Analysis of a Burst-Trapping Error Correction Procedure

By W. K. PEHLERT, JR.

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This paper presents an analysis technique for determining upper and lower bounds on the performance of a burst-trapping error control procedure. The analysis is valid for random, burst, or compound channels provided that a block interleaving degree $\ell \geqq 1$ can be found for the channel such that error patterns in blocks spaced ℓ blocks apart occur approximately independently. Good agreement is achieved between the theoretical performance of codes on telephone channels and the performance obtained by computer simulation.

I. INTRODUCTION

A burst-trapping error correction procedure for error control on compound channels such as the telephone channel has been described by S. Y. Tong.[1] An evaluation of the burst-trapping procedure by computer simulation of its performance on recorded telephone channel error data is presented in the companion paper.[2] An analysis technique for determining the performance of the burst-trapping procedure is presented here. The analysis technique permits determination of upper and lower bounds on the probability of block error. It is valid for random, burst, or compound channels provided that a block interleaving degree $\ell \geqq 1$ can be found for the channel such that error patterns in blocks spaced ℓ blocks apart occur (approximately) independently.

Expressions for upper and lower bounds on the probability of block error for codes of rate $\frac{1}{2}$ and $\frac{2}{3}$ are given in Table I. For higher rate codes, numerical determination of the stationary probabilities which yield the bounds seems preferable to determination of expressions for the bounds. The performance of a rate $\frac{2}{3}$ code, the (39, 26) shortened BCH code, has been computed using the expressions for probability of block error of Table I for two sets of recorded telephone channel error data. For both sets of data the computed performance agrees well with the performance obtained by computer simulation.

TABLE I—BOUNDS ON BLOCK ERROR PROBABILITY FOR RATE 1/2 AND RATE 2/3 CODES

RATE $\frac{1}{2}$ Codes

$$P_{eg} = \frac{\pi_1}{p_0 + p_c} \{p_d[1 - (p_0 + p_w)] + (K + p_u + p_w)\},$$

where $$\pi_1 = \frac{p_0 + p_c}{1 + p_d}.$$

$$P_{eu1} = \frac{\rho_1}{p_0 + p_c} \left\{p_d[1 - (p_0 + p_w)] + (K + p_u + p_w)\left(\frac{1 + p_0}{p_0^2}\right)\right\},$$

where $$\rho_1 = \frac{p_0 + p_c}{1 + p_d + \dfrac{1 + p_0}{p_0^2}(K + p_u + p_w)}.$$

$$P_{eu2} = P_{eu1}.$$

$$P_{eL} = \frac{\rho_1}{p_0 + p_c} \{p_d[1 - (p_0 + p_w)] + (K + p_u + p_w)\},$$

where $$K = p_d\{1 - [p_0 + \alpha(p_c + p_u + p_d)]\}.$$

RATE $\frac{2}{3}$ Codes, $t = 1$, $d_m \geqq 4$

$$P_{eg} = \frac{\pi_1}{p_0 + p_c} \{p_d[1 - (p_0 + p_w)^2] + (2K + p_u + p_w)\},$$

where $$\pi_1 = \frac{p_0 + p_c}{1 + 2p_d}.$$

$$P_{eu1} = \frac{\rho_1}{p_0 + p_c} \cdot \left\{p_d[1 - (p_0 + p_w)^2] + (2K + p_u + p_w) + \frac{D(1 - 2p_0^5 + p_0^6)}{p_0^5(1 - p_0)}\right\},$$

where $$\rho_1 = \frac{p_0 + p_c}{1 + 2p_d + \dfrac{D(1 - p_0^5)}{p_0^5(1 - p_0)}}.$$

$$P_{eu2} = \frac{\rho_1}{p_0 + \rho_c} \cdot \left\{p_d[1 - (p_0 + p_w)^2] + (2K + p_u + p_w) + \frac{D(1 - 4p_0^5 + 3p_0^6)}{p_0^5(1 - p_0)}\right\}$$

where $$s_1 = p_1.$$

$$P_{eL} = \frac{\rho_1}{p_0 + p_c} \{p_d[1 - (p_0 + p_w)^2] + (2K + p_u + p_w)\},$$

where $$D = p_u + p_w + p_d\{1 - [p_0 + \alpha(p_c + p_u + p_d)]^2\}.$$

II. ENCODING AND BURST-TRAPPING DECODING PROCEDURES

The encoding procedure and burst-trapping decoding technique have been described in detail elsewhere.[1] The code is a rate $(b-1)/b$ recurrent code[3] whose parity check matrix, A, is constructed from the parity check matrix, H, of an (n, k) linear systematic block code [where $k/n = (b-1)/b$] and the $(n-k) \times (n-k)$ identity matrix I. The truncated parity check matrix, A_N, for rate $\frac{2}{3}$ codes is given in Fig. 1. The constraint length of the code is $N = [(b-1)\ell + 1]n$ where $\ell \geqq 1$ is a block interleaving constant.

Although encoding is specified by the A matrix, it is useful to interpret encoding as the interleaved encoding of ℓ subcodes in the following way. Blocks $0, \ell, 2\ell, 3\ell, \cdots$ form the first subcode. The k-tuple of information bits, $I_{i\ell}$, of the $[i\ell]$th block ($i = 0, 1, 2, \cdots$) is encoded into an n-tuple $M_{i\ell}$ which is the concatenation of $I_{i\ell}$ and a parity $(n-k)$-tuple $Q_{i\ell}$:

$$M_{i\ell} = I_{i\ell} \,||\, Q_{i\ell}. \tag{1}$$

$I_{i\ell}$ can be represented as the concatenation of $(b-1)$ equal length segments:

$$I_{i\ell} = I_{i\ell}^1 \,||\, I_{i\ell}^2 \,||\, \cdots \,||\, I_{i\ell}^{b-1}. \tag{2}$$

The parity $(n-k)$-tuple $Q_{i\ell}$ is

$$Q_{i\ell} = P_{i\ell} + I_{(i-1)\ell}^1 + I_{(i-2)\ell}^2 + \cdots + I_{(i-b+1)\ell}^{b-1} \tag{3}$$

where $P_{i\ell}$ is the parity $(n-k)$-tuple obtained by encoding $I_{i\ell}$ with the block code parity check matrix H. Thus each encoded subcode block is a block code word whose parity portion is modified by the addition of an information segment from each of the previous $(b-1)$ subcode blocks. Further the jth information segment of the $[i\ell]$th block is added to the information portion of the $[(i+j)\ell]$th block ($j = 1, 2, \cdots, b-1$).

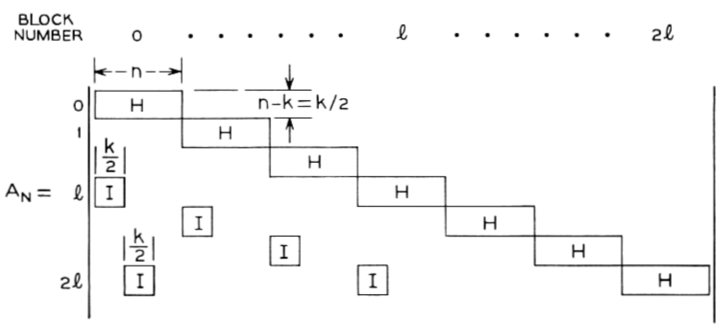

Fig. 1—A_N matrix of a rate 2/3 code.

A given block of a subcode is decoded by the burst-trapping procedure in one of three ways. Successive blocks of a subcode (blocks $0, \ell, 2\ell, \cdots$) are decoded by Random Error Decoding (RED) as long as the decoder decides that t or fewer errors occur in each block. When the decoder decides that more than t errors occur in a block, for example block $[i\ell]$, then that block is decoded by Burst-Trapping Decoding (BTD). The next $(b - 1)$ subcode blocks $\{$blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$, $[(i + b - 1)\ell]\}$ are decoded by Blind Faith Decoding (BFD). Detailed descriptions of RED, BTD, and BFD are given below. Successive subcode blocks $\{$blocks $[(i + b)\ell]$, $[(i + b + 1)\ell]$, $\cdots\}$ are decoded by RED until the decoder again decides that a block has more than t errors.

Let $M^*_{i\ell}$, $I^*_{i\ell}$, $Q^*_{i\ell}$ be the received n-tuple at the $[i\ell]$th block, the received information k-tuple portion, and the received parity $(n - k)$-tuple portion respectively. Let $P^*_{i\ell}$ be the parity $(n - k)$-tuple obtained by encoding $I^*_{i\ell}$ with the block code parity check matrix H. Let $\bar{I}_{i\ell}$ be the decoded information k-tuple of the $[i\ell]$th block regardless of how the decoding is accomplished.

