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Analytical Approximations to Approximations in the Chebyshev Sense

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This paper concerns approximation in the Chebyshev, or minimax sense such that (i) a minimax approximation implies a maximum number of zero error points separated by equal error extrema, and (ii) the approximating function can be so formulated that the disposable parameters are all the coefficients in a polynomial, which may however be part of a more complicated function the rest of which is prescribed. Weighted minimax polynomial approximations can be included, by multiplying the approximated and approximating functions by the weight factor. Analytic methods are described which yield approximately equal error extrema. They are sufficiently simple so that they may sometimes compete with currently used iterative numerical methods, especially when the degree of the disposable polynomial is large. Their most probable utility concerns explorations of available accuracies over wide ranges of design parameters such as degree of disposable polynomial, interval of approximation, and coefficients in prescribed parts of the approximating function.

I. INTRODUCTION

This paper concerns approximation in the Chebyshev sense, over a prescribed interval $x_a \leqq x \leqq x_b$ of a continuous real variable x. As defined, an approximation in the Chebyshev sense is a minimax approximation—one in which the maximum error is as small as is possible within given constraints on the approximating function. Minimax approximations in which errors are weighted by a prescribed function of the independent variable can also be treated as Chebyshev approximations, by multiplying the approximated and approximating functions by the weight function.

Frequently, but not always, approximation in the Chebyshev sense implies an error of the "equal ripple" sort illustrated in Fig. 1—that is, a sequence of equal positive and negative extrema with monotonic variations in between. General necessary and sufficient conditions for this are not known. However, the following conditions are sufficient: the p disposable parameters of the approximating function are to be such that the approximation error can be made zero at p arbitrary points within the approximating interval. Referring to Fig. 1, the arbitrary error points divide the approximation interval into $p + 1$ segments. There is to be a particular division such that the error function achieves its maximum magnitude $p + 1$ times—at the two edges of the approximation interval and once within each of the $p - 1$ interior segments. There are to be no other local extrema within the approximation interval. Generally, shrinking any one of the $p + 1$ segments (by bringing two zero error points closer together or one closer to an edge of the approximation interval) tends to reduce the corresponding error extremum. Conditions are to be such that all the $p + 1$ equal extrema can be reduced simultaneously only by shrinking all the $p + 1$ segments, which is impossible without shrinking the given approximation interval. These conditions are encountered in many practical problems and are assumed here. Thus we are concerned only with equal ripple approximations like Fig. 1.

Exactly equal ripple approximations have long been known for a very few special cases (which have been useful for example in filter design). Iterative numerical methods have been developed for the solution of various more general problems and are described in text-

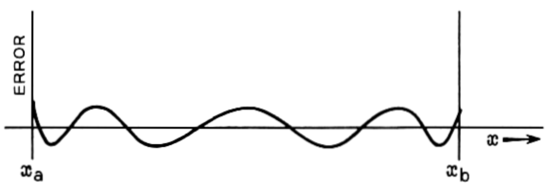

Fig. 1 — An equal ripple error function (p zeros; $p + 1$ segments; $p + 1$ extrema).

books such as Ref. 1. In contrast, this paper describes analytical procedures which yield error extrema of approximately equal amplitude. Their full range of validity has not been determined. However, they are clearly appropriate for a substantial, although poorly defined class of problems. It is characterized further later.

Useful applications are likely to concern equal ripple problems which have not been solved exactly by analysis and which involve so many disposable parameters that iterative numerical solutions are likely to be more costly. The most useful applications probably concern preliminary explorations over primary design parameters (such as intervals of approximation, magnitudes of errors, and degrees of approximating functions) before numerical refinement of specific designs. Accordingly, this paper emphasizes relatively simple means for approximating equal ripples and says little about more complicated higher order approximations.

The procedures apply only to approximating functions characterized as follows. The disposable parameters must be all the coefficients in a polynomial (which may have been obtained, however, by some sort of transformation on the original independent variable and/or the approximating function). This is referred to as the disposable polynomial. On the other hand, the disposable polynomial may be only a part of a more general approximating function the rest of which is prescribed in advance (for example, the numerator of a rational fraction with a prescribed denominator). Weighted as well as unweighted minimax approximations are included. For some problems, closed form formulas are obtained for approximate error size as functions of the degree of the disposable polynomial, usable for degrees of any size. For other problems, the error size is related to an eigenvalue of a certain matrix equation, but the order of the matrix may be small even though the degree of the disposable polynomial is arbitrarily large.

