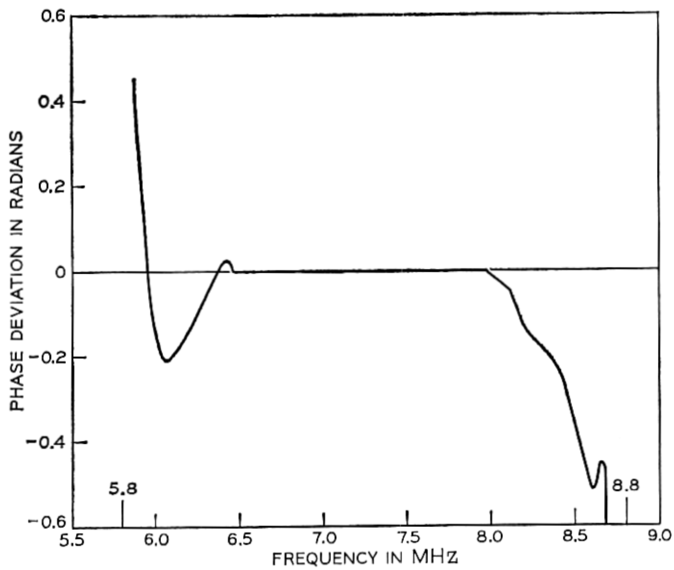

Fig. 7—Phase deviation of optimum phase.

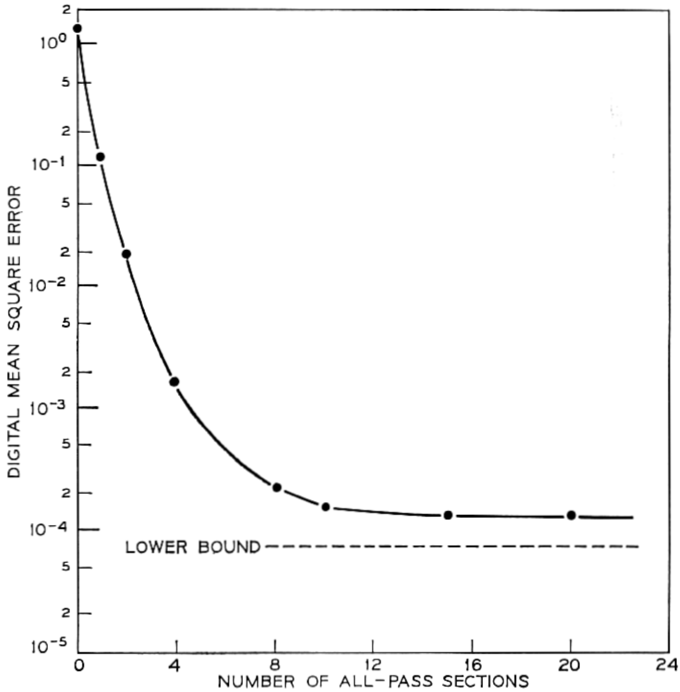

Fig. 8—Minimum digital mean squared error obtained with various numbers of all-pass sections.

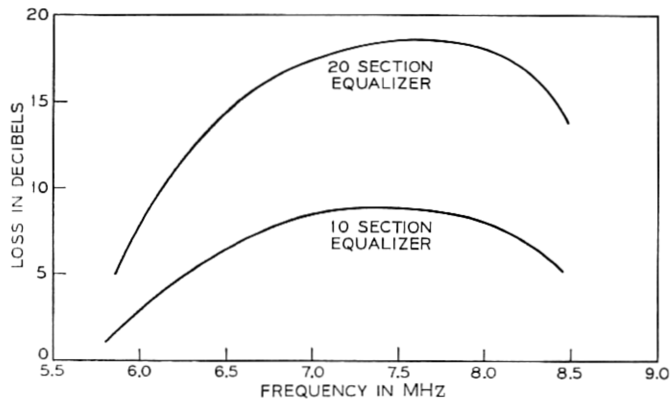

Fig. 9—Calculated loss characteristics of non-ideal phase equalizers.