The primary method of decoding is RED. RED is attempted at the $[i\ell]$th block if none of blocks $[(i - b + 1)\ell]$, $[(i - b + 2)\ell]$, $\cdots$, $[(i - 1)\ell]$ are decoded by BTD. RED is the removal of the effects of previous block information segments from the parity portion and the decoding of the block by bounded distance decoding of t or fewer errors. That is, to decode the $[i\ell]$th block the decoder computes

$$P'_{i\ell} = Q^*_{i\ell} + \bar{I}^1_{(i-1)\ell} + \bar{I}^2_{(i-2)\ell} + \cdots + \bar{I}^{b-1}_{(i-b+1)\ell} \tag{4}$$

and decodes $I^*_{i\ell} \,||\, P'_{i\ell}$ as a block code word. Following standard terminology for recurrent codes we will say that $\bar{I}^1_{(i-1)\ell}$, $\bar{I}^2_{(i-2)\ell}$, $\cdots$, $\bar{I}^{b-1}_{(i-b+1)\ell}$ are "fed back" to the $[i\ell]$th block.

The secondary method of decoding is BTD. BTD is effected when a RED attempt indicates that the block code word is detected in error but cannot be corrected by bounded distance decoding. If this occurs in block $[i\ell]$ then blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$, $[(i + b - 1)\ell]$ are necessarily decoded by BFD (to be described subsequently) and decoding of block $[i\ell]$ is delayed until block $[(i + b - 1)\ell]$ has been decoded. The information segments of block $[i\ell]$ are obtained from the corresponding parity portion of blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$, $[(i + b - 1)\ell]$ as follows

$$\bar{I}^1_{i\ell} = Q^*_{(i+1)\ell} + P^*_{(i+1)\ell} + \bar{I}^2_{(i-1)\ell} + \bar{I}^3_{(i-2)\ell} + \cdots + \bar{I}^{b-1}_{(i-b+2)\ell}\,; \tag{5}$$

$$\bar{I}^2_{i\ell} = Q^*_{(i+2)\ell} + P^*_{(i+2)\ell} + \bar{I}^1_{(i+1)\ell} + \bar{I}^3_{(i-1)\ell} + \cdots + \bar{I}^{b-1}_{(i-b+3)\ell}\,; \tag{6}$$

$$\bar{I}^3_{i\ell} = Q^*_{(i+3)\ell} + P^*_{(i+3)\ell} + \bar{I}^1_{(i+2)\ell} + \bar{I}^2_{(i+1)\ell} + \bar{I}^4_{(i-1)\ell} + \cdots + \bar{I}^{b-1}_{(i-b+4)\ell} ; \quad (7)$$

$$\vdots$$

$$\bar{I}^{b-1}_{i\ell} = Q^*_{(i+b-1)\ell} + P^*_{(i+b-1)\ell} + \bar{I}^1_{(i+b-2)\ell} + \bar{I}^2_{(i+b-3)\ell} + \cdots + \bar{I}^{b-2}_{(i+1)\ell} . \quad (8)$$

In equation (6), for example, we will say that $(Q^*_{(i+2)\ell} + P^*_{(i+2)\ell})$ and $\bar{I}^1_{(i+1)\ell}$ are "fed forward" and $\bar{I}^3_{(i-1)\ell}, \cdots, \bar{I}^{b-1}_{(i-b+3)\ell}$ are "fed back" to the $[i\ell]$th block.

The third method of decoding is BFD. BFD is effected at the $[i\ell]$th block if one of blocks $[(i - b + 1)\ell]$, $[(i - b + 2)\ell]$, $\cdots$, $[(i - 1)\ell]$ has been decoded by BTD. BFD of $[i\ell]$th block is the use of the received information k-tuple as the decoded information k-tuple. That is

$$\bar{I}_{i\ell} \equiv I^*_{i\ell} . \quad (9)$$

III. COMMUNICATION CHANNEL

We consider channels where each block word (binary n-tuple) is subjected to the component-wise modulo-two addition of an error pattern (binary n-tuple) during transmission. We define a partition of the set of 2^n possible error patterns as follows.

C_0—is the set of one element—the pattern with no errors.
C_w—is the set of channel error patterns which are identical to nonzero code words.
C_c—is the set of nonzero channel error patterns which are correctable by RED (bounded distance decoding of t or fewer errors).
C_u—is the set of channel error patterns which are both uncorrectable by RED and undetectable (but are not nonzero code words).
C_d—is the set of channel error patterns which are detectable (but not correctable by RED).

We consider channels where error patterns separated by ℓ or more blocks occur independently. On burst or compound channels, proper design of the burst-trapping procedure requires sufficient block interleaving that the requirement of independent error patterns is approximately met. The nth power[4] of the binary symmetric channel with any $\ell \geqq 1$ is included in the class of channels under consideration.

Let p_0, p_w, p_c, p_u, p_d be the probability that the channel error pattern for a particular block is in set C_0, C_w, C_c, C_u, C_d respectively. Let α be the probability that the information bits of a received word are unaffected by the channel error pattern given that the channel error pattern is either correctable, detectable, or undetectable. For the class of communication channels under consideration the probability that the error patterns in successive blocks of a subcode are in sets C_i and C_j is $p_i p_j$, $i, j \epsilon \{0, w, c, u, d\}$.

IV. ANALYSIS OF PERFORMANCE

Since each subcode is independently and identically decoded it suffices to analyze the decoding of one subcode. The decoding of successive blocks of a subcode can be described by a set of states and a set of transition probabilities between states which form a Markov Chain. The set of states is partitioned into a set of "normal" states and a set of "anomalous" states. Given the block code parameters, the sequence of states is determined by the error pattern sequence. The block code parameters which are required are d_m, the minimum distance of the block code, and t, the amount of bounded distance random-error-correction done in RED.

Before enumerating the states we give a general description of the two sets of states. In all following discussion we assume the decoding of a single subcode. A block is decoded in a normal state *only* if all the blocks containing information segments fed back to that block are correctly decoded. Thus a block decoded in a normal state can only be in error due to its own error pattern or due to errors which are fed forward. A block is decoded in an anomalous state if one or more of the blocks containing information segments to be fed back to that block are incorrectly decoded. Once a block is decoded in an anomalous state the decoder is affected by fed back errors in addition to the channel error pattern sequence. Successive blocks are assumed to be decoded in anomalous states until a run of V blocks with no channel errors occurs. V is termed the recovery space for a subcode. V is determined in terms of d_m and t (independent of details of the specific code) so that if no channel errors occur in a period exceeding $(V - 1)$ blocks, then no errors are fed back to subsequent blocks regardless of the previous channel history. The first block after a run of V blocks free of channel errors is decoded in a normal state. A block decoded in an anomalous state can be in error due to its own error pattern, due to errors which are fed forward, and/or errors which are fed back. Error propagation which is due to fed back errors is bounded and can occur only in anomalous states.

Each normal state is numbered by means of the following notation:

The first digit represents the method of decoding

1—Random Error Decoding (RED),
2—Burst-Trapping Decoding (BTD),
3, 4, $\cdots$, $(b + 1)$—Blind Faith Decoding (BFD).

States whose first digit is 1 or 2 have a two digit number. States whose first digit is j, $3 \leqq j \leqq (b + 1)$, have a j digit number. The last digit is 0(1) to designate that the block is correctly (incorrectly) decoded. Interior digits for blocks with $3 \leqq j \leqq (b + 1)$ digit numbers designate the decoding history of the $(j - 2)$ previous blocks. An interior digit is 0(1) to designate that its corresponding block is correctly (incorrectly) decoded. For rate $(b - 1)/b$ codes there are 2^{b+1} normal states. Normal state diagrams illustrating this notation are given in Figs. 2 and 3 for rate $\frac{1}{2}$ and $\frac{2}{3}$ codes respectively. Throughout this paper a rate $\frac{2}{3}$ code will be used in a running example. The analysis technique is easily used for any rate $(b - 1)/b$ code.

Since the error pattern sequence determines the state sequence the transition probabilities between normal states can be expressed in terms of p_0, p_w, p_c, p_u, p_d and α. Transition probabilities between normal states are given in Figs. 2 and 3 and Appendix A for rate $\frac{1}{2}$ and $\frac{2}{3}$ codes. For simplicity of notation in writing the one step transition probabilities, $p_{i,j}$, from state i to state j the normal states are renumbered as in Figs. 2 and 3.

