Asymptotic Analysis of a Nonlinear Autonomous Vibratory System

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A system consisting of a spring, dashpot, and mass upon which is mounted an eccentric driven by a motor with a linear torque-speed characteristic, is analyzed by perturbation procedures based on small reciprocal of rotational inertia. Periodic solutions of the third order system, which arises when the angular position of eccentric mass is taken as the new independent variable, are constructed, and their stability is analyzed. An asymptotic solution is also obtained which is more general than a periodic solution, in that the averaged rotational speed is a slowly varying function, rather than a constant. The results are applicable to the determination of the interaction between the rotational motion of a flexibly mounted motor and the translational vibratory motion of its frame.

I. INTRODUCTION

In a recent paper Senator analyzed a system consisting of a spring, dashpot, and mass upon which is mounted a rotating eccentric weight driven by a motor with a linear torque-speed characteristic.[1] This system has been analyzed by several authors, under different assumptions on the values of the parameters of the system (see Refs. 5 through 9), and Senator discusses their results. The system is a model for the interaction between the rotational motion of a motor driving an eccentric and the translational vibratory motion of the frame, which is caused by this rotation.

In Ref. 1, Senator constructed periodic (rotational) solutions by means of a perturbation technique, based on small reciprocal of rotational inertia. However, he did not analyze the stability of the periodic solutions directly, but proceeded in a somewhat different manner. Thus, he introduced a van der Pol type transformation, but imposed a subsidiary condition on the slowly varying functions of time which differs from the one usually imposed in the method of averaging. He then made assumptions regarding the order of smallness of various derivatives, and dropped all second order terms from the equations for the slowly varying quantities, obtaining what he called averaged equations. The stationary solutions of these averaged equations correspond to periodic solutions of the original system, and he analyzed the stability of the stationary solutions on the basis of the corresponding linearized variational equations.

It is the purpose of this paper to show how the stability condition obtained by Senator may be derived rigorously for sufficiently large values of rotational inertia. This is done by taking the angular position of eccentric mass as the new independent variable, constructing periodic solutions of the resulting third order system, and then analyzing the linearized variational equations corresponding to them. Perturbation procedures, based on the small reciprocal of rotational inertia, are used.

An asymptotic solution is also obtained which is more general than a periodic solution, in that the averaged rotational speed is a slowly varying function, rather than constant. However, this asymptotic solution is not completely general, in that the transients in the translational motion, which decay on a much faster scale, are not included. The asymptotic solution nevertheless provides insight into the manner in which a stable periodic solution is approached, and the analytical results are borne out by some numerical calculations.

II. PERIODIC SOLUTIONS

The equations of motion, in dimensionless form, for the system under consideration are (from Ref. 1),

$$\frac{d^2u}{d\tau^2} + 2\zeta \frac{du}{d\tau} + u = \alpha\left[\left(\frac{d\theta}{d\tau}\right)^2 \sin\theta - \frac{d^2\theta}{d\tau^2}\cos\theta\right] \tag{1}$$

$$\frac{d^2\theta}{d\tau^2} = \epsilon\left(p - b\frac{d\theta}{d\tau} - \alpha\frac{d^2u}{d\tau^2}\cos\theta\right). \tag{2}$$

Here τ is dimensionless time, α, b, p and $\zeta > 0$ are constants, and $\epsilon > 0$, the reciprocal of dimensionless inertia, is a small parameter. Also, u is the dimensionless translational displacement, and θ is the angular position of eccentric mass. Instead of dealing with the fourth order system (1) and (2), as did Senator, it turns out to be more convenient to take θ as the new independent variable. Accordingly, defining

$$\Omega = \frac{d\theta}{d\tau} \tag{3}$$

the third order system

$$\Omega^2 \frac{d^2u}{d\theta^2} + 2\zeta\Omega \frac{du}{d\theta} + u + \Omega \frac{d\Omega}{d\theta}\left(\frac{du}{d\theta} + \alpha \cos \theta\right) = \alpha\Omega^2 \sin \theta \tag{4}$$

$$\Omega \frac{d\Omega}{d\theta}\left(1 + \epsilon\alpha \cos \theta \frac{du}{d\theta}\right) + \epsilon\alpha\Omega^2 \cos \theta \frac{d^2u}{d\theta^2} = \epsilon(p - b\Omega) \tag{5}$$

is obtained.

