Laser Speckle Pattern— A Narrowband Noise Model

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We represent by an electrical model the imaging of a one-dimensional coherently illuminated and diffusely reflecting surface by an optical system with a rectangular aperture. We then obtain the statistical properties of the image intensity from the statistical properties of the square of the envelope of a narrowband noise signal in the electrical model. The analysis is simple because use can be made of results known in communication theory. The results agree with those obtained in a direct way.

I. INTRODUCTION

The speckle pattern in the image of a coherently illuminated and diffusely reflecting object has been analyzed by Enloe[1]. Enloe's results show a remarkable similarity to some results occurring in the theory of narrowband noise (See Ref. 2, pp. 397 ff.). This similarity prompts the question whether Enloe's results can be derived by the use of an electrical model analogy involving narrowband noise.* In this paper we will show that this is indeed possible for the special case of a one-dimensional subject and an optical system with a rectangular aperture. The analysis is simple because we can use results known in communication theory and the results agree with those of Enloe[1].

II. THE OPTICAL MODEL

The optical model is shown in Fig. 1. In plane P_1 there is a coherently illuminated row of scatterers which scatter with random phase (a one-dimensional diffusely reflecting surface). At a distance d in plane P_2 there is a lens. In front of the lens there is a rectangular aperture with an amplitude transmission $H(x_2, y_2)$. The distance d is large compared to the focal length f of the lens and therefore to a good degree of approxima-

* Such an analogy was already suggested by Rigden and Gordon.[3]

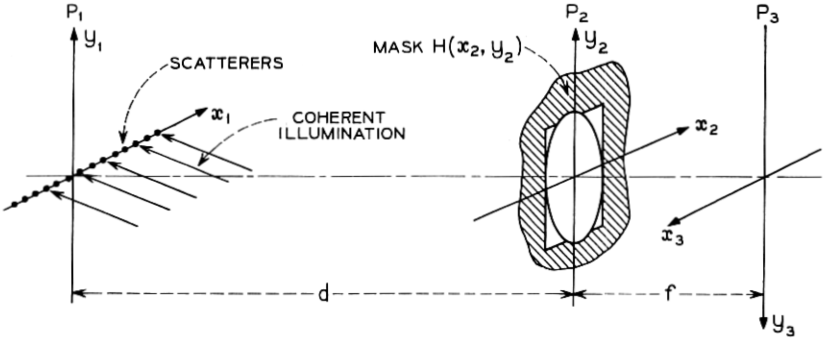

Fig. 1 — Optical model.

tion the image of the scatterers in plane P_1 is situated in plane P_3 at a distance f from the lens. Enloe computed the intensity as well as the autocorrelation of the intensity in the image plane.[1]

As a help in understanding the following electrical model we make the following comments regarding the optical model. Suppose that the instantaneous value of the electric field $e(x_1, t)$ in the object plane P_1 is given by

$$e(x_1\,,\,t) = a\left(\frac{x_1}{\lambda d}\right) \exp\,(-2\pi j\nu_0 t). \tag{1}$$

$e(x_1, t)$ is zero for $y_1 \neq 0$. The time-independent phasor is

$$a\left(\frac{x_1}{\lambda d}\right) = a(\alpha_1), \tag{2}$$

where we have introduced the spatial frequency coordinate

$$\alpha_1 = \frac{x_1}{\lambda d}. \tag{3}$$

(We write a( ) as a function of the spatial frequency $x_1/\lambda d$ and not of x_1 in order to simplify the following computation.) For later use we also introduce

$$\beta_1 = \frac{y_1}{\lambda d}. \tag{4}$$

Since the lens is situated in the far field of the object, the phasor $A(x_2, y_2)$ in plane P_2 in front of the aperture is the Fourier transform of $a(\alpha_1)$,

$$A(x_2\,,\,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(\alpha_1)\exp\,(2\pi j\alpha_1 x_2)\exp\,(2\pi j\beta_1 y_2)\,d\alpha_1\,d\beta_1\,. \tag{5}$$