The normal state diagrams represent the possible state transitions until an undetected error occurs (state 11) or a BTD and its associated $(b - 1)$ BFD's occur [states $(b + 1)00 \cdots 0$ to $(b + 1)11 \cdots 1$]. Transitions from state $(b + 1)00 \cdots 0$ or state $(b + 1)10 \cdots 0$ (states marked with ϕ in Figs. 2 and 3) are the same as those from state 10 since the

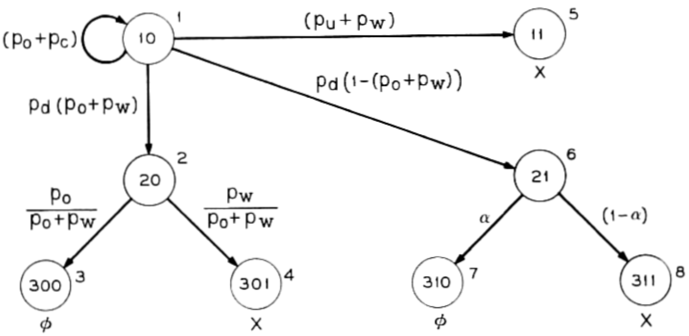

Fig. 2—Normal state diagram for rate 1/2 codes.

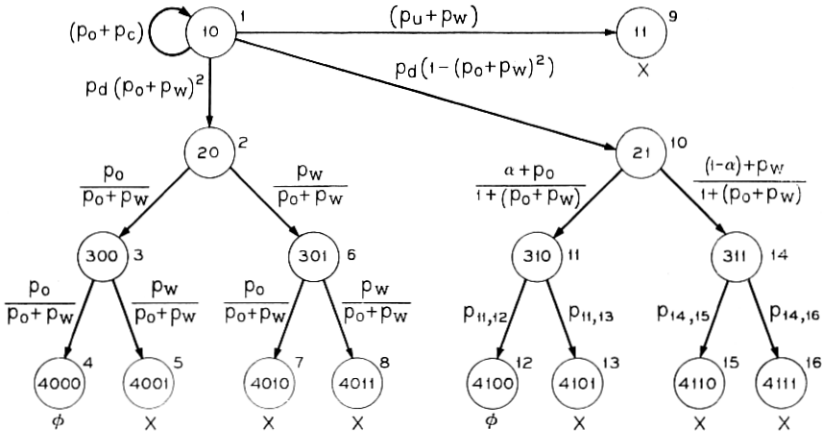

Fig. 3—Normal state diagram for rate 2/3 codes.

last $(b - 1)$ blocks containing information block segments to be fed back are correctly decoded. Transitions from other normal states (marked with X in Figs. 2 and 3) are to anomalous states since one or more of the last $(b - 1)$ blocks containing information block segments to be fed back are incorrectly decoded. The enumeration of the anomalous states is presented later.

V. AN OPTIMISTIC ESTIMATE OF THE PROBABILITY OF BLOCK ERROR

If we assume, for purposes of obtaining an optimistic estimate of performance, the presence of a Genie Decoder which corrects all fed back information, then transitions from normal states marked X (in Figs. 2 and 3) are the same as those from state 10 or the two states marked ϕ. Note that with the Genie Decoder blocks can still be incorrectly decoded and incorrect information can be fed forward but only correct information can be fed back. The assumption of a Genie Decoder in effect eliminates error propagation (which arises from erroneous feedback) and the only possible states are normal states. The entire transition probability matrix $[p_{i,j}]$ is then known from the normal state diagram (for example, Figs. 2 and 3) where $p_{i,j}$ is the one step transition probability from renumbered state i to renumbered state j. The Markov Chain is regular and therefore the stationary probabilities exist.[5] The stationary probability π_i of being in renumbered state i, $i = 1, 2, \cdots, 2^{b+1}$, can be obtained from

$$\pi_i = \sum_{j=1}^{2^{b+1}} \pi_j p_{j,i} \tag{10}$$

and

$$\sum_{i=1}^{2^{b+1}} \pi_i = 1. \tag{11}$$

The probability of block error with a Genie Decoder, P_{eg}, is the probability of being in a normal state whose identifying number has a last digit of 1. For rate $\frac{2}{3}$ codes,

$$P_{eg} = \pi_5 + \pi_6 + \pi_8 + \pi_9 + \pi_{10} + \pi_{13} + \pi_{14} + \pi_{16}. \tag{12}$$

Expressions for P_{eg} for rate $\frac{1}{2}$ and $\frac{2}{3}$ codes are given in Table I. For higher rate codes numerical determination of the stationary probabilities which yield P_{eg} seems preferable to determination of an expression for P_{eg}.

5.1 *Upper Bound on the Probability of Block Error*

An upper bound on the probability of block error can be obtained by considering worst case error propagation. The set of anomalous states is used to represent the decoder when error propagation can occur. Assume that one or more of blocks $[(i - b + 2)\ell]$ to $[i\ell]$ is incorrectly decoded and that block $[i\ell]$ has channel errors but all succeeding blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$ have no channel errors. Error propagation is the effect that, although no channel errors occur beyond block $[i\ell]$, decoding errors in block $[i\ell]$ and/or previous blocks may, through feedback, cause decoding errors to occur in some succeeding blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$.

The amount of error propagation, a, is the number of blocks after the last block with channel errors that have decoding errors. The length of error propagation, λ, is the number of blocks after the last block with channel errors up to and including the last block with decoding errors. Upper bounds A and Λ on a and λ respectively are derived in Appendix B and given in Table II for codes with $d_m \geqq 2t + 2$. In Table II the notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

Another required quantity, the recovery space, is the minimum number of blocks after the last block with channel errors that must be free of channel errors to guarantee that error propagation ceases and to guarantee that the decoder has completed any BTD decoding and associated BFD decodings resulting from error propagation. Specifically, the recovery space, V, is defined as the minimum number of consecutive blocks free of channel errors (blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$, $[(i + V)\ell]$ required to guarantee that the decoder will return to one of normal states $\{10, 11, 20, 21\}$ at block $[(i + V + 1)l]$ regardless of the channel error sequence prior to block $[(i + 1)l]$. The

TABLE II—RECOVERY SPACE AND BOUNDS ON THE AMOUNT AND LENGTH OF ERROR PROPAGATION FOR CODES WITH $d_m \geq 2t + 2$

$b = 2$	$\Lambda = 1$ $A = 1$ $V = 2$		
$b = 3,\ t = 1$	$\Lambda = 2$ $A = 2$ $V = 5$		
	$d_m > b$	$d_m \leqq b$	
$b \geqq 4,\ t = 1$	$\Lambda = 3b - 8$ $A = b$ $V = 4b - 7$	$\Lambda = 3b - 8$ $A = (b - 1) + \left\lfloor \frac{b - 4}{d_m - 2} + 1 \right\rfloor$ $V = 4b - 7$	
	$d_m > bt$	$d_m \leqq bt$	
		$d_m \leqq 3t$	$d_m > 3t$
$b \geqq 3,\ t \geqq 2$	$\Lambda = 3b - 6$ $A = b$ $V = 4b - 7$	$\Lambda = (2b - 3) + \left\lfloor \frac{(b - 2)t - 2}{d_m - 2t} \right\rfloor$ $A = (b - 1) + \left\lfloor \frac{(b - 2)t - 2}{d_m - 2t} + 1 \right\rfloor$ $V = (3b - 4) + \left\lfloor \frac{(b - 2)t - 2}{d_m - 2t} \right\rfloor$	$\Lambda = 3b - 6$ $A = (b - 1) + \left\lfloor \frac{(b - 2)t - 2}{d_m - 2t} + 1 \right\rfloor$ $V = 4b - 7$

recovery space for codes with $d_m \geqq 2t + 2$ is derived in Appendix B and is given in Table II. Values for A, Λ, and V are presented in Table III for typical values of b, t, and d_m.

The decoder can be represented by $(V + 1)$ anomalous states, as in Fig. 4, when it is not in a normal state. State A_0 exists whenever channel errors occur in a block. States $A_1, A_2, \cdots, A_V$ represent successive blocks free of channel errors. In Fig. 4 state X represents any normal state marked X in Figs. 2 and 3 from which the transition from normal state to anomalous state occurs. A complete state diagram includes both normal and anomalous states. The transition probability matrix is implicitly given in Figs. 2(3) and 4 for rate $\frac{1}{2}(\frac{2}{3})$ codes. The Markov

Chain is regular and therefore the stationary probabilities exist.[5] Let ρ_i, $i = 1, 2, \cdots, 2^{b+1}, A_0, A_1, \cdots, A_V$ be the stationary probabilities of being in the respective states.