A periodic solution to (4) and (5) is sought in the form

$$u = \tilde{u}(\theta, \epsilon) \equiv u_0(\theta) + \epsilon u_1(\theta) + \epsilon^2 u_2(\theta) + \cdots \tag{6}$$

$$\Omega = \tilde{\Omega}(\theta, \epsilon) \equiv \omega_0 + \epsilon\Omega_1(\theta) + \epsilon^2\Omega_2(\theta) + \cdots \tag{7}$$

where $u_i(\theta)$ and $\Omega_i(\theta)$ are periodic in θ, with period 2π, and ω_0 is a constant. Substitution of (6) and (7) into (4) and (5), and comparison of the lowest powers of ϵ, leads to

$$\omega_0^2 \frac{d^2u_0}{d\theta^2} + 2\zeta\omega_0 \frac{du_0}{d\theta} + u_0 = \alpha\omega_0^2 \sin \theta \tag{8}$$

$$\omega_0 \frac{d\Omega_1}{d\theta} + \alpha\omega_0^2 \cos \theta \frac{d^2u_0}{d\theta^2} = (p - b\omega_0). \tag{9}$$

The periodic solution of (8) is

$$u_0 = \alpha\omega_0^2\Delta_0[(1 - \omega_0^2) \sin \theta - 2\zeta\omega_0 \cos \theta] \tag{10}$$

where

$$\Delta_0 = [(1 - \omega_0^2)^2 + 4\zeta^2\omega_0^2]^{-1}. \tag{11}$$

In order that $\Omega_1(\theta)$ should be periodic, it is necessary from (9) that

$$p = b\omega_0 + \alpha\omega_0^2\left\langle \cos \theta \frac{d^2u_0}{d\theta^2}\right\rangle_{av} = b\omega_0 + \alpha^2\zeta\omega_0^5\,\Delta_0 \equiv p^*(\omega_0) \tag{12}$$

using (10). $\langle\ \rangle_{av}$ denotes average over a period 2π of θ. Equation (12) gives a relationship between ω_0 and p, the dimensionless stall torque, and this relationship is depicted graphically in the figure for $\alpha = 0.707$, $\zeta = 0.2$, $b = 0$. It is noted that ω_0 is a triple valued function of p in part of the range. Senator concluded from his analysis that the middle branch corresponds to unstable periodic solutions, while the outer branches correspond to stable ones, a result verified in this paper.

Now, from (9), (10), and (12) it follows that

$$\Omega_1 = \{\omega_1 - \tfrac{1}{4}\alpha^2\omega_0^3\Delta_0[(1 - \omega_0^2) \cos 2\theta + 2\zeta\omega_0 \sin 2\theta]\} \tag{13}$$

where ω_1 is a constant, which is to be determined from the condition that $\Omega_2(\theta)$ should be periodic. It is clear as to how the higher order terms in the expansions in (6) and (7) may be obtained, but they will not be needed in the subsequent analysis. The periodic solutions in (6) and (7) are equivalent to those derived by Senator as periodic solutions of (1) and (2). It is necessary, of course, to perform a quadrature of equation (3) in order to obtain a relationship between θ and τ.

III. STABILITY ANALYSIS

The variational equations corresponding to the periodic solution $\tilde{u}$, $\tilde{\Omega}$, given by (6) and (7), are formed by substituting

$$u = (\tilde{u} + \xi), \qquad \Omega = (\tilde{\Omega} + \eta) \tag{14}$$

in (4) and (5), and linearizing in ξ and η. Thus,

$$\tilde{\Omega}^2 \frac{d^2\xi}{d\theta^2} + 2\tilde{\Omega}\frac{d^2\tilde{u}}{d\theta^2}\eta + 2\zeta\left(\tilde{\Omega}\frac{d\xi}{d\theta} + \frac{d\tilde{u}}{d\theta}\eta\right) + \xi$$
$$+ \tilde{\Omega}\frac{d\tilde{\Omega}}{d\theta}\frac{d\xi}{d\theta} + \left(\frac{d\tilde{u}}{d\theta} + \alpha\cos\theta\right)\left(\tilde{\Omega}\frac{d\eta}{d\theta} + \frac{d\tilde{\Omega}}{d\theta}\eta\right) = 2\alpha\tilde{\Omega}\sin\theta\eta, \tag{15}$$

$$\left(1 + \epsilon\alpha\cos\theta\frac{d\tilde{u}}{d\theta}\right)\left(\tilde{\Omega}\frac{d\eta}{d\theta} + \frac{d\tilde{\Omega}}{d\theta}\eta\right) + \epsilon\alpha\tilde{\Omega}\frac{d\tilde{\Omega}}{d\theta}\cos\theta\frac{d\xi}{d\theta}$$
$$+ 2\epsilon\alpha\tilde{\Omega}\cos\theta\frac{d^2\tilde{u}}{d\theta^2}\eta + \epsilon\alpha\tilde{\Omega}^2\cos\theta\frac{d^2\xi}{d\theta^2} + \epsilon b\eta = 0. \tag{16}$$

