Limiting Behaviors of Randomly Excited Hyperbolic Tangent Systems

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We investigate the steady-state probability density distribution of a large class of random processes by solving the governing Fokker–Planck equation. The random response statistics of a nonlinear single-degree-of-freedom mechanical model with hyperbolic tangent stiffness are discussed in some detail. The probability density of such systems is of the sech-power type which belongs to a class of distributions whose behaviors are carefully examined at the limits where the system parameter b approaches zero and infinity. Other important response statistics such as the mean square response, zero crossings, and peak distributions are also studied.

I. INTRODUCTION

In recent years, random vibrations of nonlinear systems have attracted considerable attention among engineers.[1] In this paper we investigate the Fokker–Planck equations[2,3] associated with a class of random processes whose steady-state probability density distributions, of the Liapunov potential function type.

The random response statistics of a nonlinear single-degree-of-freedom model having a hyperbolic tangent stiffness function can be described as a softening spring whose force-deflection relationship is asymptotic to some maximum force level. Such a model can be used to represent an elastic-perfect-plastic system, material often encountered in classical mechanics. Limiting situations for a class of probability density functions such as those obtained in this study are examined. We show that the limiting behavior of the steady-state output probability density function of a system having a generalized hyperbolic tangent stiffness function, $F(u) = (k_0/b^{\alpha-1}) \tanh bu$, is closely related to the range of the parameter α. At the limit $b \to \infty$, the probability density function becomes a Dirac delta (impulse) function or an exponential distribution, or identically approaches zero for all u, depending upon whether α

is less than, equal to, or greater than 1. At the limit $b \to 0$, it vanishes identically for all u and becomes a normal distribution or a Dirac delta function, depending upon whether α is less than, equal to, or greater than 2. In addition, we study statistics of other response parameters such as the mean square output, zero crossings and peak output distribution, which are relevant to the control of the failure modes of the system.

The motion of a dynamic system under purely random disturbance is described by a Markoff process $\mathbf{y}(t) = [y_1(t), y_2(t), \cdots, y_n(t)]$ in the n-dimensional phase space. It can be shown[3,4] that for the initial condition

$$p(\mathbf{y}_o) = \prod_{i=1}^{n} \delta(y_i - y_{io})$$

where $\mathbf{y}_o$ is the initial state of $\mathbf{y}(t)$ and δ is the Dirac delta function, the conditional probability density function $p(\mathbf{y} \mid \mathbf{y}_o, t)$ of the process $\mathbf{y}(t)$ satisfies the forward Fokker–Planck equation,

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{n} \frac{\partial G_i(\mathbf{y})}{\partial y_i}, \qquad t \geqq 0 \tag{1}$$

where

$$G_i(\mathbf{y}) = A_i(\mathbf{y})p - \tfrac{1}{2} \sum_{j=1}^{n} \frac{\partial}{\partial y_j} [B_{ij}(\mathbf{y})p] \tag{2}$$

is the component of the probability current vector $p(\mathbf{y} \mid \mathbf{y}_o, t)$ in which

$$A_i(\mathbf{y}) = \lim_{\Delta t \to 0} \langle y_{i,\Delta t} - y_i \rangle \tag{3}$$

and

$$B_{ij}(\mathbf{y}) = \lim_{\Delta t \to 0} \langle (y_{i,\Delta t} - y_i)(y_{j,\Delta t} - y_j) \rangle \tag{4}$$

are intensity coefficients depending on the input and the properties of the system (the bracket indicating ensemble averaging).

We are interested in the solution of the steady-state equation (1), that is when $\partial p / \partial t = 0$, for cases where all generalized response variables of a system in the $2n$ phase-space coordinates are independent of one another. For this type of motion it is sometimes possible to find appropriate partial operators which, when linearly operated on functions of the type $g_i(y_i)p + h_i(y_i)(\partial p / \partial y_i)$, generate an equation equivalent to (1). More specifically, the steady-state equation (1) can be put in the form

$$\sum_{i=1}^{2n} L_i \left[g_i(y_i)p + h_i(y_i) \frac{\partial p}{\partial y_i} \right] = 0 \tag{5}$$

where the coefficients L_i are arbitrary first-order partial-differential operators. If there exists a $p(y)$ independent of initial conditions and satisfying each

$$g_i(y_i)p + h_i(y_i)\frac{\partial p}{\partial y_i} = 0, \qquad (i = 1, 2, \cdots, 2n),$$

then by Gray's uniqueness theorem such $p(y)$ is the unique solution of equation (5).[5] Such a solution is

$$p_{st}(\mathbf{y}) = C \prod_{i=1}^{2n} \exp\left[-\int_0^{y_i} \frac{g_i(\lambda_i)}{h_i(\lambda_i)}\, d\lambda_i\right] \tag{6}$$

and C is the normalization factor.

Equations (5) and (6) will be used in the following sections to analyze a class of nonlinear systems.