An upper bound on the probability of block error, P_{eu1}, is obtained by summing the stationary probabilities that the decoder is in a normal state whose last digit is 1 or in an anomalous state A_0 through A_{V-1}. If the decoder reaches state A_V then correct decodings are assured in states $A_{\Lambda+1}$ to A_V, but if the decoder returns to state A_0 after state $A_{\Lambda+j}$, $j = 1, 2, \cdots, (V - \Lambda - 1)$ correct decodings are not assured in states $A_{\Lambda+1}$ to $A_{\Lambda+j}$. A tighter bound which accounts for this difference will be obtained next. For rate $\frac{2}{3}$ codes with $t = 1$ and $d_m \geqq 4$

$$P_{eu1} = \rho_5 + \rho_6 + \rho_8 + \rho_9 + \rho_{10} + \rho_{13} + \rho_{14} + \rho_{16} + \rho_{A_0} + \rho_{A_1} + \rho_{A_2} + \rho_{A_3} + \rho_{A_4}. \tag{13}$$

Expressions for P_{eu1} for rate $\frac{1}{2}$ and $\frac{2}{3}$ codes are given in Table I.

A tighter upper bound, P_{eu2}, is obtained by using $2V$ anomalous states as in Fig. 5. State B_0 exists whenever channel errors occur. States $B_1, B_2, \cdots, B_{V-1}$ represent successive blocks free of channel errors when the total number of consecutive blocks free of channel errors is less than V. States $C_1, C_2, \cdots, C_V$ represent successive blocks free of channel errors when the total number of consecutive blocks

TABLE III—A, Λ, V FOR TYPICAL VALUES OF CODE PARAMETERS

Code Parameters			A	Λ	V
$b = 2$,		$d_m \geq 2t + 2$	1	1	2
$b = 3$,	$t = 1$,	$d_m \geq 4$	2	2	5
	$t = 2$,	$d_m \geq 6$	3	3	5
	$t = 3$,	$d_m \geq 8$	3	3	5
	$t = 4$,	$d_m = 10$	4	4	6
		$d_m \geq 11$	3	3	5
$b = 4$,	$t = 1$,	$d_m \geq 4$	4	4	9
	$t = 2$,	$d_m = 6$	5	6	9
		$d_m \geq 7$	4	6	9
	$t = 3$,	$d_m = 8$	6	7	10
		$d_m = 9$	5	6	9
		$d_m = 10$	5	6	9
		$d_m \geq 11$	4	6	9
	$t = 4$,	$d_m = 10$	7	8	11
		$d_m = 11$	6	7	10
		$d_m = 12$	5	6	9
		$d_m = 13$	5	6	9
		$d_m = 14$	5	6	9
		$d_m \geq 15$	4	6	9

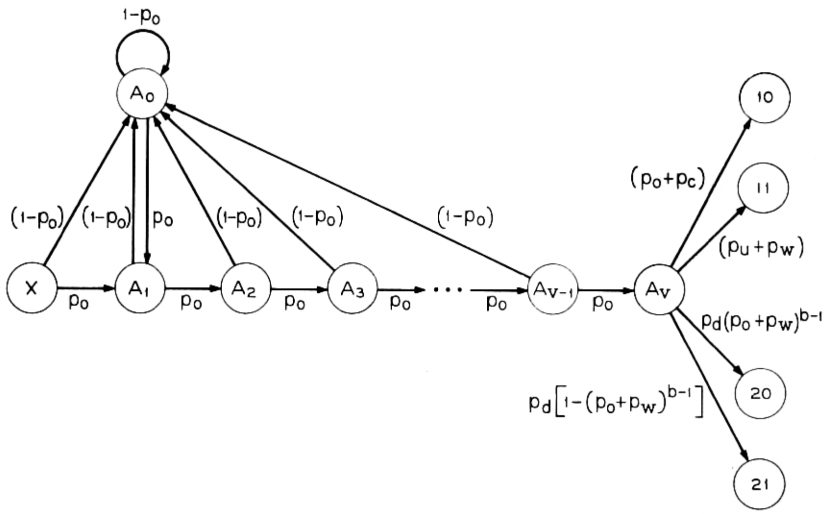

Fig. 4—$(V + 1)$ anomalous states for P_{eu1} bound on probability of block error.

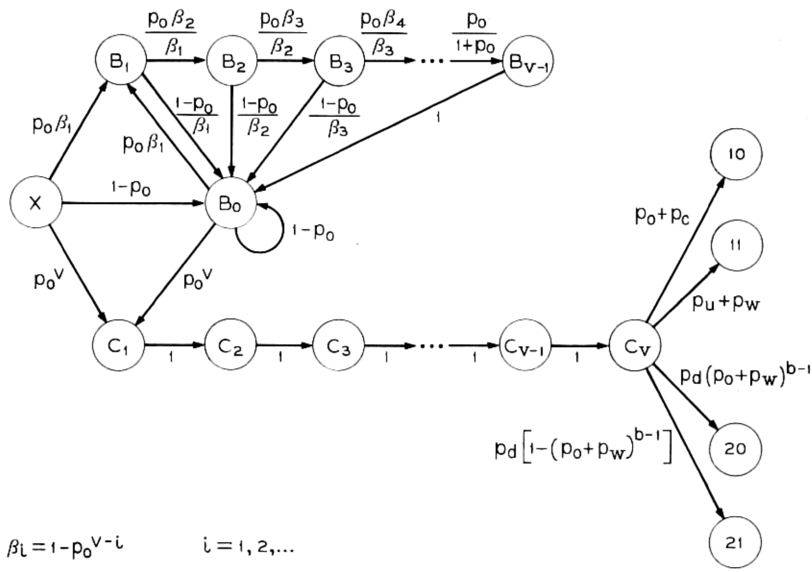

Fig. 5—$(2V)$ anomalous states for P_{eu2} bound on probability of block error.

free of channel errors is at least V. In Fig. 5 state X represents any normal state marked X in Figs. 2 and 3 from which the transition from normal state to anomalous state occurs. A complete state diagram includes both normal and anomalous states. The transition probability matrix is implicitly given in Figs. 2(3) and 5 for rate $\frac{1}{2}(\frac{2}{3})$ codes. The Markov Chain is regular and therefore the stationary probabilities exist.[5] Let s_i, $i = 1, 2, \cdots, 2^{b+1}, B_0, B_1, \cdots, B_{V-1}, C_1, C_2, \cdots, C_V$ be the stationary probabilities of being in the respective states.

P_{eu2} is obtained by summing the stationary probabilities that the decoder is in a normal state whose last digit is 1 or in an anomalous state B_0 through B_{V-1} or C_1 through C_A. For rate $\frac{2}{3}$ codes with $t = 1$ and $d_m \geqq 4$

$$P_{eu2} = s_5 + s_6 + s_8 + s_9 + s_{10} + s_{13} + s_{14} + s_{16} + s_{B_0} + s_{B_1} + s_{B_2} + s_{B_3} + s_{B_4} + s_{C_1} + s_{C_2}. \tag{14}$$

Expressions for P_{eu2} for rate $\frac{1}{2}$ and $\frac{2}{3}$ codes are given in Table I.

5.2 *Lower Bound on the Probability of Block Error*

A lower bound on the probability of block error, P_{eL}, is obtained by summing the stationary probabilities that the decoder is in a normal state whose last digit is 1. That is, it is assumed that no decoding errors are made when the decoder is in anomalous states. For rate $\frac{2}{3}$ codes with $t = 1$ and $d_m \geqq 4$

$$P_{eL} = \rho_5 + \rho_6 + \rho_8 + \rho_9 + \rho_{10} + \rho_{13} + \rho_{14} + \rho_{16}. \tag{15}$$

Expressions for P_{eL} are given in Table I.

VI. PERFORMANCE OF TWO CODES

The performance of a rate $\frac{1}{2}$ (18, 9) code of minimum distance $d_m = 6$ has been evaluated on the binary symmetric channel ($\ell = 1$). This code is obtained by extending the (17, 9) quadratic residue code of minimum distance $d_m = 5$ by one bit. Upper and lower bounds on the probability of block error for the (18, 9) code used in burst-trapping with $t = 1$ and $t = 2$ versus the binary symmetric channel transition probability, p, are given in Fig. 6. The probability of block error for maximum likelihood decoding of the (18, 9) code and the probability of block error for an uncoded 9 bit block are presented for comparison. It is interesting to note that the performance of the code is better with burst-trapping decoding ($t = 2$) than with maximum likelihood