A primary concern here is the distinction between simple truncation of infinite series of Chebyshev polynomials and approximation in the Chebyshev or minimax sense. The functions which we are to approximate can be expanded into infinite series of Chebyshev polynomials. Approximations with polynomials of degree n can be obtained by simply truncating the infinite series after the terms of degree n. However, simple truncation does not usually give an approximation of the minimax sort. A polynomial of degree n which approximates the given function in the minimax manner can be represented as a linear combination of Chebyshev polynomials, but the coefficients are usually different from those in the truncated infinite series.

One way to approach approximation in the Chebyshev sense is to start with the truncated series of Chebyshev polynomials. Then corrections to the coefficients are determined, to obtain equal ripple error functions. Such a procedure has been used before, for example in Refs. 2 through 4, and is used here. Departures from the previous work known to the author include simple approximations to ideal solutions formulated for more general approximating functions and for weighted as well as unweighted minimax approximations, as opposed to more rigorous analyses of more restricted problems.

Sometimes truncation of an infinite series of Chebyshev polynomials yields an approximately equal ripple error function without further adjustment of the coefficients. The procedures for adjusting the coefficients, described herein, sometimes also give an initial insight into whether or not adjustments are needed.

It is interesting to note that some 35 years ago a conference was held in the office of T. C. Fry, at Bell Telephone Laboratories, to consider some filter patents offered for sale by W. Cauer. One of the patents disclosed Cauer's equal ripple image impedance and transfer functions, which soon became famous among circuit theorists, but did not include proofs or derivations. At the conference, S. A. Schelkunoff asserted a very simple principle which enabled him to confirm and interpret Cauer's formulas. However, it did not explain how Cauer might have derived or discovered the formulas. The principle applies also to more general equal ripple approximations. It does not, by itself, solve the approximation problem, but it does furnish a starting point from which to develop procedures which do. We call it Schelkunoff's principle.

Section II describes Schelkunoff's principle. Section III solves two problems for which exactly equal ripple solutions are easily found. Section IV develops general procedures, whereby approximate solutions can be obtained for a large class of problems. Section V further clarifies the general procedures by means of examples.

Various aspects of the procedures described here bear some relation to other work. Section VI notes some of these relationships. Finally, Section VII reviews and summarizes the general conclusions, including a comment on the possibility of generalizations to disposable rational fractions.

II. SCHELKUNOFF'S PRINCIPLE

Consider first a function $\mathbf{T}_n(x)$ proportional to a Chebyshev polynomial, defined by

$$\mathbf{T}_n(x) = \frac{K_n}{2^{n-1}} \cos (n \cos^{-1} x). \tag{1}$$

It is illustrated in Fig. 2a, for $n = 4$. It may be regarded as an "equal ripple" approximation to zero, over the interval $-1 \leqq x \leqq +1$, by a polynomial of degree n in which the coefficient of x^n is required to be K_n. Let

$$x = \cos \varphi. \tag{2}$$

Substitution in equation (1) gives

$$\mathbf{T}_n(x) = T_n(\varphi) = \frac{K_n}{2^{n-1}} \cos n\varphi. \tag{3}$$

The new function is illustrated in Fig. 2b, again for $n = 4$. Note that x is periodic in φ with period 2π and $T_n(\varphi)$ is periodic in φ with period $2\pi/n$. Thus there are n periods of $T_n(\varphi)$ in each period of x.