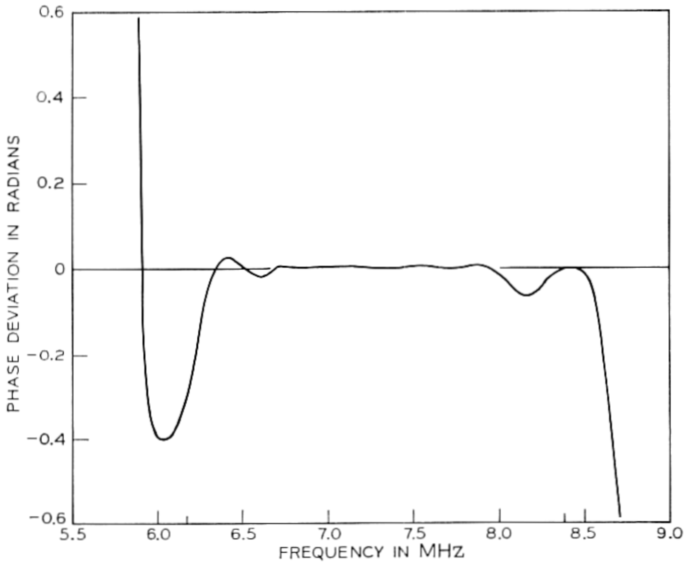

Fig. 10—Theoretical phase deviation of 10-section optimum design.

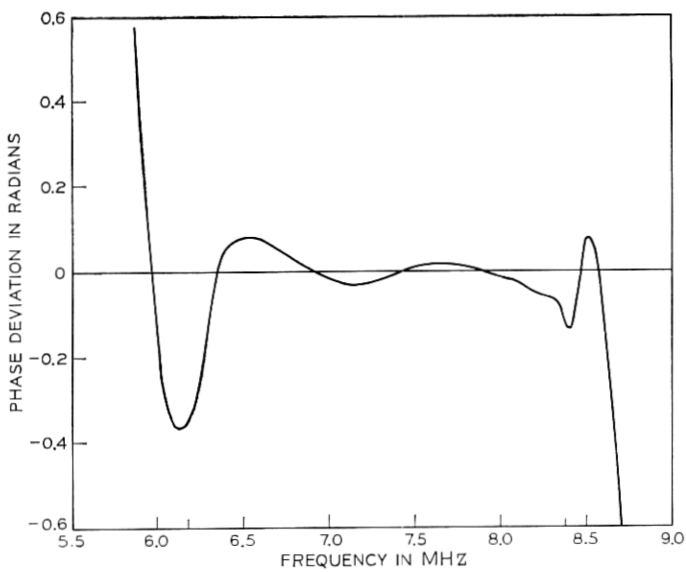

Fig. 11—Phase deviation of optimum 10-section design as determined from measured characteristics.

TABLE I—THE DIGITAL MEAN SQUARE ERROR FOR THE TWO PHASE EQUALIZER DESIGNS

	Theoretical	Measured
Original Design (20 Sections)	1.45×10^{-4}	1.05×10^{-2}
New Design (10 Sections)	1.41×10^{-4}	1.99×10^{-3}

were equal to about 20) was built and tested. The phase deviation of the theoretical design is shown in Fig. 10 with a corresponding error of 1.41×10^{-4}, while the phase deviations determined from the measurements of J. S. Ronne are shown in Fig. 11. This corresponds to an error of 1.99×10^{-3}. Table I summarizes the results.

A P_E test was made on both the 10-section and 20-section designs at

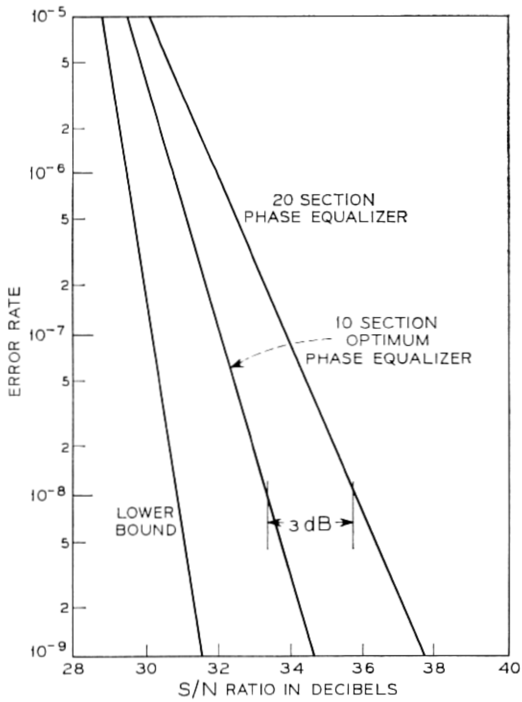

Fig. 12—P_E vs S/N for the 10- and 20-section phase equalizers.