II. HYPERBOLIC TANGENT STIFFNESS MODEL

The mechanical system considered in this investigation is a single-degree-of-freedom oscillator with a mass m, a linear viscous damping c, and a nonlinear spring function $F(u)$. When the system is subjected to a base acceleration excitation $\ddot{x}_b(t)$, its response is characterized by the displacement $u(t)$ relative to the base. The equation of motion of the system is

$$\ddot{u} + 2\beta\dot{u} + \mathfrak{F}(u) = a(t) \tag{7}$$

where

$$\beta = \frac{c}{2m}, \qquad \mathfrak{F}(u) = \frac{F(u)}{m},$$

and

$$a(t) = -\ddot{x}_b(t).$$

Let $a(t)$ be a gaussian, stationary white noise with zero mean; that is, with the properties

$$\langle a(t)\rangle = 0$$

$$\langle a(t_1)a(t_2)\rangle = 2S_o\,\delta(t_1 - t_2)$$

where S_o is the constant power spectral density of $a(t)$. Then the associated steady-state Fokker–Planck equation for $\mathbf{u}(t) = [u(t), \dot{u}(t)]$ is

$$S_o \frac{\partial^2}{\partial \dot{u}^2} p(u, \dot{u}) - \frac{\partial}{\partial u}[\dot{u}p(u, \dot{u})] + \frac{\partial}{\partial \dot{u}}\{[2\beta\dot{u} + \mathfrak{F}(u)]\}p(u, \dot{u}) = 0. \tag{8}$$

For this two-dimensional case ($n = 2$), according to equations (5) and (6), the solution can be written down readily,

$$p(u, \dot{u}) = C \exp\left\{-\frac{2\beta}{\pi S_o}\left[\frac{\dot{u}^2}{2} + \int_0^u \mathfrak{F}(\xi)\, d\xi\right]\right\} \tag{9}$$

where C is the normalization factor determined by

$$\iint p(u, \dot{u})\, du\, d\dot{u} = 1.$$

A special kind of softening spring described by a hyperbolic tangent function will now be considered. The force-deflection characteristic is shown in Fig. 1 and given as follows:

$$F(u) = \frac{k_o}{b} \tanh bu, \qquad k_o\,,\, b > 0, \tag{10}$$

where k_o is the initial stiffness, and b is the rate of convergence of the force-deflection curve.

It should be noted that the spring force $F(u)$ developed during the motion is bounded between k_o/b and $-k_o/b$. Therefore k_o/b may be regarded as yielding force and $1/b$ the corresponding yielding displacement. The stiffness function $F(u)$ described in equation (10) then provides a good representation of the elastic-perfect-plastic behavior often encountered in the fields of classical mechanics and structural engineering.

Let $\omega_o^2 = k_o/m$ where ω_o represents the natural frequency of the linear

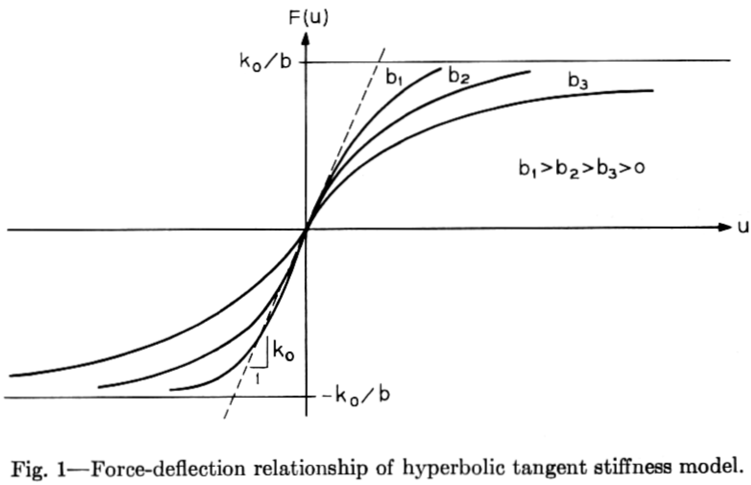

Fig. 1—Force-deflection relationship of hyperbolic tangent stiffness model.

oscillator with stiffness k_o; then substitution of equation (10) into (9) yields

$$p(u, \dot{u}) = C \exp\left\{-\frac{\dot{u}^2}{2\omega_o^2\sigma_o^2} - \frac{1}{b^2\sigma_o^2} \ln \cosh bu\right\} \tag{11}$$

where $\sigma_o^2 = (\pi S_o/2\beta\omega_o^2)$ is the variance of the linear response [that is, if $F(u) = k_o u$].