Stated with a little more detail, we have this situation: The original function $\mathbf{T}_n(x)$ has "equal ripples" in the sense of equal extrema. However, the extrema are not uniformly spaced and hence the ripples differ as to width. The periodic transformation from x to φ has two important properties. As φ increases, x sweeps back and forth across the approximation interval, $-1 \leqq x \leqq +1$. In each interval in which x varies monotonically from ± 1 to ∓ 1 the φ scale is a distortion of the x scale such that the ripples of $T_n(\varphi)$ are uniformly spaced and are

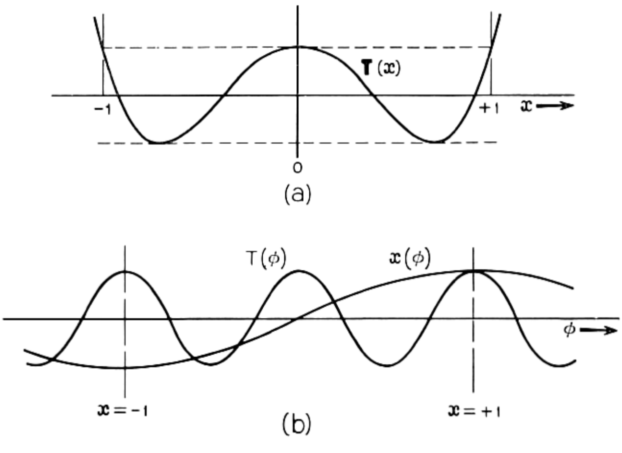

Fig. 2 — Illustrating Schelkunoff's principle.

also otherwise identical. The ripples could not be made identical by any distortion of the x scale alone if $\mathbf{T}_n(x)$ had *unequal* maxima or *unequal* minima. This is a special case of Schelkunoff's principle.

More generally, let $\mathbf{E}(x)$ be a function of x with the following properties over an interval $x_a \leqq x \leqq x_b$: The function $\mathbf{E}(x)$ is real and single valued; there are a number of local maxima, all equal; there are a number of local minima, all equal; the equal extrema for the interval include the end points $\mathbf{E}(x_a)$, $\mathbf{E}(x_b)$ (at which $d\mathbf{E}/dx$ need not $=0$).* Then Schelkunoff's principle asserts the existence of a transformation

$$x = \Gamma(\varphi) \tag{4}$$

with the following properties: The original variable x is periodic in the new variable φ; as φ increases x sweeps back and forth over the given interval $x_a \leqq x \leqq x_b$, monotonically each way once each period; the periodic function

$$E(\varphi) = \mathbf{E}[\Gamma(\varphi)] \tag{5}$$

has a number of periods in each period of x, equal to one less than the number of extrema of $\mathbf{E}(x)$ in the given interval of x (including the end points). In applications to approximation in the Chebyshev sense, $\mathbf{E}(x)$ and $E(\varphi)$ represent the equal ripple error, as functions of x and φ.

The transformation $x = \Gamma(\varphi)$ clearly is not unique, for there are obvious transformations on φ itself which retain the desired character of $E(\varphi)$. For example, φ can be replaced by $\varphi + q(\varphi)$, where $q(\varphi)$ is periodic with the same period as $E(\varphi)$ and is such that $\varphi + q(\varphi)$ is monotonic in φ. When $\mathbf{E}(x)$ is continous (in the given interval of x), a particular $\varphi + q(\varphi)$ will make $E(\varphi)$ sinusoidal.

We do not attempt a very general, rigorous proof of Schelkunoff's principle; we merely use it as a guide to a strategy for solving minimax problems. However, a demonstration of the principle for a specific class of problems will be implicit in what follows, for we shall find transformations which do in fact change our equal ripple errors into sinusoidal errors.

In later sections we will again use the transformatlon (2), or a generalization for end points other than $x = \pm 1$. Usually, however, it will not be a Schelkunoff transformation. We will use it to transform the disposable polynomial in x into a finite Fourier series in φ. The coef-

* Problems can be found such that minimax approximations have equal extrema which do not include both end points. Then the number of local extrema for the approximation interval is abnormally large when the end points are counted. Such problems are not considered here.

ficients of the Fourier series are to be chosen in such a way that the overall approximation is approximately sinusoidal on a *distortion* of the φ scale. Similar strategies have been used before, for example in Refs. 2 through 4.

Means for determining the distortion of the φ scale and the adjustment of the Fourier coefficients are introduced by means of two examples in the next section.