Since $a(\alpha_1)$ is zero for $\beta_1 \neq 0$ the above two-dimensional Fourier transform operation reduces to the one-dimensional Fourier transform operation

$$A(x_2, y_2) = \int_{-\infty}^{\infty} a(\alpha_1) \exp(2\pi j \alpha_1 x_2)\, d\alpha_1 . \tag{6}$$

It is seen that $A(x_2, y_2)$ is a function of x_2 only and we will therefore write $A(x_2)$. The phasor behind the aperture, $B(x_2, y_2)$ is obtained as the product of $A(x_2)$ and the aperture transmission function $H(x_2, y_2)$

$$B(x_2, y_2) = A(x_2) \cdot H(x_2, y_2). \tag{7}$$

As was mentioned we assume a rectangular aperture function. We assume that $H(x_2, y_2)$ can be written as

$$H(x_2, y_2) = H_1(x_2) \cdot H_2(y_2). \tag{8}$$

Since our object is one-dimensional we are only interested in the image intensity at $y_3 = 0$. The image intensity at $y_3 = 0$ is not influenced by $H_2(y_2)$ except for a factor which remains constant over all x_3. For equation (7) we can therefore write

$$B(x_2) = A(x_2) \cdot H_1(x_2) \tag{9}$$

with the understanding that $B(x_2)$ does vary in the y_2 direction but that this variation is of no interest to us. So much for the optical model. We will now present the electrical model.

III. THE ELECTRICAL MODEL

The electrical model is shown in Fig. 2. The electric field in the object plane is scanned by a detector whose output voltage is proportional to the instantaneous value of the electric field in the object plane. (In the optical region there are no such detectors available. This should not present any conceptual difficulties since we can scale up our model to a longer wavelength where there are such detectors.)

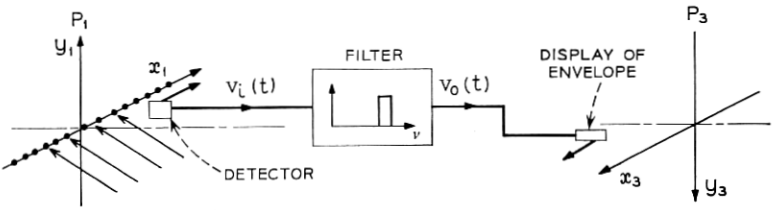

Fig. 2 — Electrical narrowband noise model.

The electric field $e(x_1, t)$ in plane P_1 is given by equation (1) and the output voltage $v_i(t)$ of the scanning detector is

$$v_i(t) = c_1 a\left(\frac{v_1 t}{\lambda d}\right) \exp(-2\pi j \nu_0 t). \tag{10}$$

v_1 is the scanning velocity and we have

$$x_1 = v_1 t. \tag{11}$$

c_1 denotes a constant of proportionality which is of no interest. The Fourier transform of $v_i(t)$ of equation (10), $FT[v_i(t)]$ is given by

$$\begin{aligned} FT[v_i(t)] &= c_1 \frac{\lambda d}{v_1} A\left[\frac{\lambda d}{v_1}(\nu - \nu_0)\right] \\ &= c_2 A\left[\frac{\lambda d}{v_1}(\nu - \nu_0)\right]. \end{aligned} \tag{12}$$

$A(\nu)$ is the Fourier transform of $a(t)$. c_2 is a constant of proportionality. The Fourier transform or spectrum of $v_i(t)$ is centered at ν_0 as shown in Fig. 3. In the optical model the spectrum $A(x_2)$ is multiplied by the aperture transmission function $H_1(x_2)$ in plane P_2. See equation (9). In order to simulate this in our electrical model we have to pass the voltage $v_i(t)$ through a temporal frequency filter with the frequency response $H_1[(\lambda d/v_1)(\nu - \nu_0)]$. The spectrum $B[(\lambda d/v_1)(\nu - \nu_0)]$ at the output of the filter is then given by

$$B\left[\frac{\lambda d}{v_1}(\nu - \nu_0)\right] = A\left[\frac{\lambda d}{v_1}(\nu - \nu_0)\right] H_1\left[\frac{\lambda d}{v_1}(\nu - \nu_0)\right]. \tag{13}$$