Equation (11) shows that u and $\dot{u}$ are statistically independent. The probability density function for velocity $\dot{u}$ is normal with zero mean and variance $\sigma_o^2\omega_o^2$, that is,

$$p(\dot{u}) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_o\omega_o} \exp\left(-\frac{\dot{u}^2}{2\sigma_o^2\omega_o^2}\right), \qquad -\infty < u < \infty. \tag{12}$$

The probability density function for the displacement u is

$$p(u) = C_1(b)[\text{sech } bu]^{1/\sigma_o^2 b^2} \tag{13}$$

where

$$C_1(b) = \left(\int_{-\infty}^{\infty} \text{sech}^{1/\sigma_o^2 b^2}\, b\xi\, d\xi\right)^{-1}. \tag{14}$$

Because $(\text{sech } b\xi)^{1/b^2\sigma_o^2}$ converges to zero very rapidly as $\xi \to \infty$, $C_1(b)$ in equation (14) can be evaluated numerically for any positive b. If $1/b^2\sigma_o^2$ is an integer, equation (14) then becomes

$$C_1(b) = \frac{b}{2^D(D-1)!} \prod_{k=0}^{D-1} (2D - 2k - 1) \tag{15}$$

where $2D = 1/b^2\sigma_o^2$ are integers.[6] It is interesting to see that, if tanh bu is expanded into a power series, equation (13) then becomes

$$p(u) = C_1(b) \exp\left[-\frac{1}{2\sigma_o^2}\left(u^2 - \frac{b^2}{6}u^4 + \cdots\right)\right], \qquad |u| \leqq \frac{\pi}{2b},$$

which indicates that a cubic softening spring with nonlinear coefficient $k_o b^2/3$ is the first approximation of the hyperbolic tangent spring.

Values of $p(u)$ given by equation (13) for various $1/b^2\sigma_o^2$ are shown in Figs. 2 and 3.

III. LIMITING SITUATIONS OF $p(u)$

In connection with the examination of the limiting behaviors of $p(u)$ in equation (13), where the parameter b approaches zero and infinity alternately, three useful theorems are presented.

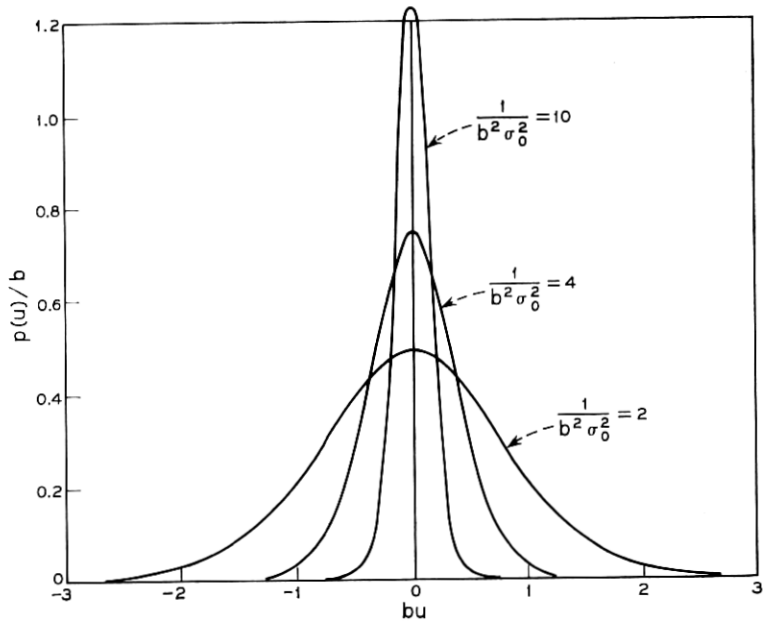

Fig. 2—Sech-power probability density distributions.

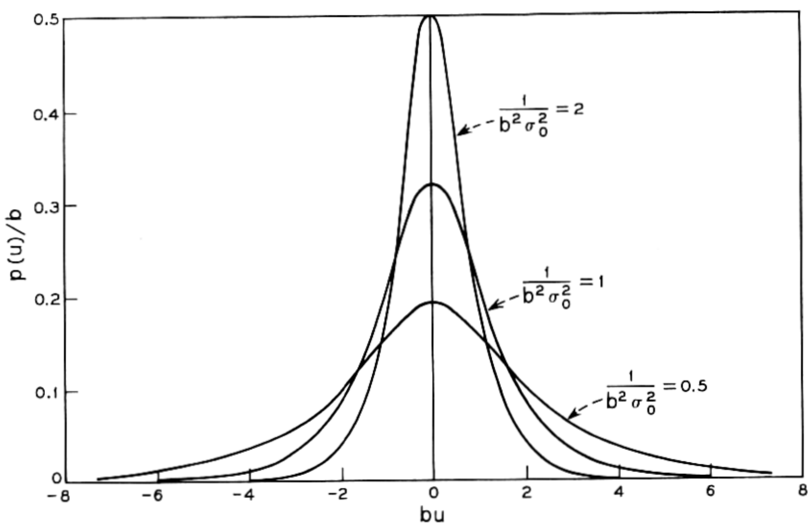

Fig. 3—Sech-power probability density distributions.