Equations (15) and (16) are linear equations with periodic coefficients, and the form of solution is known from Floquet theory.[2] Moreover, if all the characteristic exponents of the variational equations have negative real parts, then the periodic solution $\tilde{u}$, $\tilde{\Omega}$ is asymptotically stable. The behavior of the characteristic exponents will be analyzed for $0 < \epsilon \ll 1$.

The limiting case $\epsilon \to 0+$ will first be considered. In this case, from (6) and (7), $\tilde{\Omega} = \omega_0$ and $\tilde{u} = u_0(\theta)$ so that, from (15) and (16), $d\eta/d\theta = 0$ and

$$\omega_0^2\frac{d^2\xi}{d\theta^2} + 2\zeta\omega_0\frac{d\xi}{d\theta} + \xi = 2\left(\alpha\omega_0\sin\theta - \omega_0\frac{d^2u_0}{d\theta^2} - \zeta\frac{du_0}{d\theta}\right)\eta. \tag{17}$$

Hence one of the characteristic exponents is $\lambda_0 = 0$, and the remaining two characteristic exponents satisfy

$$(\omega_0\lambda_0)^2 + 2\zeta(\omega_0\lambda_0) + 1 = 0 \tag{18}$$

and hence have negative real parts, since $\zeta > 0$ and $\omega_0 > 0$. For sufficiently small ϵ, these real parts will remain negative, so that it suffices to investigate the characteristic exponent which vanishes as $\epsilon \to 0$.

In the light of Floquet theory, a solution of (15) and (16) is sought in the form

$$\xi = e^{\lambda\theta}P(\theta), \qquad \eta = e^{\lambda\theta}Q(\theta) \tag{19}$$

where P and Q are periodic in θ, with period 2π, and

$$\lambda = \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots \tag{20}$$

$$P(\theta) = P_0(\theta) + \epsilon P_1(\theta) + \epsilon^2 P_2(\theta) + \cdots \tag{21}$$

$$Q(\theta) = Q_0(\theta) + \epsilon Q_1(\theta) + \epsilon^2 Q_2(\theta) + \cdots . \tag{22}$$

It is a straightforward matter to substitute from (6), (7), and (19)–(22) into (15) and (16), and to compare like powers of ϵ. In particular, it is found from (16) that $dQ_0/d\theta = 0$. Omitting a multiplicative constant and taking $Q_0 = 1$, it is then found that

$$\omega_0^2 \frac{d^2P_0}{d\theta^2} + 2\omega_0 \frac{d^2u_0}{d\theta^2} + 2\zeta\omega_0 \frac{dP_0}{d\theta} + 2\zeta \frac{du_0}{d\theta} + P_0 = 2\alpha\omega_0 \sin\theta \tag{23}$$

and

$$\omega_0\left(\frac{dQ_1}{d\theta} + \lambda_1\right) + \frac{d\Omega_1}{d\theta} + 2\alpha\omega_0 \cos\theta \frac{d^2u_0}{d\theta^2} + \alpha\omega_0^2 \cos\theta \frac{d^2P_0}{d\theta^2} + b = 0. \tag{24}$$

Now, in order for $Q_1(\theta)$ to be periodic, it is necessary from (24) that

$$\omega_0\lambda_1 + b + \alpha\omega_0\left\langle \cos\theta \frac{d^2R_0}{d\theta^2}\right\rangle_{\text{av}} = 0 \tag{25}$$

where

$$R_0 = (\omega_0 P_0 + 2u_0) \tag{26}$$

is periodic in θ, with period 2π. But, from (23),

$$\omega_0^2 \frac{d^2R_0}{d\theta^2} + 2\zeta\omega_0 \frac{dR_0}{d\theta} + R_0 = 2\left(\alpha\omega_0^2 \sin\theta + \zeta\omega_0 \frac{du_0}{d\theta} + u_0\right) \tag{27}$$

and u_0 is given by (10) and (11). Straightforward calculations lead to

$$R_0 = 2\alpha\omega_0^2\, \Delta_0^2\{[(1-\omega_0^2)^2(2-\omega_0^2) + 4\zeta^2\omega_0^2(1-2\omega_0^2)]\sin\theta - \zeta\omega_0[(1-\omega_0^2)(5-\omega_0^2) + 12\zeta^2\omega_0^2]\cos\theta\}. \tag{28}$$