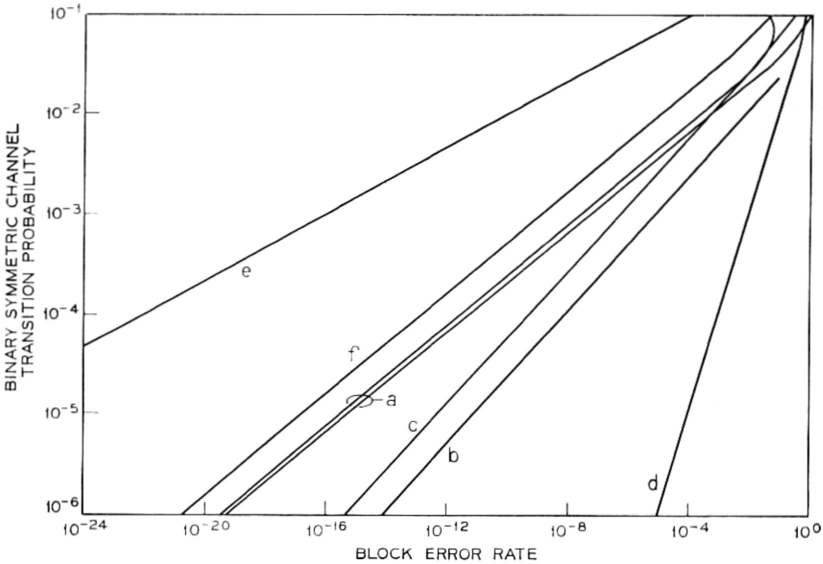

Fig. 6—Performance of (18, 9) code on binary symmetric channel. a. Upper and lower bounds for burst-trapping procedure, $t = 2$; b. Upper bound for burst-trapping procedure, $t = 1$; c. Maximum likelihood decoding; d. Uncoded 9-bit block; e. Decision feedback, $t = 0$; f. Decision feedback, $t = 2$.

decoding for values of p where the rate $\frac{1}{2}$ is much less than the channe capacity. Channel capacity is $\frac{1}{2}$ when $p = 0.11$. The probability of block error for two decision feedback systems using the (18, 9) code are also presented for comparison. In one system, ($t = 0$), repeat requests are made on all received n-tuples but code words. In the other system, ($t = 2$), single and double error corrections are made and repeat requests are made on all received n-tuples but code words or code words perturbed by one or two errors.

The performance of a rate $\frac{2}{3}$ (39, 26) shortened BCH code of minimum distance $d_m = 6$ with $t = 1$ has been computed on the basis of recorded telephone error data. One set of recorded telephone error data is the Vestigial-Sideband (VSB) data.[6,7] A selected set of 85 calls was used. Each call is an error sequence of about 3×10^6 bits recorded at 3600 b/s (4-level operation of the VSB modem). The 85 calls were divided into groups of calls of similar bit error rate as shown in Table IV and $P(m, 39)$ statistics[8] were determined for each group. $P(m, 39)$, $m = 1, 2, \cdots, 39$, is the probability that m errors occur in a block of 39 bits. Estimates of p_0, p_e, p_w, p_d, p_u, and α were obtained for each group by using

TABLE IV—GROUPING OF 85 VSB CALLS BY BIT ERROR RATE TO OBTAIN ESTIMATES OF p_0, p_c, p_w, p_d, p_u and α
[For (39, 26) Code with $d_m = 6$ and $t = 1$.]

Number of Calls in Group	Range of Bit Error Rate of Calls in Group	p_0	p_c	p_w	p_d	p_u	α
6	$4.30 \times 10^{-4} < p < 7.67 \times 10^{-4}$	0.982964	1.538×10^{-2}	5.407×10^{-8}	1.652×10^{-3}	2.173×10^{-6}	3.063×10^{-1}
6	$1.56 \times 10^{-4} < p < 3.83 \times 10^{-4}$	0.994869	4.418×10^{-3}	5.976×10^{-8}	7.106×10^{-4}	2.353×10^{-6}	2.899×10^{-1}
6	$7.01 \times 10^{-4} < p < 1.52 \times 10^{-4}$	0.996415	3.349×10^{-3}	7.652×10^{-9}	2.363×10^{-4}	3.103×10^{-7}	3.145×10^{-1}
6	$4.50 \times 10^{-5} < p < 6.81 \times 10^{-5}$	0.998643	1.078×10^{-3}	9.329×10^{-9}	2.785×10^{-4}	3.790×10^{-7}	2.757×10^{-1}
6	$2.20 \times 10^{-5} < p < 4.19 \times 10^{-5}$	0.999220	6.505×10^{-4}	4.221×10^{-9}	1.295×10^{-4}	1.646×10^{-7}	2.882×10^{-1}
6	$1.19 \times 10^{-5} < p < 1.79 \times 10^{-5}$	0.999604	3.170×10^{-4}	8.473×10^{-10}	7.863×10^{-5}	3.305×10^{-8}	2.844×10^{-1}
6	$7.86 \times 10^{-6} < p < 1.07 \times 10^{-5}$	0.999703	2.552×10^{-4}	5.065×10^{-10}	4.147×10^{-5}	1.975×10^{-8}	2.967×10^{-1}
6	$4.95 \times 10^{-6} < p < 7.85 \times 10^{-6}$	0.999842	1.138×10^{-4}	2.836×10^{-10}	4.413×10^{-5}	1.106×10^{-8}	2.623×10^{-1}
6	$1.91 \times 10^{-6} < p < 3.48 \times 10^{-6}$	0.999914	7.441×10^{-5}	0.0	1.191×10^{-5}	0.0	2.960×10^{-1}
7	$6.03 \times 10^{-7} < p < 1.87 \times 10^{-6}$	0.999957	2.990×10^{-5}	0.0	1.308×10^{-5}	0.0	2.607×10^{-1}
7	$p \sim 3.33 \times 10^{-7}$						
17	$p < 3.33 \times 10^{-7}$						

approximate code and coset weight spectra. A plot of the distribution of the percent of calls (quantized by the grouping of calls) with respect to raw (39 bit) block error rate and the upper and lower bounds on the distribution with respect to decoded block error rate is given in Fig. 7. The distribution of the percent of calls with respect to decoded block error rate as estimated by computer simulation[2] on the same 85 call sample is also given for comparison. For the comparison simulation a block interleaving degree $\ell = 117$ was used as this amount of interleaving attained approximate independence between successive blocks of the subcodes.

Another set of recorded telephone error data is the Alexander-Gryb-Nast (AGN) data.[9] AGN data consists of about 1000 calls recorded at data rates of 600 b/s and 1200 b/s on an FM data modem. These calls are from three classes of calls of approximately 3.6×10^5, 7.2×10^5, or 2.16×10^6 bits in length. The 294 calls with bit error rate greater than 10^{-5} were divided into groups as shown in Table V and estimates of p_0, p_c, p_w, p_d, p_u, and α were obtained as for the VSB data. A

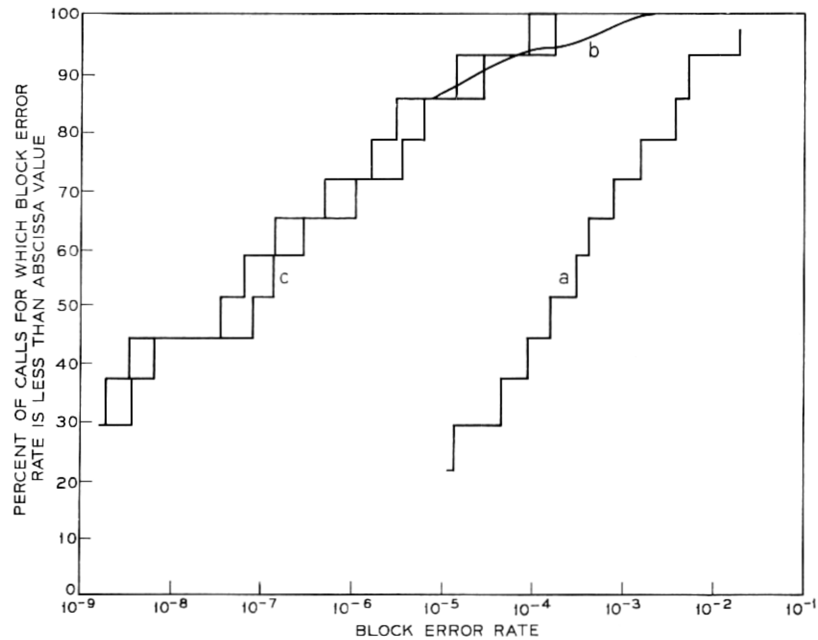

Fig. 7—Distribution of 85 call VSB sample with respect to block error rate. a. Uncoded 39 bit blocks; b. Simulation of (39, 26) code interleaved to degree 117; c. Upper and lower bounds for (39, 26) code.

plot of the distribution of the percent of calls (quantized by the grouping of calls) with respect to raw (39 bit) block error rate and the upper and lower bounds on the distribution with respect to decoded block error rate is given in Figs. 8 and 9. The performance as estimated by computer simulation[2] on the same call sample is also presented for comparison.

For both sets of data there is good agreement between the performance estimates obtained by the theory and simulation. Two advantages of the theoretical technique over the simulation technique are the ability to obtain the tail of the distribution and the ability to obtain the distribution from reduced data. This latter advantage is due to the fact that p_0, p_c, p_w, p_d, p_u, and α can be approximately determined from reduced data such as $P(m, n)$ statistics.