III. TWO GENERALIZATIONS OF CHEBYSHEV POLYNOMIALS

The two problems described below are solved exactly. The form of the solutions suggests approximate solutions to more general problems.

3.1 *A Rational Function Generalization of Chebyshev Polynomials*

Consider the following generalization of Chebyshev polynomials: Let

$$\mathbf{T}_{Dn}(x) = \frac{\mathbf{P}(x)}{\mathbf{D}(x)} \tag{6}$$

in which $\mathbf{P}(x)$ is a polynomial of degree n and $\mathbf{D}(x)$ is a polynomial of degree $\leqq n$. Suppose $\mathbf{D}(x)$ is prescribed in advance and $\mathbf{P}(x)$ is to be chosen in such a way that $\mathbf{T}_{Dn}(x)$ has equal ripples like those of a Chebyshev polynomial in the interval $-1 \leqq x \leqq +1$. More specifically, require that $\mathbf{T}_{Dn}(x) = \pm 1$ at $n - 1$ local extrema within the interval $-1 \leqq x \leqq +1$ and at the end points $x = \pm 1$. Real zeros of $\mathbf{D}(x)$ are to be excluded from the interval $-1 \leqq x \leqq +1$. The conditions on the extrema insure that all n of the zeros of $\mathbf{P}(x)$ will be in the interval.

Let φ be defined again by equation (2) and note that the real axis in the φ plane corresponds to the real interval $-1 \leqq x \leqq +1$ in the x plane. If $\cos \varphi$ is written in terms of exponentials, the polynomial $\mathbf{P}(x)$ can be related to φ by

$$\mathbf{P}(x) = P(e^{i\varphi}) + P(e^{-i\varphi}) \tag{7}$$

in which $P(\cdot)$ is a polynomial of the same degree, n, as $\mathbf{P}(\cdot)$. Given the coefficients of $P(\cdot)$ it is a simple matter to compute the coefficients of $\mathbf{P}(\cdot)$. We shall consider our problem solved when we have found the coefficients of $P(\cdot)$ required for our equal ripple conditions.

It is convenient to relate the prescribed denominator $\mathbf{D}(x)$ to φ in the following slightly different way:

$$\mathbf{D}(x) = D(e^{i\varphi})\, D(e^{-i\varphi})$$

in which $D(\cdot)$ is a polynomial of the same degree, $\leqq n$, as $\mathbf{D}(\cdot)$. If $\mathbf{D}(\cdot)$ and $D(\cdot)$ are written in factored form there is a one to one correspondence between factors. Thus if $(x_\sigma - x)$ is a factor of $\mathbf{D}(x)$ and $(1 - \gamma_\sigma e^{i\varphi})$ a corresponding factor of $D(e^{i\varphi})$,

$$x_\sigma - x = M_\sigma(1 - \gamma_\sigma e^{i\varphi})(1 - \gamma_\sigma e^{-i\varphi}) \tag{9}$$

in which M_σ is a constant scale factor.

By equation (2), $e^{i\varphi} = \pm 1$ at $x = \pm 1$, and hence

$$x_\sigma \pm 1 = M_\sigma(1 \pm \gamma_\sigma)^2. \tag{10}$$

Given x_σ, two solutions for γ_σ are easily obtained, for which $|\gamma_\sigma|$ is respectively < 1 and > 1. (Exclusion of zeros x_σ of $\mathbf{D}(x)$ from the real interval $-1 \leqq x \leqq +1$ removes the possibility of $|\gamma_\sigma| = 1$.) We need the solution for which $|\gamma_\sigma| < 1$, for reasons which will soon be apparent. From equation (11), with the sign of the square root chosen for $|\gamma_\sigma| < 1$,

$$\frac{1 - \gamma_\sigma}{1 + \gamma_\sigma} = \left[\frac{x_\sigma - 1}{x_\sigma + 1}\right]^{\frac{1}{2}}, \qquad \mathrm{Re}\left[\frac{x_\sigma - 1}{x_\sigma + 1}\right]^{\frac{1}{2}} > 0. \tag{11}$$

The scale factor M_σ need not concern us at this time.