Thus, from (25),

$$\omega_0\lambda_1 + b + \alpha^2\zeta\omega_0^4\,\Delta_0^2[(1 - \omega_0^2)(5 - \omega_0^2) + 12\zeta^2\omega_0^2] = 0. \tag{29}$$

However, as may be verified from (12), using (11), equation (29) may be written in the form

$$\omega_0\lambda_1 + \frac{dp^*}{d\omega_0} = 0. \tag{30}$$

Since the sign of λ in (20) is determined by the sign of λ_1, for sufficiently small $\epsilon > 0$, it follows that the periodic solution $\tilde{u}$, $\tilde{\Omega}$ is asymptotically stable if $dp^*/d\omega_0 > 0$, and is unstable if $dp^*/d\omega_0 < 0$. That is, the middle branch of the figure corresponds to unstable periodic solutions, while the outer branches correspond to asymptotically stable ones, provided that $\epsilon > 0$ is sufficiently small.

IV. MORE GENERAL SOLUTIONS

In this section a more general solution of (4) and (5) is constructed, for $0 < \epsilon \ll 1$. Thus an asymptotic solution is sought in the form

$$u = v_0(\omega, \theta) + \epsilon v_1(\omega, \theta) + \epsilon^2 v_2(\omega, \theta) + \cdots \tag{31}$$

$$\Omega = \omega + \epsilon w_1(\omega, \theta) + \epsilon^2 w_2(\omega, \theta) + \cdots \tag{32}$$

where $v_i(\omega, \theta)$ and $w_i(\omega, \theta)$ are periodic in θ, with period 2π, and

$$\frac{d\omega}{d\theta} = \epsilon g_1(\omega) + \epsilon^2 g_2(\omega) + \cdots . \tag{33}$$

This procedure may be regarded as a variant of the method of averaging.[3] The above solution is more general than the periodic solutions constructed previously, since the latter correspond to the case in which ω is constant, rather than a slowly varying function of θ. However, this solution is not completely general, since the initial transients in (4) are not taken into account.

Substituting (31) and (32) into (4) and (5), using (33), and comparing the lowest powers of ϵ, it follows that

$$\omega^2 \frac{\partial^2 v_0}{\partial \theta^2} + 2\zeta\omega \frac{\partial v_0}{\partial \theta} + v_0 = \alpha\omega^2 \sin\theta \tag{34}$$

and

$$\omega\left(\frac{\partial w_1}{\partial \theta} + g_1\right) + \alpha\omega^2 \cos\theta \frac{\partial^2 v_0}{\partial \theta^2} = (p - b\omega). \tag{35}$$

The periodic solution of (34) is

$$v_0 = \alpha\omega^2\Delta(\omega)[(1 - \omega^2) \sin \theta - 2\zeta\omega \cos \theta] \tag{36}$$

where

$$\Delta(\omega) = [(1 - \omega^2)^2 + 4\zeta^2\omega^2]^{-1}. \tag{37}$$

In order that w_1 should be periodic, it is necessary from (35), using (36), that

$$\omega g_1(\omega) = [p - b\omega - \alpha^2\zeta\omega^5\Delta(\omega)]. \tag{38}$$

Then $w_1(\omega, \theta)$ may be found from (35), to within an arbitrary function of ω. This arbitrariness is usual in averaging procedures, and may be removed by requiring $\langle w_i(\omega, \theta)\rangle_{av} = 0$, so that, from (32), ω is the averaged value of Ω. Higher order terms in the asymptotic expansion (31) and (32) may be obtained in a systematic manner.