VII. CONCLUSIONS

A technique for analyzing the performance of a burst-trapping error correction procedure has been described. The criterion of performance is the probability of block error. The analysis technique for burst-trapping procedures with block interleaving degree ℓ is valid on channels where error patterns in blocks separated by ℓ blocks are (approximately) independent. Thus the analysis technique is valid for random error channels or for properly designed burst-trapping procedures on burst or compound channels such as the telephone channel.

The performance of a rate $\frac{2}{3}$ code, the (39, 26) shortened BCH code, has been computed for two sets of recorded error data. For both sets of data the computed performance agrees with the performance obtained by computer simulation. An advantage of the theoretical technique over simulation is the ability to estimate performance on the basis of reduced data [$P(m, n)$ statistics] rather than on extensive error sequence data.

VIII. ACKNOWLEDGMENTS

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APPENDIX A

Transition Probabilities for Normal State Diagrams

For simplicity the development of the one step transition probabilities $p_{i,j}$ from renumbered state i to renumbered state j is demonstrated for

TABLE V—GROUPING OF 294 AGN CALLS BY BIT ERROR RATE TO OBTAIN ESTIMATES OF p_0, p_c, p_w, p_d, p_u AND α
[For (39, 26) Code with $d_m = 6$ and $t = 1$.]

Number of Calls in Group	Range of Bit Error Rate of Calls in Group	p_0	p_c	p_w	p_d	p_u	α
5	$1.306 \times 10^{-3} < p < 3.799 \times 10^{-3}$	0.959926	2.779×10^{-2}	3.328×10^{-7}	1.227×10^{-2}	1.318×10^{-5}	0.2512
10	$6.338 \times 10^{-4} < p < 9.444 \times 10^{-4}$	0.992275	3.186×10^{-3}	2.188×10^{-7}	4.530×10^{-3}	8.849×10^{-6}	0.1560
10	$2.667 \times 10^{-4} < p < 5.300 \times 10^{-4}$	0.990216	8.122×10^{-3}	1.851×10^{-8}	1.661×10^{-3}	7.356×10^{-7}	0.2901
10	$1.700 \times 10^{-4} < p < 2.662 \times 10^{-4}$	0.996270	2.725×10^{-3}	4.133×10^{-8}	1.003×10^{-3}	1.640×10^{-6}	0.2587
10	$1.335 \times 10^{-4} < p < 1.630 \times 10^{-4}$	0.997023	2.117×10^{-3}	1.720×10^{-8}	8.597×10^{-4}	7.081×10^{-7}	0.2528
10	$1.005 \times 10^{-4} < p < 1.278 \times 10^{-4}$	0.996567	2.855×10^{-3}	2.939×10^{-9}	5.777×10^{-4}	1.305×10^{-7}	0.2913
10	$7.964 \times 10^{-5} < p < 9.770 \times 10^{-5}$	0.997792	1.834×10^{-3}	9.263×10^{-9}	3.740×10^{-4}	3.816×10^{-7}	0.2859
10	$6.659 \times 10^{-5} < p < 7.691 \times 10^{-5}$	0.998481	1.251×10^{-3}	1.058×10^{-8}	2.677×10^{-4}	4.205×10^{-7}	0.2844
10	$6.111 \times 10^{-5} < p < 6.565 \times 10^{-5}$	0.998498	1.208×10^{-3}	6.632×10^{-9}	2.937×10^{-4}	2.727×10^{-7}	0.2789
10	$5.410 \times 10^{-5} < p < 6.110 \times 10^{-5}$	0.998618	1.028×10^{-3}	4.837×10^{-9}	3.537×10^{-4}	2.177×10^{-7}	0.2603
10	$4.447 \times 10^{-5} < p < 5.290 \times 10^{-5}$	0.998998	6.673×10^{-4}	5.683×10^{-9}	3.350×10^{-4}	2.376×10^{-7}	0.2415
10	$3.611 \times 10^{-5} < p < 4.446 \times 10^{-5}$	0.998900	8.475×10^{-4}	1.572×10^{-9}	2.524×10^{-4}	7.452×10^{-8}	0.2728
10	$3.240 \times 10^{-5} < p < 3.607 \times 10^{-5}$	0.999076	7.674×10^{-4}	2.129×10^{-9}	1.569×10^{-4}	9.580×10^{-8}	0.2860
10	$2.914 \times 10^{-5} < p < 3.117 \times 10^{-5}$	0.999087	7.702×10^{-4}	7.433×10^{-10}	1.431×10^{-4}	2.899×10^{-8}	0.2953
10	$2.779 \times 10^{-5} < p < 2.914 \times 10^{-5}$	0.999529	3.182×10^{-4}	4.958×10^{-9}	1.521×10^{-4}	2.033×10^{-7}	0.2408
10	$2.505 \times 10^{-5} < p < 2.778 \times 10^{-5}$	0.999377	4.590×10^{-4}	2.613×10^{-9}	1.640×10^{-4}	1.106×10^{-7}	0.2613
20	$2.223 \times 10^{-5} < p < 2.504 \times 10^{-5}$	0.999367	4.710×10^{-4}	1.319×10^{-9}	1.620×10^{-4}	5.776×10^{-8}	0.2657
20	$1.953 \times 10^{-5} < p < 2.223 \times 10^{-5}$	0.999462	4.277×10^{-4}	1.878×10^{-9}	1.107×10^{-4}	7.775×10^{-8}	0.2772
20	$1.714 \times 10^{-5} < p < 1.950 \times 10^{-5}$	0.999471	4.119×10^{-4}	3.444×10^{-10}	1.170×10^{-4}	1.756×10^{-8}	0.2768
20	$1.528 \times 10^{-5} < p < 1.671 \times 10^{-5}$	0.999706	1.628×10^{-4}	1.944×10^{-9}	1.309×10^{-4}	8.102×10^{-8}	0.2084
20	$1.340 \times 10^{-5} < p < 1.526 \times 10^{-5}$	0.999622	2.918×10^{-4}	6.556×10^{-10}	8.591×10^{-5}	2.819×10^{-8}	0.2722
20	$1.113 \times 10^{-5} < p < 1.340 \times 10^{-5}$	0.999660	2.571×10^{-4}	1.516×10^{-10}	8.320×10^{-5}	1.046×10^{-8}	0.2682
19	$1.017 \times 10^{-5} < p < 1.112 \times 10^{-5}$	0.999735	2.021×10^{-4}	1.175×10^{-9}	6.251×10^{-5}	4.758×10^{-8}	0.2687

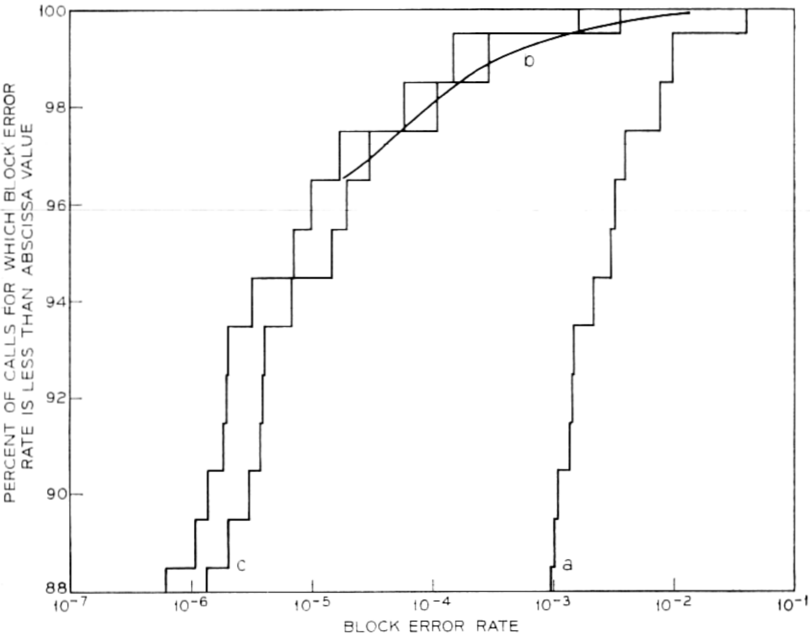

Fig. 8—Distribution of AGN call sample with respect to block error rate. a. Uncoded 39 bit blocks; 2. Simulation of (39, 26) code interleaved to degree 117; c. Upper and lower bounds for (39, 26) code.