The filter $H_1[(\lambda d/v_1)(\nu - \nu_0)]$ is shown in the electrical model of Fig. 2 and its frequency response is sketched in Fig. 3. (The frequency response $H_1[(\lambda d/v_1)(\nu - \nu_0)]$ in the electrical model corresponding to the rectangular aperture transmission function of the optical system can only be

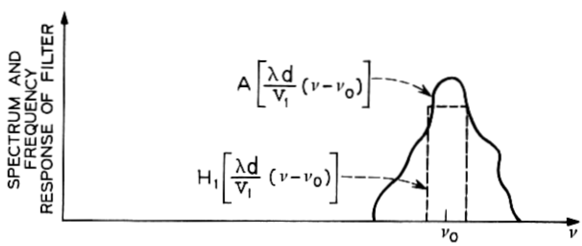

Fig. 3 — Spectrum of the electrical signal and frequency response of the filter.

realized with a time lag, but this need not disturb us.) At the output of the filter we have the voltage $v_o(t)$

$$v_o(t) = c_3 b\left(\frac{v_1 t}{\lambda d}\right) \exp(-2\pi j \nu_0 t), \tag{14}$$

where $b(t)$ is the Fourier transform of $B(\nu)$. c_3 again is a constant of proportionality of no interest. The output device scans the image plane P_3 with a velocity v_2 and

$$\frac{v_2}{v_1} = \frac{f}{d} \tag{15}$$

because in the optical model the image is demagnified by a factor d/f with respect to the object. The inversion of the optical image with respect to the object is accounted for by inverting the coordinate system in the image plane P_3, see Fig. 1. We can now write for equation (14)

$$\begin{aligned} v_o(t) &= c_3 b\left(\frac{v_2 t}{\lambda f}\right) \exp(-2\pi j \nu_0 t) \\ &= c_3 b\left(\frac{x_3}{\lambda f}\right) \exp(-2\pi j \nu_0 t). \end{aligned} \tag{16}$$

In the optical model we detect the intensity in the image plane, that is, the square of the absolute value of the phasor. Therefore in our electrical model the output device displays $| b(x_3/\lambda f) |^2$.

The voltage $v_o(t)$ at the output of the filter is narrowband compared to ν_0. We now consider $v_o(t)$ as a narrowband noise voltage. Another equivalent representation is (see, for example, Ref. 2, p. 397)

$$v_o(t) = N_c(t) \cos 2\pi\nu_0 t + N_s(t) \sin 2\pi\nu_0 t, \tag{17}$$

where

$$\begin{aligned} N_c(t) &= \operatorname{Re}\left[b\left(\frac{v_2 t}{\lambda f}\right)\right], \\ N_s(t) &= \operatorname{Im}\left[b\left(\frac{v_2 t}{\lambda f}\right)\right]. \end{aligned} \tag{18}$$

Re [ ] means "real part of" and Im [ ] means "imaginary part of". We now assume that the filter is so narrowband that its impulse response is much wider than the time over which $v_i(t)$ shows any appreciable correlation. (This assumption is the equivalent of Enloe's assumption that the optical spread function is much wider than the average distance between the images of scatterers with independent phase.) If this assumption holds, the voltage $v_o(t)$ results from many independent values of $v_i(t)$. According to the central limit theorem $v_o(t)$ and therefore $N_c(t)$ and $N_s(t)$ show a Gaussian distribution. The voltage $v_o(t)$ therefore has the statistical properties of narrowband Gaussian noise which are discussed for example in Ref. 2, pp. 397 ff. As was mentioned, the square of the envelope corresponds to the optical intensity. The envelope $E(t)$ is obtained as

$$\begin{aligned} E(t) &= [N_c^2(t) + N_s^2(t)]^{\frac{1}{2}} \\ &= \left(\left\{\mathrm{Re}\left[b\left(\frac{v_2 t}{\lambda f}\right)\right]\right\}^2 + \left\{\mathrm{Im}\left[b\left(\frac{v_2 t}{\lambda f}\right)\right]\right\}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{19}$$