The function $\mathbf{T}_{Dn}(x)$ can now be mapped into a function $T_{Dn}(\varphi)$ in terms of equations (7) and (8).

$$\mathbf{T}_{Dn}(x) = T_{Dn}(\varphi) = \frac{P(e^{i\varphi}) + P(e^{-i\varphi})}{D(e^{i\varphi})\, D(e^{-i\varphi})}, \tag{12}$$

By Schelkunoff's principle, our requirements on the extrema of $\mathbf{T}_{Dn}(x)$ imply that $T_{Dn}(\varphi)$ has the following special form

$$\begin{aligned} T_{Dn}(\varphi) &= \frac{P(e^{i\varphi}) + P(e^{-i\varphi})}{D(e^{i\varphi})\, D(e^{-i\varphi})}, \\ &= \tfrac{1}{2}e^{i[n\varphi + f(\varphi)]} + \tfrac{1}{2}e^{-i[n\varphi + f(\varphi)]}. \end{aligned} \tag{13}$$

The variable $\varphi + (1/n)f(\varphi)$ is a distortion of the φ scale for Schelkunoff's principle, for which $f(\varphi)$ is to be periodic in φ with period 2π and $\varphi + (1/n)f(\varphi)$ is to vary monotonically with φ.

Given $f(\varphi)$ one can easily find $P(e^{i\varphi})$ by equation (13). The problem is to find an $f(\varphi)$ for which $P(\cdot)$ is a polynomial of degree n. From equation (13)

$$P(e^{i\varphi}) + P(e^{-i\varphi}) = \begin{Bmatrix} \tfrac{1}{2}e^{i[n\varphi + f(\varphi)]}\, D_n(e^{i\varphi})\, D_n(e^{-i\varphi}) \\ + \tfrac{1}{2}e^{-i[n\varphi + f(\varphi)]}\, D_n(e^{i\varphi})\, D_n(e^{-i\varphi}) \end{Bmatrix}. \tag{14}$$

If $P(e^{i\varphi})$ is to have no terms in $e^{i\sigma\varphi}$ with $\sigma > n$, $e^{if(\varphi)}$ needs to cancel out $D_n(e^{i\varphi})$. This suggests

$$e^{if(\varphi)} = \frac{D(e^{-i\varphi})}{D(e^{i\varphi})} \tag{15}$$

(and note that this does make $f(\varphi)$ real when φ is real). Substitution in equation (14) gives

$$P(e^{i\varphi}) + P(e^{-i\varphi}) = \tfrac{1}{2}e^{in\varphi}\, D^2(e^{-i\varphi}) + \tfrac{1}{2}e^{-in\varphi}\, D^2(e^{i\varphi}). \tag{16}$$

Expanding the right side gives a polynomial in $e^{i\varphi}$. When polynomial $D(\cdot)$ is of degree $\leqq n$ (as assumed) there will be no powers of $e^{i\varphi}$ outside the range $-n$ to $+n$. Collecting positive powers (and half the constant term) gives $P(e^{i\varphi})$.

The function $f(\varphi)$ determined by equation (15) is periodic in φ with period 2π provided the zeros of $D(\lambda)$ lie outside the unit circle in the λ plane, which is assumed by equation (11). It is easily shown that the same condition makes $\varphi + (1/n)f(\varphi)$ monotone in φ. Once $P(\cdot)$ is known it is a simple matter to find $\mathbf{P}(x)$ by means of equations (2) and (7). It is probably simplest to omit the scale factor M_σ of equation (10) in the initial formulation. This does not affect the ratio in equation (15), but only the scale factor of the polynomial $\mathbf{P}(x)$, which can be corrected later on [for example to meet the condition $\mathbf{T}_{Dn}(1) = 1$].

Obvious generalizations of the problem include the following: For extrema $A \pm J$ (instead 0 ± 1) use

$$\mathbf{F}(x) = A \pm J\mathbf{T}_{Dn}(x). \tag{17}$$

In a more general interval of x, say $x_a \leqq x \leqq x_b$, replace equation (2) by

$$x = \frac{x_b + x_a}{2} + \frac{x_b - x_a}{2} \cos \varphi \tag{18}$$

and change equation (10) to

$$x_\sigma - x_a = M_\sigma(1 - \gamma_\sigma)^2, \qquad x_\sigma - x_b = M_\sigma(1 + \gamma_\sigma)^2. \tag{19}$$

The function $\mathbf{F}(x)$ defined by equation (17) has long been used by filter theorists, but previous derivations have been quite different.[5] The form of equation (16) suggests a similar solution to the problem described below.