rate $\frac{2}{3}$ codes (Fig. 3). The technique is easily generalized to rate $(b-1)/b$ codes, $b \geqq 2$. Obviously,

$$p_{1,1} = p_0 + p_c ; \tag{16}$$

$$p_{1,9} = p_u + p_w ; \tag{17}$$

$$p_{1,2} + p_{1,10} = p_d . \tag{18}$$

Let $p_{i,j}(b)$ be the b step transition probability from renumbered state i to renumbered state j. Due to the structure of the normal state diagram it is easily seen that,

$$p_{1,2} = p_{1,4}(3) + p_{1,5}(3) + p_{1,7}(3) + p_{1,8}(3); \tag{19}$$

$$p_{2,3} = (1 - p_{2,6}) = \frac{p_{1,4}(3) + p_{1,5}(3)}{p_{1,2}} ; \tag{20}$$

$$p_{3,4} = (1 - p_{3,5}) = \frac{p_{1,4}(3)}{p_{1,2}p_{2,3}} ; \tag{21}$$

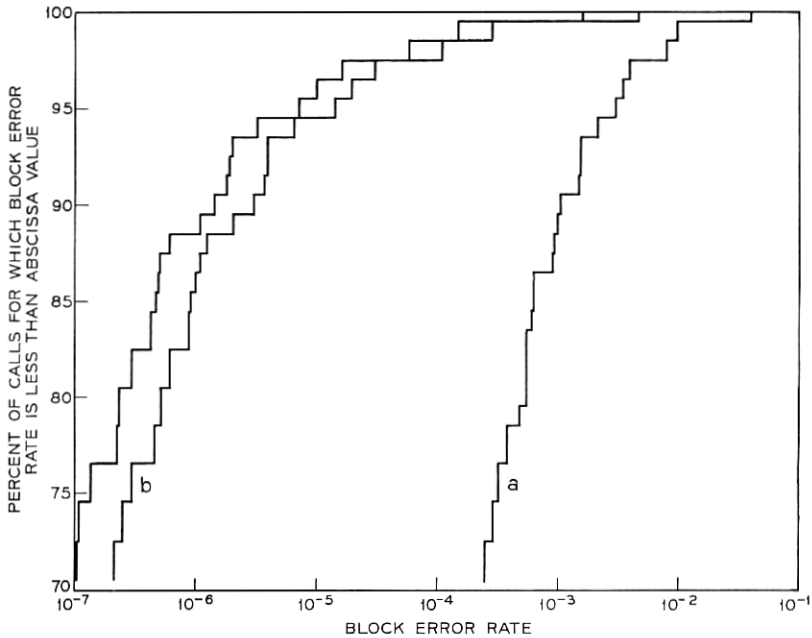

Fig. 9—Distribution of AGN call sample with respect to block error rate. a. Uncoded 39 bit block; b. Upper and lower bounds for (39, 26) code.

$$p_{6,7} = (1 - p_{6,8}) = \frac{p_{1,7}(3)}{p_{1,2}p_{2,6}}. \tag{22}$$

The three step transition probabilities $p_{1,4}(3)$, $p_{1,5}(3)$, $p_{1,7}(3)$, $p_{1,8}(3)$ can be computed by summing the probabilities of all combinations of three successive error patterns which cause transition from normal state 10 to normal states 4000, 4001, 4010, and 4011 respectively.

$$p_{1,4}(3) = p_d p_0^2 \; ; \tag{23}$$

$$p_{1,5}(3) = p_d p_0 p_w \; ; \tag{24}$$

$$p_{1,7}(3) = p_d p_w p_0 \; ; \tag{25}$$

$$p_{1,8}(3) = p_d p_w^2 \; . \tag{26}$$

Further,

$$p_{1,10} = p_{1,12}(3) + p_{1,13}(3) + p_{1,15}(3) + p_{1,16}(3); \tag{27}$$

$$p_{10,11} = (1 - p_{10,14}) = \frac{p_{1,12}(3) + p_{1,13}(3)}{p_{1,10}}; \tag{28}$$

$$p_{11,12} = (1 - p_{11,13}) = \frac{p_{1,12}(3)}{p_{1,10}p_{10,11}}; \tag{29}$$

$$p_{14,15} = (1 - p_{14,16}) = \frac{p_{1,15}(3)}{p_{1,10}p_{10,14}}. \tag{30}$$

The three step transition probabilities $p_{1,12}(3)$, $p_{1,13}(3)$, $p_{1,15}(3)$, $p_{1,16}(3)$ can be computed by summing the probabilities of all combinations of three successive error patterns which cause transition from normal state 10 to normal states 4100, 4101, 4110, 4111 respectively.

$$p_{1,12}(3) = p_d\{\alpha^2(p_c + p_u + p_d)^2 + 2p_0\alpha(p_c + p_u + p_d)\}; \tag{31}$$

$$p_{1,13}(3) = p_d\{\alpha(1 - \alpha)(p_c + p_u + p_d)^2 + p_0(1 - \alpha)(p_c + p_u + p_d) + \alpha(p_c + p_u + p_d)p_w\}; \tag{32}$$

$$p_{1,15}(3) = p_{1,13}(3); \tag{33}$$

$$p_{1,16}(3) = p_d\{(1 - \alpha)^2(p_c + p_u + p_d)^2 + 2p_w(1 - \alpha)(p_c + p_u + p_d)\}. \tag{34}$$

The transition probabilities for rate $\frac{1}{2}$ codes are given in Fig. 2. The transition probabilities for rate $\frac{2}{3}$ codes with the exception of

$$p_{11,12} = \frac{\alpha^2(p_c + p_u + p_d) + 2\alpha p_0}{\alpha + p_0}; \tag{35}$$

$$p_{11,13} = \frac{(1 - \alpha)\alpha(p_c + p_u + p_d) + p_0(1 - \alpha) + \alpha p_w}{\alpha + p_0}; \tag{36}$$

$$p_{14,15} = \frac{(1 - \alpha)\alpha(p_c + p_u + p_d) + p_0(1 - \alpha) + \alpha p_w}{(1 - \alpha) + p_w}; \tag{37}$$

$$p_{14,16} = \frac{(1 - \alpha)^2(p_c + p_u + p_d) + 2(1 - \alpha)p_w}{(1 - \alpha) + p_w}; \tag{38}$$

are given in Fig. 3.

APPENDIX B

Bounds on the Amount and Length of Error Propagation and Determination of the Recovery Space

We consider codes in which $d_m \geqq 2t + 2$. Let $0 \leqq e_i \leqq k$ be the maximum number of errors possible in the information part of the decoded word at block $[i\ell]$. The value of e_i depends on the method of decoding the block and the error pattern sequence. We will say that

$$f_i = \sum_{j=i-b+2}^{i} e_j \tag{39}$$

is the maximum number of errors "available" for feedback to blocks after block $[i\ell]$. As in the text, assume that one or more of blocks $[(i - b + 2)\ell]$ to $[i\ell]$ is incorrectly decoded and that block $[i\ell]$ has channel errors, but all succeeding blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$, have no channel errors. We consider three mutually exclusive and exhaustive cases. We use the fact that a block code of minimum distance d_m is simultaneously capable of correcting t errors and detecting $d \geqq t$ errors if $d_m = t + d + 1$.

B.1 *Case 1*

Assume that no error detections (BTD's) occur in blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$. Under this hypothesis error propagation cannot occur beyond block $[j\ell]$ if $f_j < d_m - t$. The maximum length of error propagation is then the smallest integer Λ_1 such that $f_{i+\Lambda_1} < d_m - t$. Since $f_i = (b - 1)k$, blocks $[(i + 1)\ell]$ to $[(i + b - 1)\ell]$ are subject to decoding error due to error propagation and therefore $f_{i+b-1} = (b - 1)t$. Each subsequent block, $[(i + b)\ell]$, $[(i + b + 1)\ell]$, $\cdots$, which is correctly (incorrectly) decoded removes t $(d_m - 2t)$ errors from the maximum number available for feedback. Therefore

$$\Lambda_1 = \begin{cases} (b - 1) + \left\lfloor \dfrac{(b-2)t}{d_m - 2t} \right\rfloor, & \text{if} \quad d_m - 2t \leqq t \\[2ex] (b - 1) + \left\lfloor \dfrac{bt - 2(d_m - 2t)}{t} \right\rfloor, & \text{if} \quad d_m - 2t > t \end{cases} \tag{40}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

The maximum amount of error propagation is

$$A_1 = (b - 1) + \left\lfloor \frac{(b-2)t}{d_m - 2t} \right\rfloor. \tag{41}$$

Blocks $[(i + \Lambda_1 + b)\ell]$, $[(i + \Lambda_1 + b + 1)\ell]$, $\cdots$ are correctly decoded by RED in normal state 10.

B.2 *Case 2*

Assume that an error detection (BTD) occurs in block $[(i + j)\ell]$, $1 \leqq j \leqq b - 1$. Under this hypothesis error propagation cannot occur beyond block $[(i + j)\ell]$. The maximum length of error propagation is

$$\Lambda_2 = b - 1 \tag{42}$$

and the maximum amount of error propagation is

$$A_2 = b - 1. \tag{43}$$

Blocks $[(i + \Lambda_2 + b)\ell]$, $[(i + \Lambda_2 + b + 1)\ell]$, $\cdots$, are correctly decoded by RED in normal state 10.