Theorem 1: *Let $f_n(x)$ be a sequence of nonnegative density functions integrable on $[-\infty, \infty]$. Suppose there exists a sequence of positive integrable $g_n(x)$ such that*

$$g_n(x) \geqq F_n(x) = f_n(x) \Big/ \int_{-\infty}^{\infty} f_n(s)\, ds$$

and

$$\lim_{n\to\infty} \left[\int_{\epsilon}^{\infty} g_n(x)\, dx + \int_{-\infty}^{-\epsilon} g_n(x)\, dx \right] = 0 \quad \text{for every} \quad \epsilon > 0.$$

Then

$$\lim_{n\to\infty} F_n(x) = \delta(x), \tag{16}$$

the Dirac delta function.

Proof: We must show that, for every h in $C_o^\infty(R)$, the space of test functions

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(x) h(x)\, dx = h(0).$$

By the mean value theorem the following relationship holds:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(x) h(x)\, dx$$

$$= \lim_{n\to\infty} \int_{\epsilon}^{\infty} F_n(x) h(x)\, dx + \lim_{n\to\infty} \int_{-\infty}^{-\epsilon} F_n(x) h(x)\, dx + \lim_{n\to\infty} h(\xi) \cdot \int_{-\epsilon}^{\epsilon} F_n(x)\, dx$$

where ξ is some member of $[-\epsilon, \epsilon]$, depending on ϵ and n. The first two limits on the right side of the previous equation are zero by a comparison test; therefore, one can show that

$$\lim_{n\to\infty} \int_{-\epsilon}^{\epsilon} F_n(x)\, dx = 1.$$

Then

$$\lim_{n\to\infty} h(\xi) = \lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(x) h(x)\, dx.$$

But the right side is independent of ϵ. Thus, letting ϵ approach zero, we deduce that

$$h(0) = \lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(x)h(x)\,dx,$$

which completes the proof.

Instead of considering $p(u)$ of equation (13), we shall investigate its more general form as

$$p_{(b,\alpha)}(u) = \frac{[\text{sech}\,(bu)]^{1/b^\alpha A}}{\int_{-\infty}^{\infty} [\text{sech}\,(by)]^{1/b^\alpha A}\,dy},$$

which is the steady-state displacement density function corresponding to a generalized hyperbolic tangent stiffness function

$$F(u) = \frac{k_o}{b^{\alpha-1}} \tanh bu \tag{17}$$

if $A = \sigma_o^2$.

Theorem 2: *Let* $p_\alpha(u) = \lim_{b\to\infty} p_{(b,\alpha)}(u)$, *then*

(*i*) $\alpha > 1$ *implies* $p_\alpha(u) = 0$,

(*ii*) $\alpha = 1$ *implies* $p_\alpha(u) = \left(\frac{1}{2A}\right) e^{-|u|/A}$,

and

(*iii*) $\alpha < 1$ *implies* $p_\alpha(u) = \delta(u)$.

Proof: First suppose $\alpha > 1$. We observe that

$$|\,p_{(b,\alpha)}(u)\,| \leqq \frac{1}{\int_{-\infty}^{\infty} [\text{sech}\,(by)]^{1/b^\alpha A}\,dy} \quad \text{for all} \quad u,$$

but

$$[\text{sech (by)}]^{1/b^\alpha A} \geqq \exp\,(-|\,y\,|/b^{\alpha-1}A),$$

thus

$$\int_{-\infty}^{\infty} [\text{sech}\,(by)]^{1/b^\alpha A}\,dy \geqq \int_{-\infty}^{\infty} \exp\,(-|\,y\,|/b^{\alpha-1}A)\,dy = 2b^{\alpha-1}A.$$

Thus, since

$$|\,p_{(b,\alpha)}(u)\,| \leqq 1/(2b^{\alpha-1}A) \tag{18}$$

for all u, we conclude that $\lim_{b\to\infty} p_{(b,\alpha)}(u) = 0$.

Now suppose that $\alpha = 1$. Then

$$\lim_{b\to\infty} \ln [\operatorname{sech}(bu)]^{1/bA} = \lim_{b\to\infty} (1/bA) \ln (\operatorname{sech} bu)$$
$$= \lim_{b\to\infty} (-1/bA) \ln (\cosh bu)$$
$$= \lim_{b\to\infty} [-\tanh (bu)/A] = -| u |/A.$$

Thus,

$$\lim_{b\to\infty} [\operatorname{sech}(bu)]^{1/bA} = \exp(-| u |/A).$$

By the Lebesgue dominated-convergence theorem

$$\lim_{b\to\infty} \int_{-\infty}^{\infty} [\operatorname{sech}(by)]^{1/bA}\, dy = \int_{-\infty}^{\infty} \exp(-| y |/A)\, dy = 2A.$$