a signal-to-noise (S/N) ratio of 40 dB. The error rate of the 20-section was 2000 times that of the 10-section. The two designs were also tested under the circumstances in which they would be used. Attached to the system is an adaptive equalizer which automatically adjusts to partly compensate for distortions in the system. The P_E of both designs were taken as a function of S/N, with the adaptive equalizer included, and the results are presented in Fig. 12. These measurements indicate that the new design yields 3 dB more S/N margin at $P_E = 10^{-8}$ than the previous design. The lower bound on P_E obtained from additive noise considerations (assuming a Nyquist pulse shape) by F. S. Hill[11] indicates that there is little room for additional improvement through manipulation of the pulse shape.

The advantages of the new design can be summarized as follows: (*i*) it is less expensive (fewer sections), (*ii*) it is easier to manufacture, and, (*iii*) most importantly, it yields better performance.

VII. CONCLUSIONS

The use of the digital mean square error, D, as a design criterion has proved to be of significant value in the design of the phase equalization for the transmission of digital information over the L-4 channel. Further studies indicate that comparable advantages exist in applying this technique to L-5 and other L-4 network designs. The use of the digital mean square error criterion is not limited to the phase equalization described here. Not mentioned above, but necessary, is amplitude equalization to compensate for the amplitude distortion introduced by the physical filters and all-pass sections. The design of this equalization is done optimally using the same techniques.

With conventional techniques, the design of the band limiting filters considers amplitude characteristics alone. The passband of these filters should be designed to minimize D, in which case both the amplitude and phase influence the result. The next step, therefore, is the jointly optimum design of these filters, together with equalization, on a digital mean square error basis.

VIII. ACKNOWLEDGMENTS

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APPENDIX A

The Gradient

The error D can be written

$$D(\mathbf{f}, \mathbf{b}) = 2T \int_0^{1/2T} M(f) \mid \epsilon(f, \mathbf{f}, \mathbf{b}) \mid^2 df \tag{43}$$

where, for the band limited $V(f)[\mid V(f) \mid = 0, \ \mid f \mid > 1/T)]$

$$\epsilon(f) \equiv V_1(f) + V_1\left(f - \frac{1}{T}\right) - 1 \tag{44}$$

and

$$\begin{aligned} V_1(f) \equiv g\Big\{ &\mid R(f + f_c) \mid \exp\left[j\left(\phi(f + f_c) - \sum_{n=1}^{N} \beta(f + f_c, f_n, b_n) + \theta\right)\right] \\ &+ \mid R(-f + f_c) \mid \\ &\cdot \exp\left[-j\left(\phi(-f + f_c) - \sum_{n=1}^{N} \beta(-f + f_c, f_n, b_n) + \theta\right)\right]\Big\} \\ &\cdot \exp(j2\pi f\tau) \operatorname{rect}\left(\frac{fT}{2}\right). \end{aligned}$$

The gradient $\mathbf{G}$ is

$$\mathbf{G} \equiv \begin{Bmatrix} \dfrac{\partial D}{\partial b_1} \\ \dfrac{\partial D}{\partial b_2} \\ \vdots \\ \dfrac{\partial D}{\partial b_N} \\ \dfrac{\partial D}{\partial f_1} \\ \vdots \\ \dfrac{\partial D}{\partial f_N} \end{Bmatrix} \tag{45}$$

where

$$\frac{\partial D}{\partial b_n} = 4T \int_0^{1/2T} M(f) \operatorname{Re} \left\{ \epsilon(f) \left[\frac{\partial V_1^*(f)}{\partial b_n} + \frac{\partial V_1^*\left(f - \frac{1}{T}\right)}{\partial b_n} \right] \right\} df,$$

$$\frac{\partial D}{\partial f_n} = 4T \int_0^{1/2T} M(f) \operatorname{Re} \left\{ \epsilon(f) \left[\frac{\partial V_1^*(f)}{\partial f_n} + \frac{\partial V_1^*\left(f - \frac{1}{T}\right)}{\partial f_n} \right] \right\} df, \tag{46}$$

and

$$\begin{Bmatrix} \dfrac{\partial V_1}{\partial b_n} \\ \dfrac{\partial V_1}{\partial f_n} \end{Bmatrix} = g \exp(j2\pi f\tau) \Bigg[\,|\, R(f + f_c) \,|\, \exp \Bigg(j \Bigg\{ \phi(f + f_c) - \phi(f_c)$$