B.3 *Case 3*

Assume that an error detection (BTD) occurs in block $[(i + j)\ell]$, $b \leqq j \leqq J$, where J is the maximum value of j such that block $[(i + j)\ell]$ can be detected in error when the last channel errors occur in block $[i\ell]$ and blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$ are free of channel errors. Under this hypothesis, error propagation cannot occur beyond block $[(i + j)\ell]$. For block $[(i + J)\ell]$ to be detected in error it must have $t_1 \geqq t + 1$ errors fed back to it. For $b = 2$ (rate $\frac{1}{2}$ codes) $J = b - 1$, and Case 3 cannot occur.

As in Case 1, $f_i = (b - 1)k$, and blocks $[(i + 1)\ell]$ to $[(i + b - 1)\ell]$ are subject to decoding error due to error propagation. Therefore, $f_{i+b-1} = (b - 1)t$. Each subsequent block, $[(i + b)\ell]$, $[(i + b + 1)\ell]$, $\cdots$, which is correctly (incorrectly) decoded removes $t(d_m - 2t)$ errors from the maximum number available for feedback. Let H_a be the largest integer such that $f_{i+H_a} \geqq t + 1$. Then when $b \geqq 3$

$$H_a = \begin{cases} (b - 1) + \left\lfloor \dfrac{(b - 2)t - 1}{d_m - 2t} \right\rfloor, & \text{if } d_m - 2t \leqq t; \\ 2b - 4 & \text{if } d_m - 2t > t. \end{cases} \tag{44}$$

Since errors must be fedback to block $[(i + J)\ell]$ from at least two blocks to get $t_1 > t$ feedback errors,

$$J = H_a + (b - 2). \tag{45}$$

B.3.1 *Case 3a*

Assume block $[(i + j)\ell]$ is correctly decoded by BTD. Then

$$\Lambda_3 = H_a ; \tag{46}$$

$$A_3 = (b - 1) + \left\lfloor \frac{(b - 2)t - 1}{d_m - 2t} \right\rfloor. \tag{47}$$

Blocks $[(i + J + b)\ell]$, $[(i + J + b + 1)\ell]$, $\cdots$ are correctly decoded in normal state 10.

B.3.2 *Case 3b*

Assume block $[(i + j)\ell]$ is incorrectly decoded by BTD. Now $b \leqq j \leqq G \leqq J$, where G is the maximum value of j such that block $[(i + j)\ell]$ can be detected in error and incorrectly decoded due to error propagation when the last channel errors occur at block $[i\ell]$ and blocks $[(i + 1)\ell]$, $[(i + 2)\ell]$, $\cdots$ are free of channel errors. For block $[(i + G)\ell]$ to be detected in error it must have $t_1 \geqq t + 1$ errors fed back to it to cause the detection and $t_2 \geqq 1$ errors fed back during the BTD to cause erroneous decoding.

If $f_{i+b-1} = (b - 1)t < t + 2$, $G = b - 1$ and Case 3b cannot occur. Let H_b be the largest integer such that $f_{i+H_b} > t + 1$. Then when $b \geqq 3$ and $(b - 2)t \geqq 2$

$$H_b = \begin{cases} (b - 1) + \left\lfloor \dfrac{(b - 2)t - 2}{d_m - 2t} \right\rfloor, & \text{if} \quad d_m - 2t \leqq t; \\ (b - 1) + \left\lfloor \dfrac{(b - 2)t - 2}{t} \right\rfloor, & \text{if} \quad d_m - 2t > t. \end{cases} \tag{48}$$

Since errors must be fed back to block $[(i + G)\ell]$ from at least two blocks (three blocks if $t = 1$) to get $t_1 + t_2 \geqq t + 2$ feedback errors

$$\Lambda_4 = G = \begin{cases} H_b + (b - 2), & \text{if} \quad t \geqq 2; \\ H_b + (b - 3), & \text{if} \quad t = 1. \end{cases} \tag{49}$$

$$A_4 = (b - 1) + \left\lfloor \frac{(b - 2)t - 2 + d_m - 2t}{d_m - 2t} \right\rfloor. \tag{50}$$

Blocks $[(i + \Lambda_4 + b)\ell]$, $[(i + \Lambda_4 + b + 1)\ell]$, $\cdots$ are correctly decoded in normal state 10.

Table I is constructed by taking

$$\Lambda = \max \{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\} \tag{51}$$

and

$$A = \max \{A_1, A_2, A_3, A_4\}.$$

By definition the recovery space is the minimum number of consecutive blocks free of channel errors required to guarantee the return of the decoder to one of normal states {10, 11, 20, 21} at block $[(i + V+1)\ell]$ regardless of the channel error sequence prior to block $[(i+1)\ell]$. We say that V is a necessary recovery space if fewer than V consecutive blocks free of channel errors do not guarantee transition to one of normal states {10, 11, 20, 21} at block $[(i + V + 1)\ell]$. We say that

V is a sufficient recovery space if V consecutive blocks free of channel errors guarantee transition to one of normal states {10, 11, 20, 21} at block $[(i + V + 1)\ell]$. For Cases 1, 2, and 4 $V_i = \Lambda_i + (b - 1)$, $i = 1, 2$, and 4, is a necessary and sufficient recovery space. For Case 3 $V_3 = \Lambda_3 + (2b - 3)$ is a necessary and sufficient recovery space. Since in all cases $V \geqq V_i$, $i = 1, 2, 3, 4$, is a sufficient recovery space $V = \max \{V_1, V_2, V_3, V_4\}$ is a necessary and sufficient recovery space.

APPENDIX C

List of Symbols

ℓ —block interleaving constant
b —parameter determining rate of recurrent code
A —recurrent code parity check matrix
H —component block code parity check matrix
N —constraint length of recurrent code
A_N —recurrent code truncated parity check matrix
n —block length of component block code
k —number of information bits per block of component block code
$I_{i\ell}$ —k-tuple of information bits of $[i\ell]$th block
$M_{i\ell}$ —n-tuple of transmitted bits of $[i\ell]$th block
$Q_{i\ell}$ —$(n - k)$-tuple of parity bits of $[i\ell]$th block
$I^j_{i\ell}$ —jth segment of $I_{i\ell}$
$M^*_{i\ell}$ —n-tuple of received bits of $[i\ell]$th block
$I^*_{i\ell}$ —k-tuple of received information bits of $[i\ell]$th block
$Q^*_{i\ell}$ —$(n - k)$-tuple of received parity bits of $[i\ell]$th block
$P^*_{i\ell}$ —parity $(n - k)$-tuple obtained by encoding $I^*_{i\ell}$ with H
$\bar{I}_{i\ell}$ —k-tuple of decoded information bits of $[i\ell]$th block
$P'_{i\ell}$ —$Q^*_{i\ell}$ as modified by fed back information segments
$\bar{I}^j_{i\ell}$ —jth segment of $\bar{I}_{i\ell}$
C_0 —the set of one element—the pattern with no errors
C_w —the set of channel error patterns which are identical to nonzero code words
C_c —the set of nonzero channel error patterns which are correctable by RED
C_u —the set of channel error patterns which are both uncorrectable by RED and undetectable
C_d —the set of channel error patterns which are detectable
α —the probability that the information bits of a received word are unaffected by the channel error pattern given that the channel error pattern is correctable, detectable, or undetectable

p_0 —the probability that the channel error pattern is in set C_0
p_w —the probability that the channel error pattern is in set C_w
p_c —the probability that the channel error pattern is in set C_c
p_u —the probability that the channel error pattern is in set C_u
p_d —the probability that the channel error pattern is in set C_d
d_m —minimum distance of the component block code
t —amount of error correction done in RED
V —recovery space
$p_{i,j}$ —single step transition probability from state i to state j
π_i —stationary probability of being in state i with Genie Decoder
P_{eg} —probability of block error with Genie Decoder
a —amount of error propagation
A —upper bound on a
λ —length of error propagation
Λ —upper bound on λ
A_i —designation of anomalous states for upper bound one on block error probability
ρ_i —stationary probability of being in state i for upper bound one
P_{eu1} —upper bound one on probability of block error
B_i ,C_i —designation of anomalous states for upper bound two on block error probability
s_i —stationary probability of being in state i for upper bound two
P_{eu2} —upper bound two on probability of block error
P_{eL} —lower bound on probability of block error
$p_{i,j}(3)$—three step transition probability from state i to state j
e_i —maximum number of errors possible in decoded information k-tuple at block $[i\ell]$
f_i —maximum number of errors available for feedback to blocks after block $[i\ell]$
d —amount of error detection capability of component block code

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