Thus,

$$\lim_{b\to\infty} p_{(b,1)}(u) = (2A)^{-1} \exp(-| u |/A).$$

Finally, we suppose that $\alpha < 1$. Let

$$g_{(b,\alpha)}(u) = \frac{2^{1/b^\alpha A} \exp(-| u |\, b^{(1-\alpha)}/A)}{\int_{-\infty}^{\infty} \exp[-(b^{(1-\alpha)}\ | y |)/A]\, dy}$$

or, equivalently,

$$g_{(b,\alpha)}(u) = \left(\frac{2^{1/b^\alpha A} b^{(1-\alpha)}}{2A}\right) \exp(-| u |\, b^{(1-\alpha)}/A).$$

Then, since

$$\lim_{b\to\infty} \int_{\epsilon}^{\infty} g_{(b,\alpha)}(y)\, dy = \lim_{b\to\infty} \int_{-\infty}^{-\epsilon} g_{(b,\alpha)}(y)\, dy = 0$$

for every $\epsilon > 0$ and

$$g_{(b,\alpha)}(u) \geqq p_{(b,\alpha)}(u) \quad \text{for every} \quad u,$$

we conclude from Theorem 1 that

$$\lim_{b\to\infty} p_{(b,\alpha)}(u) = \delta(u),$$

and the proof is completed.

Theorem 3: *Let* $p_\alpha'(u) = \lim_{b\to 0} p_{(b,\alpha)}(u)$, *then* (*i*) $\alpha > 2$ *implies* $p'_\alpha(u) = \delta(u)$,

(*ii*) $\alpha = 2$ *implies* $p'_\alpha(u)$ = *normal distribution with variance* σ_o^2, and

(*iii*) $\alpha < 2$ *implies* $p'_\alpha(u) = 0$ for all u.

Proof: The case $\alpha > 2$ implies $p'_\alpha(u) = \delta(u)$ is proved in Appendix A in which we also show that there exists for every $\gamma \in (0, 1)$ a $C_\gamma > 0$ such that

$$g_b(u) = C_\gamma \exp\left(-\frac{u^2}{b^{\alpha-2}A}\right)(2b^{\alpha-1}A)$$

$$\geqq \frac{[\operatorname{sech}(bu)]^{1/b^\alpha A}}{\int_{-\infty}^{\infty}[\operatorname{sech}(by)]^{1/b^\alpha A}\,dy} \quad \text{for} \quad |u| \leqq \frac{\gamma\pi}{2b}.$$

It follows from the above that

$$p'_\alpha(u) = 0 \quad \text{for} \quad 1 < \alpha < 2.$$

From equation (18) we immediately have $p'(u) = 0$ for $\alpha < 1$. Now we have only to consider the cases when $\alpha = 2$ and $\alpha = 1$. In Appendix B we show that when $\alpha = 2$, $p'_\alpha(u)$ is a normal distribution with variance σ_o^2. In Appendix C we show that when $\alpha = 1$, $p'_\alpha(u) = 0$ for all u. Therefore the proof is completed.

According to Theorem 3, for $p(u)$ given by equations (13) and (14), it follows that

$$\lim_{b\to 0} p(u) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_o} \exp\left[-\frac{u^2}{2\sigma_o^2}\right], \qquad \text{a normal distribution,}$$

and according to Theorem 2

$$\lim_{b\to\infty} p(u) = 0. \tag{20}$$

At the limit $b \to \infty$, the yielding force $k_o/b \to 0$, that is, the system becomes perfect plastic. Thus one may expect an equal probability for all u on $[-\infty, \infty]$, as equation (20) indicated. As $b \to 0$, then $k_o/b \to \infty$, the system remains elastic on $[-\infty, \infty]$ with the initial stiffness k_o. It is well known that for linear systems the response probability distribution is gaussian, which agrees with the result of equation (19).

It is of interest to note that a similar force-deflection relationship as shown in Fig. 1 and as described by the hyperbolic tangent stiffness function given in equation (10) can be described by a full-wave smooth limiter which is

$$G(u) = k_0 \int_0^u \exp\left(-\frac{\eta^2}{2d^2}\right) d\eta$$

in which $d^2 = 2/\pi b^2$.

It will be noted that in the above equation, $G(u)$ is proportional to the integral of a gaussian probability curve. Function $G(u)$ can also be used to evaluate the probability density function if made equivalent to $F(u)$ as given by equation (10) when both $G(u)$ and $F(u)$ have the same initial slope and spring resistance limits.

IV. OTHER IMPORTANT RESPONSE STATISTICS

The failure modes of a mechanical system are generally controlled by response parameters such as the mean square displacement, zero crossings, or the peak displacement distributions. These response statistics are closely related to $p(u)$ and will be briefly discussed.