3.2 *An Irrational Generalization of Chebyshev Polynomials*

Now let

$$\mathbf{T}_{Sn}(x) = \frac{\mathbf{P}(x)}{[\mathbf{S}(x)]^{\frac{1}{2}}} \tag{20}$$

in which $\mathbf{P}(x)$ is again a polynomial of degree n but $\mathbf{S}(x)$ is a polynomial of degree $\leqq 2n$. Suppose $\mathbf{S}(x)$ is prescribed and that $\mathbf{T}_{Sn}(x)$ is to meet the same conditions as to extrema as $\mathbf{T}_{Dn}(x)$ in the previous subsection. In place of equation (8), we can now use

$$\mathbf{S}(x) = S(e^{i\varphi})S(e^{-i\varphi})$$

to determine a polynomial $S(\cdot)$. We can then replace equation (15) by

$$e^{if(\varphi)} = \left[\frac{S(e^{-i\varphi})}{S(e^{i\varphi})}\right]^{\frac{1}{2}} \tag{21}$$

and then equation (16) by

$$P(e^{i\varphi}) + P(e^{-i\varphi}) = \tfrac{1}{2}e^{in\varphi}S(e^{-i\varphi}) + \tfrac{1}{2}e^{-in\varphi}S(e^{i\varphi}). \tag{22}$$

This makes $P(\cdot)$ again a polynomial of degree n. Note that $\mathbf{T}_{Sn}(x)$ cannot be used in place of $\mathbf{T}_{Dn}$ in equation (17), with $A \neq 0$, without changing the polynomial character of the numerator.

IV. GENERAL FORMULATIONS

This section shows how a large class of minimax approximations can be approximated by generalizing the manipulations described above. In Section V, we clarify the general procedures further by providing examples.

4.1 *Unweighted Minimax Approximations*

Let

$$\mathbf{P}(x) = \mathbf{F}(x) + \boldsymbol{\varepsilon}(x), \qquad x_a \leqq x \leqq x_b \tag{23}$$

in which $\mathbf{P}(x)$ is a disposable polynomial of degree n, $\mathbf{F}(x)$ is a given function to be approximated by $\mathbf{P}(x)$ in the interval $x_a \leqq x \leqq x_b$, and $\boldsymbol{\varepsilon}(x)$ is the error in the approximation. For what $\mathbf{P}(x)$ is $\boldsymbol{\varepsilon}(x)$ smallest in the minimax sense? We assume that the minimax $\boldsymbol{\varepsilon}(x)$ has the equal ripple form (Fig. 1) and we seek only approximations to equal ripples. We also restrict the class of applicable functions by certain further assumptions which can best be introduced a little later.

As before, let x and φ be related by equation (18), so that $x_a \leqq x \leqq x_b$ maps into real φ, and replace $\mathbf{P}(x)$ by

$$\mathbf{P}(x) = P(e^{i\varphi}) + P(e^{-i\varphi}). \tag{24}$$

If $P(\cdot)$ is again a polynomial of degree n, it is uniquely determined by $\mathbf{P}(\cdot)$ [and x_a, x_b in equation (18)]. Now, however, we find it expedient to permit $P(z)$ to include negative powers of z, up to z^{-n}. Thus, in equation (24),

$$P(e^{i\varphi}) = \sum_{\sigma=-n}^{n} P_\sigma e^{i\sigma\varphi}. \tag{25}$$

This $P(\cdot)$ is not uniquely determined by $\mathbf{P}(\cdot)$. However, $\mathbf{P}(\cdot)$ *is* uniquely determined by $P(\cdot)$, and it is still a polynomial of degree n. We solve the approximation problem by finding a suitable $P(\cdot)$, from which $\mathbf{P}(\cdot)$ can be easily determined.