Now, from (33) and (38),

$$\frac{d\omega}{d\theta} = \frac{\epsilon}{\omega} [p - p^*(\omega)] + O(\epsilon^2) \tag{39}$$

where

$$p^*(\omega) = b\omega + \alpha^2\zeta\omega^5\Delta(\omega). \tag{40}$$

As previously remarked, the case in which ω is constant corresponds to the periodic solutions constructed earlier. From (11), (12), (37), and (40), it follows that ω_0 is the lowest order approximation to a stationary solution of (39). If $\omega \neq \omega_0$, equation (39) determines, to lowest order, the slow variation of ω with θ. The direction in which ω changes is determined, to lowest order in ϵ, by the sign of $[p - p^*(\omega)]$, and is illustrated in Fig. 1 for $p = 0.45$, to which there correspond three values of ω_0, denoted by ω_{0l}, ω_{0m}, and ω_{0r}.

Under more general initial conditions similar results should hold, for sufficiently small ϵ, provided that the initial value of Ω is not too close to ω_{0m}. This is because Ω does not change significantly, for sufficiently small ϵ, during the time in which the initial transients in the translational motion die out.

A partial check of these analytical results was made by Senator,[4] who carried out some numerical solutions of (1) and (2). With $\alpha = 0.707$, $\zeta = 0.2$, $b = 0$ and $\epsilon = 0.1$, he chose initial conditions consistent with the unstable periodic solution corresponding to $p = 0.425$, that is, the periodic solution corresponding to $p^*(\omega_{0m}) = 0.425$. He then carried out numerical solutions of (1) and (2) for $p = 0.45$ and $p = 0.4$. He found that for $p = 0.45$ the solution approaches the periodic solution corresponding to ω_{0r} in the figure, that is, to $p^*(\omega_{0r}) = 0.45$, while for

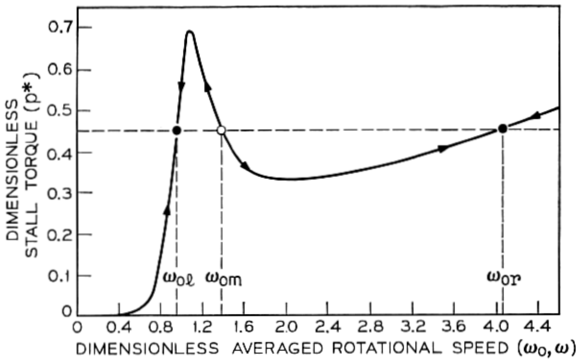

Fig. 1 — Stall torque vs. averaged rotational speed.

$p = 0.4$ the solution approaches the periodic solution corresponding to $p^*(\omega_{0l}) = 0.4$. These results are consistent with our analytical results. Moreover, the number of cycles required before the solution settles down to the appropriate periodic solution was somewhat larger in the case $p = 0.45$ than in the case $p = 0.4$. This is consistent with (39), and the presence of the factor $(1/\omega)$ multiplying $[p - p^*(\omega)]$ therein.

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REFERENCES

1. Senator, M., "Limit Cycles and Stability of a Nonlinear Two-Degree of Freedom Autonomous Vibratory System," Journal of Engineering for Industry, Trans. ASME, *91,* Series B, No. 4 (November 1969), pp. 959–966.
2. Minorsky, N., *Nonlinear Oscillations,* Princeton: Van Nostrand, 1962, pp. 127–129.
3. Bogoliubov, N. N., and Mitropolsky, Y. A., *Asymptotic Methods in the Theory of Nonlinear Oscillations,* New York: Gordon and Breach, 1962, p. 40.
4. Senator, M., unpublished work.
5. Rocard, Y., *General Dynamics of Vibrations,* New York: Ungar, 1960, trans. 3rd French ed., pp. 362–369.
6. Hoekstra, T. B., "The Response of a Nonlinear Two Degree of Freedom System," Ph.D. dissertation, University of Michigan, Dept. of Eng. Mechanics, 1966, Chapter 5.
7. Kononenko, V. O., "Some Autonomous Problems of the Theory of Nonlinear Oscillations," (in Russian) Trudy Mezhdunarodnovo Simpoziums po Nelineinym Kolebaniyam Izdatel'stvo AN SSSR, *3,* 1963, pp. 151–178.
8. Kononenko, V. O., *Kolebatel'nye Sistemy s Ogranichennym Vozbuzhdeniem,* Moscow, 1964, (L.C. No. QA 871 K7), pp. 51–79.
9. Mazet, R., *Mecanique Vibratoire,* Paris: Dunod, 2nd ed., 1966, pp. 308–318.