The mean value of displacement response u vanishes because $p(u)$ in equation (13) is an even function. The mean square or the variance of the displacement is given by

$$\begin{aligned} \sigma_u^2(b) = \langle u^2 \rangle &= \int_{-\infty}^{\infty} u^2 p(u)\, du \\ &= 2C_1(b) \int_0^{\infty} u^2 \operatorname{sech}^{1/b^2\sigma_0^2} bu\, du, \end{aligned} \tag{21}$$

which can be evaluated in the following manner*: Let

$$J(a) = \int_{-\infty}^{\infty} \frac{e^{2ax}\, dx}{(\cosh x)^{2\nu}},$$

then it can be shown[6] that

$$J(0) = \frac{(\pi)^{\frac{1}{2}}\Gamma(\nu)}{\Gamma(\nu + \frac{1}{2})},$$

and

$$\int_{-\infty}^{\infty} \frac{x^2\, dx}{(\cosh x)^{2\nu}} = \left[\frac{1}{4}\frac{\partial^2}{\partial a^2} J(a)\right]_{a=0} = \tfrac{1}{2} J(0)\psi'(\nu),$$

where $\psi'(\nu) = (d/d\nu)[\Gamma'(\nu)/\Gamma(\nu)]$ is the "trigamma" function and has been numerically tabulated.[7]

* This is pointed out by S. O. Rice.

From the above results and setting $x = bu$ and $\nu = b\sigma_o$ in equation (21) we finally obtain

$$\sigma_u^2(b) = \frac{1}{2b^3} C_1(b) J(0) \psi'(b\sigma_o). \tag{22}$$

Again according to Theorem 3, it is noted that

$$\lim_{b \to 0} \sigma_u^2(b) = \sigma_o^2 ,$$

and from Theorem 2 that

$$\lim_{b \to \infty} \sigma_u^2(b) = \infty .$$

Thus the mean square response $\sigma_u^2(b)$ with such limiting behavior can be illustrated as in Fig. 4.

The expected number of zero crossings ν_o^+ with positive slope per unit time (that is, the expected frequency) can be evaluated according to Rice,[8]

$$\nu_o^+(b) = \int_0^\infty \dot{u} p(0, \dot{u}) \, d\dot{u} = \frac{C_1(b)}{2} \left(\frac{\pi}{\beta}\right)^{\frac{1}{2}} \tag{23}$$

where $C_1(b)$ is given by equation (14).

Also according to Theorem 3, it can be shown that

$$\lim_{b \to 0} \nu_o^+(b) = \frac{\omega_o}{2\pi} , \tag{24}$$

which is the frequency of the linear system.

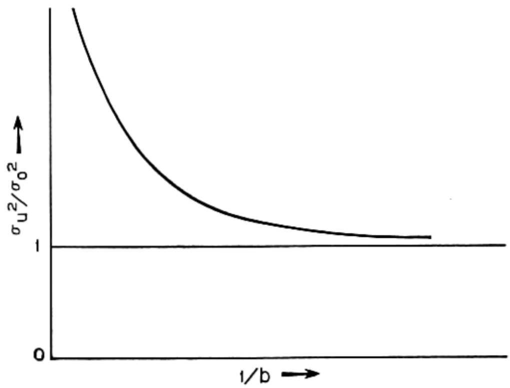

Fig. 4—Variation of mean-square displacement response.

The probability density of the peak amplitude of $u(t)$, from equation (13), is given by

$$p(a) = -\frac{\left|\frac{dp(u)}{du}\right|_{u=a}}{[p(u)]_{u=0}}$$

$$= \frac{1}{\sigma_o^2 b} (\operatorname{sech} ba)^{1/\sigma_o^2 b^2} \tanh ba. \tag{25}$$

By the same argument used in the proof of Theorem 2 it can be shown that

$$\lim_{b \to \infty} p(a) = \delta(a), \tag{26}$$

and it follows from Theorem 3 that

$$\lim_{b \to 0} p(a) = \lim_{b \to 0} \left(\frac{\tanh ba}{\sigma_o^2 b}\right) \lim_{b \to 0} \operatorname{sech}^{1/\sigma_o^2 b^2} ba$$

$$= \frac{a}{\sigma_o^2} \exp\left(\frac{-a^2}{2\sigma_o^2}\right), \tag{27}$$

which is the Rayleigh distribution as expected because at this limit ($b \to 0$) the system becomes linear. The peak probability density distribution $p(a)$ for various b in equation (25) is illustrated in Fig. 5. Notice that for all cases $p(a)$ approaches zero at large a; however, the rate of fall of $p(a)$ is reduced as b is increased.

It should be noted that when $\alpha = 1$, the forcing function described by equation (17) approaches a sgn function as $b \to \infty$, that is

$$\lim_{b \to \infty} (k_o \tanh bu) = k_o \operatorname{sgn} u.$$

Therefore, by taking appropriate limits to the density function previously obtained for second-order systems with a general hyperbolic tangent forcing function, we obtain the steady-state solution for the response density of systems governed by the following equation

$$\ddot{u} + 2\beta\dot{u} + k_o \operatorname{sgn} u = a(t).$$

The response density for the above equation is given precisely in statement (*ii*) of Theorem 2, which can also be verified by using equation (9).