According to Ref. 2, equation (9.18), $E(t)$ has a Rayleigh probability distribution density $W(E)$

$$W(E) = \frac{E \exp(-E^2/2\psi)}{\psi}. \tag{20}$$

The average value of the square of the envelope $\langle E^2 \rangle_{av}$ which is equal to the average value of the optical intensity $\langle I \rangle_{av}$ is given by [Ref. 2, Eq. (9.21a)]

$$\langle E^2 \rangle_{av} = \langle I \rangle_{av} = 2\psi. \tag{21}$$

We are now also interested in the autocorrelation $R(s)$ of the intensity I

$$\begin{aligned} R(s) &= \langle I(x_3) I(x_3 + s) \rangle_{av} \\ &= \langle E^2(x_3) E^2(x_3 + s) \rangle_{av} . \end{aligned} \tag{22}$$

According to Ref. 2, equation (9.24), $R(s)$ is given by the following expression

$$\begin{aligned} R(s) &= 4\psi^2[1 + k_0^2(s)] \\ &= \langle I^2 \rangle_{av}[1 + k_0^2(s)], \end{aligned} \tag{23}$$

where we have used equation (21). As explained in Ref. 2, [equations (9.10b and 9.12b)] $k_0(s)$ is the autocorrelation of the spread function corresponding to a filter with the frequency response $H_1((\lambda d/v_1)\nu)$. This is the frequency response of the filter in our electrical model (see Fig. 2), *but centered at zero frequency*. The frequency response $H_1((\lambda d/v_1)\nu)$ is shown in Fig. 4. $k_0(s)$ is normalized to one at $s = 0$. (The above discussion holds when the frequency response of the filter is symmetrical about the origin as is true for our case.) The spread function is the

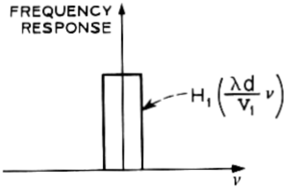

Fig. 4 — Frequency response of the filter $H_1[(\lambda d/v_1)\nu]$.

Fourier transform of the frequency response and so we have

$$FT\left[H_1\left(\frac{\lambda d}{v_1}\nu\right)\right] = c_4 h\left(\frac{v_2 t}{\lambda f}\right) \tag{24}$$

where $H_1(\nu)$ and $h(t)$ are Fourier transform pairs. c_4 is a constant of no interest. Therefore we obtain for $k_0(s)$

$$\begin{aligned} k_0(s) &= \frac{1}{\rho(0)} \int_{-\infty}^{+\infty} h^*\left(\frac{v_2 t}{\lambda f}\right) h\left(\frac{v_2 t}{\lambda f} + \frac{v_2 \tau}{\lambda f}\right) dt \\ &= \frac{\rho\left(\frac{v_2\tau}{\lambda f}\right)}{\rho(0)} = \frac{\rho\left(\frac{s}{\lambda f}\right)}{\rho(0)}, \end{aligned} \tag{25}$$

where

$$\rho(u) = \int h^*(t) h(t+u)\, dt. \tag{26}$$

Using equations (23) and (25) we finally obtain for $R(s)$

$$R(s) = \langle I^2 \rangle_{av}\left[1 + \frac{\rho^2\left(\frac{s}{\lambda f}\right)}{\rho^2(0)}\right]. \tag{27}$$

This last equation is the same as Enloe's equation (14) if the last term in that equation is disregarded. The last term in Enloe's equation (14) vanishes if the spread function of the filter is much broader than the average distance between images of scatterers with independent phase. The power spectrum follows from the autocorrelation function in the same way as in Enloe's analysis and this derivation will not be repeated here.

In summary: We have derived the statistical properties of the intensity in the image of a one-dimensional coherently illuminated and diffusely reflecting subject with the help of a narrowband noise model. Use was made of results known in communications theory.

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