$$- \sum_{n=1}^{N} [\beta(f + f_c, f_n, b_n) - \beta(f_c, f_n, b_n)] + \theta_0 \Bigg\} \Bigg)$$

$$\cdot j \begin{Bmatrix} \dfrac{-f_n(f + f_c)[(f + f_c)^2 - f_n^2]}{[f_n(f + f_c)]^2 + \dfrac{b_n^2}{4}[f_n^2 - (f + f_c)^2]^2} + \dfrac{f_n f_c (f_c^2 - f_n^2)}{(f_n f_c)^2 + \dfrac{b_n^2}{4}[f_n^2 - f_c^2]^2} \\ \dfrac{b_n(f + f_c)[f_n^2 + (f + f_c)^2]}{[f_n(f + f_c)]^2 + \dfrac{b_n^2}{4}[f_n^2 - (f + f_c)^2]^2} - \dfrac{b_n f_c (f_n^2 + f_c^2)}{(f_n f_c)^2 + \dfrac{b_n^2}{4}[f_n^2 - f_c^2]^2} \end{Bmatrix}$$

$$+ \,|\, R(-f + f_c) \,|\, \exp \Bigg(-j \Bigg\{ \phi(-f + f_c) - \phi(f_c)$$

$$- \sum_{n=1}^{N} [\beta(-f + f_c, f_n, b_n) - \beta(f_c, f_n, b_n)] + \theta_0 \Bigg\} \Bigg)$$

$$\cdot j \begin{Bmatrix} \dfrac{f_n(-f + f_c)[(-f + f_c)^2 - f_n^2]}{[f_n(-f + f_c)]^2 + \dfrac{b_n^2}{4}[f_n^2 - (-f + f_c)^2]^2} - \dfrac{f_n f_c (f_c^2 - f_n^2)}{(f_n f_c)^2 + \dfrac{b_n^2}{4}[f_n^2 - f_c^2]} \\ \dfrac{-b_n(-f + f_c)[f_n^2 + (-f + f_c)^2]}{[f_n(-f + f_c)]^2 + \dfrac{b_n^2}{4}[f_n^2 - (-f + f_c)^2]^2} + \dfrac{b_n f_c (f_n^2 + f_c^2)}{(f_n f_c)^2 + \dfrac{b_n^2}{4}[f_n^2 - f_c^2]} \end{Bmatrix} \Bigg]$$

$$\cdot \operatorname{rect}\left[\frac{fT}{2}\right]. \tag{47}$$

APPENDIX B

Phase Deviation

The phase deviation is defined as the deviation of the phase from the linear phase which best (in a mean square error sense) fits the phase.

Therefore, if $\psi(f)$ is the phase and $af + b$ is the best linear fit between f_1 and f_2 then the phase deviation ϕ_D is

$$\phi_D(f) = \psi(f) - (af + b)$$

where

$$a = \frac{\int_{f_1}^{f_2} f\psi(f)\,df - \frac{1}{f_2 - f_1}\left(\int_{f_1}^{f_2} \psi(f)\,df\right)\left(\int_{f_1}^{f_2} f\,df\right)}{\int_{f_1}^{f_2} f^2\,df - \frac{1}{f_2 - f_1}\left(\int_{f_1}^{f_2} f\,df\right)^2}$$

and

$$b = \frac{1}{f_2 - f_1}\left[\int_{f_1}^{f_2} \psi(f)\,df - a\int_{f_1}^{f_2} f\,df\right].$$

Discretizing the above equations, we obtain

$$a = \frac{\sum_{k=k_1}^{k_1+n} f_n\psi(f_n) - \frac{\left(\sum_{k=k_1}^{k_1+n} f_k\right)\left(\sum_{k=k_1}^{k_1+n} \psi(f_k)\right)}{n}}{\left[\sum_{k=k_1}^{k1+n} f_k^2 - \frac{\left(\sum_{k=k_1}^{k_1+n} f_k\right)^2}{n}\right]}$$

$$b = \frac{\sum_{k=k1}^{k+n} \psi(f_k) - a\sum_{k=k1}^{k1+n} f_k}{n}.$$

where $f_{k_1} = f_1$, $f_{k_1+n} = f_2$ and

$$\Delta f = f_{k_1+m} - f_{k_1+m-1} = \frac{f_2 - f_1}{n},$$

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