V. ACKNOWLEDGMENTS

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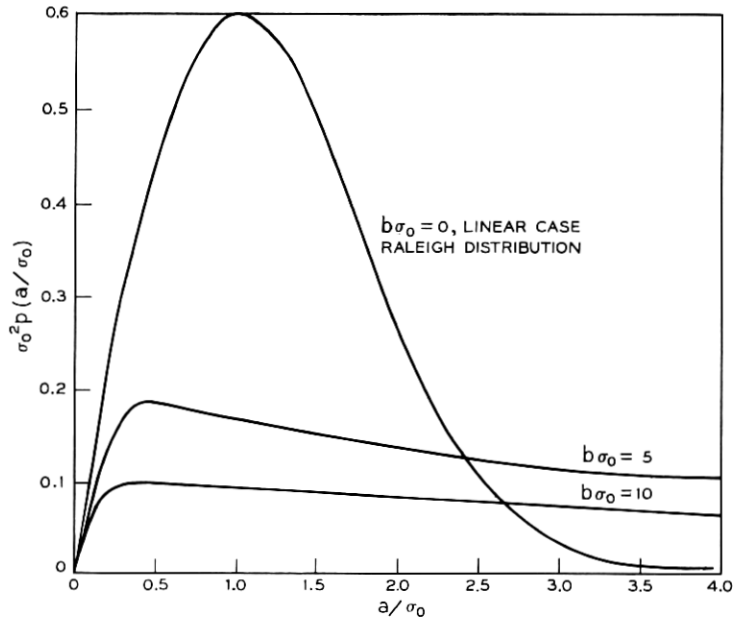

Fig. 5—Peak probability density distributions.

APPENDIX A

Partial Proof of Theorem 3 for Case $\alpha > 2$

We claim that if $\alpha > 2$, then

$$\lim_{b\to 0} \frac{\operatorname{sech}(bu)^{1/b^{\alpha}A}}{\int_{-\infty}^{\infty} \operatorname{sech}(by)^{1/b^{\alpha}A}\, dy} = \delta(u).$$

Proof: In view of Theorem 2 we have only to find functions

$$g_b(u) \geqq \frac{\operatorname{sech}(bu)^{1/b^{\alpha}A}}{\int_{-\infty}^{\infty} \operatorname{sech}(by)^{1/b^{\alpha}A}\, dy}$$

such that for every $\epsilon > 0$

$$\lim_{b\to 0} \int_{\epsilon}^{\infty} g_b(u)\, du = \lim_{b\to 0} \int_{-\infty}^{-\epsilon} g_b(u)\, du = 0$$

and such that $\sup \int_{-\infty}^{\infty} g_b(u)\, du < \infty$. We write

$$\operatorname{sech}(by)^{1/b^\alpha A} = \exp\{-\ln[\cosh(by)]/b^\alpha A\}.$$

We observe that if $|by| < \pi/2$, then

$$\ln[\cosh(by)]/b^\alpha A = \sum_{k=0}^{\infty} (1/b^\alpha A)(1/k!)\left[\lim_{\beta\to 0}\left(\frac{d}{d\beta}\right)^k \ln\cosh(\beta y)\right]b^k$$
$$= \sum_{k=2}^{\infty} (y/b^\alpha Ak!)\left[\lim_{\beta\to 0}\left(\frac{d}{d\beta}\right)^{k-1} \tanh(\beta y)\right]b^k.$$

We now make use of the fact that $|by| < 1$ implies

$$\tanh(by) = \sum_{n=1}^{\infty} \beta_n(-1)^{n+1}2^{2n}(2^{2n}-1)\frac{(by)^{2n-1}}{(2n)!}.$$

Thus, there is for every $\gamma \in (0, 1]$ a $C_\gamma > 0$ such that $|bu| < \gamma\pi/2$ implies

$$\operatorname{sech}(bu)^{1/b^\alpha A} \leqq C_\gamma \exp(-u^2/b^{\alpha-2}A).$$

Thus, since

$$\int_{-\infty}^{\infty}\left(\frac{2}{\exp(by)+\exp(-by)}\right)^{1/b^\alpha A} dy \geqq 2\int_0^{\infty}\left(\frac{2}{2\exp(by)}\right)^{1/b^\alpha A} dy$$
$$= 2b^{\alpha-1}A,$$

we therefore take

$$g_b(u) = C_\gamma \exp(-u^2/b^{\alpha-2}A)2b^{\alpha-1}A \quad \text{for} \quad |u| < \gamma\pi/2b$$

and

$$g_b(u) = 2b^{\alpha-1}A \operatorname{sech}(bu)^{1/b^\alpha A} \quad \text{for} \quad |u| > \gamma\pi/2b.$$

Note that

$$\int_{\gamma\pi/2b}^{\infty} g_b(u)\,du \leqq 2b^{\alpha-1}A \sum_{k=1}^{\infty}\left(\frac{k\gamma\pi}{2b}\right)\operatorname{sech}\left(\frac{bk\gamma\pi}{2b}\right)^{1/b^\alpha A}$$
$$= b^{\alpha-2}A\sum_{k=1}^{\infty}(k\gamma\pi)\operatorname{sech}\left(\frac{k\gamma\pi}{2}\right)^{1/b^\alpha A}$$
$$< C_2 2b^{\alpha-2}A.$$

Thus, for $\alpha > 2$

$$\lim_{b\to 0}\int_\epsilon^\infty g_b(u)\,du = \lim_{b\to 0}\int_\epsilon^{\gamma\pi/2b} C_\gamma \exp(-u^2/b^{\alpha-2}A)2b^{\alpha-1}A\,du$$
$$+ \lim_{b\to 0} 2b^{\alpha-2}A\sum_{k=1}^{\infty}(k\gamma\pi)\operatorname{sech}\left(\frac{k\gamma\pi}{2}\right)^{1/b^\alpha A}$$

$$= \lim_{b \to 0} \frac{\gamma\pi}{2b} C_\gamma \exp(-\epsilon^2/b^{\alpha-2}A) 2b^{\alpha-1}A + 0$$

$$= 0.$$

Also, we observe that

$$\lim_{b \to 0} \int_{-\infty}^{-\epsilon} g_b(u)\, du = \lim_{b \to 0} \int_{\epsilon}^{\infty} g_b(u)\, du = 0.$$

APPENDIX B

Partial Proof of Theorem 3 for Case $\alpha = 2$

Let $p_b(x) = \theta(b)(\text{sech } bx)^{1/b^2\sigma_o^2}$, $b \geqq 0$ for all x on $[-\infty, \infty]$. We will first show $\lim_{b\to 0} \theta(b) = 0$, then $\lim_{b \to 0} p_b(x)$ converges pointwisely to a normal distribution with zero mean and variance σ_o^2.

Proof: From the definition of $p_b(x)$, it follows that

$$p_b(x) = \{\exp [\ln \theta(b) b^2\sigma_o^2](\text{sech } bx)\}^{1/b^2\sigma_o^2}$$

$$\ln p_b(x) = \ln \theta(b) + \frac{\ln \text{sech } bx}{b^2\sigma_o^2}.$$

Then

$$\lim_{b \to 0} \frac{1}{b^2\sigma_o^2} \ln \text{sech } bx = \lim_{b \to 0} \frac{-x \tanh bx}{2b\sigma_o^2}$$

by L'Hopital's rule. Thus, since

$$\lim_{b \to 0} \frac{\tanh bx}{b} = x,$$

we conclude that

$$\lim_{b \to 0} \ln p_b(x) = -\frac{x^2}{2\sigma_o^2},$$

or that

$$\lim_{b \to 0} p_b(x) = \exp\left(-\frac{x^2}{2\sigma_o^2}\right), \qquad \text{the normal distribution.}$$

By using these expressions and equation (17) we can conclude that $p_\alpha'(u)$ is a normal distribution with variance σ_o^2 when $\alpha = 2$.

APPENDIX C

Partial Proof of Theorem 3 for Case $\alpha = 1$

We claim that, when

$$p_{(b,1)}(u) = \frac{\operatorname{sech}(bu)^{1/bA}}{\int_{-\infty}^{\infty} \operatorname{sech}(by)^{1/bA}\,dy},$$

then

$$\lim_{b\to 0} p_{(b,1)}(u) = 0 \quad \text{for all} \quad u.$$

Proof: We observe that

$$p_{(b,1)}(u) \leqq \frac{\operatorname{sech}(bu)^{1/bA}}{\int_{-N}^{N} \operatorname{sech}(by)^{1/bA}\,dy}$$

for all positive integers N.

We show that for every $N > 0$ there is a $\mu > 0$ such that $b < \mu$ implies

$$p_{(b,1)}(u) \leqq 1/2N.$$

We can show, using L'Hopital's rule, that

$$\lim_{b\to 0} \operatorname{sech}(bu)^{1/bA} = 1$$

for all u and A. Thus, for every $\eta > 0$ no matter how small and every $N' \geqq N(1 + \eta)/(1 - \eta)$ we can show that there is a $\mu > 0$ such that by taking $b < \mu$, we have

$$p_{(b,1)}(u) \leqq \frac{1 + \eta}{\int_{-N'}^{N'} (1 - \eta)\,d\eta}.$$

Thus, $b < \mu$ implies

$$p_{(b,1)}(u) \leqq \left(\frac{1 + \eta}{1 - \eta}\right)\left(\frac{1}{2N'}\right) \leqq \frac{1}{2N}.$$

Thus, for every $N > 0$ there is a $\mu > 0$ such that $0 < b < \mu$ implies

$$p_{(b,1)}(u) \leqq \frac{1}{2N}.$$

Hence

$$\lim_{b \to 0} p_{(b,1)}(u) = 0.$$

The proof is completed.

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