We require that the mapping from x to φ maps $\mathbf{F}(x)$ into a function of φ with a convergent Fourier series. This amounts to requiring that $\mathbf{F}(x)$ can be expanded into a convergent series of Chebyshev polynomials (defined to fit the given interval of approximation). Because equation (18) is even in φ, the Fourier series has only cosine terms. Then, replacing cosines by sums of exponentials,

$$\begin{aligned} \mathbf{F}(x) &= F(e^{i\varphi}) + F(e^{-i\varphi}) \\ F(e^{i\varphi}) &= \sum_{\sigma=0}^{\infty} C_\sigma e^{i\sigma\varphi} \end{aligned} \tag{26}$$

in which the series expansion of $F(e^{i\varphi})$ converges when φ is real.

The desired equal ripple error can be written

$$\boldsymbol{\varepsilon}(x) = \epsilon \cos\left[(n+1)\varphi + f(\varphi)\right] \tag{27}$$

in which $f(\varphi)$ is again periodic in φ and represents the distortion of the φ scale per Schelkunoff's principle. In an equivalent exponential form

$$\begin{aligned} \boldsymbol{\varepsilon}(x) &= E(e^{i\varphi}) + E(e^{-i\varphi}) \\ E(e^{i\varphi}) &= \frac{\epsilon}{2}\, e^{i[(n+1)\varphi + f(\varphi)]}. \end{aligned} \tag{28}$$

The exponent $i(n+1)\varphi$, instead of $in\varphi$ as in the previous section, reflects the following circumstances: If the Chebyshev polynomial series corresponding to equation (26) is truncated after the polynomial of degree n, the first omitted polynomial is of degree $n+1$. If all the other omitted polynomials have sufficiently small coefficients, the truncation error will approximate $\mathbf{E}(x)$ of equation (23) with $f(\varphi) = 0$. Note also that a disposable polynomial of degree n has $n+1$ disposable coefficients. These are an example of the p disposable parameters in the more general description of equal ripple errors in Section I.

Using equations (24), (26) and (28) in equation (23) gives

$$P(e^{i\varphi}) + P(e^{-i\varphi}) = F(e^{i\varphi}) + F(e^{-i\varphi}) + E(e^{i\varphi}) + E(e^{-i\varphi}). \tag{29}$$

We arbitrarily equate the terms in exp $(+i\varphi)$ separately, so that

$$P(e^{i\varphi}) = F(e^{i\varphi}) + E(e^{i\varphi}). \tag{30}$$

If equation (30) is satisfied at all real φ so is the corresponding equation in exp $(-i\varphi)$. Thus a solution of equation (30) is a solution of equation (29). But the converse is not necessarily true. Frequently, an exactly equal ripple approximation corresponds to a solution of equation (29) which is not a solution of equation (30). However, we will find that approximations with approximately equal ripples can frequently be derived from equation (30), and in a much simpler way.

In equation (30), expand $P(\cdot)$, $F(\cdot)$, $E(\cdot)$ per equations (24), (26) and (28). The result can be rearranged as follows:

$$\begin{aligned} \sum_{\lambda=1}^{2n+1} G_\lambda e^{-i\lambda\varphi} &= \frac{\epsilon}{2} e^{if(\varphi)} + \sum_{\lambda=0}^{\infty} C_{n+1+\lambda} e^{i\lambda\varphi}; \\ G_\lambda &= P_{n+1-\lambda} - C_{n+1-\lambda}, \qquad \lambda \leqq n+1 \ \lambda \leqq n+1; \\ &= P_{n+1-\lambda}, \qquad n+1 < \lambda \leqq 2n+1. \end{aligned} \tag{31}$$

In this equation, $C_{n+1+\lambda}$ is fixed by equation (26) but $P_{n+1-\lambda}$ is a disposable parameter in equation (25). Thus we seek an ϵ and exp $[if(\varphi)]$ with the following properties: First, $(\epsilon/2)$ exp $[if(\varphi)]$ is to be expandable in terms of positive and negative powers of exp $(i\varphi)$. Second, the coefficients of positive powers are to cancel the corresponding coefficients $C_{n+1+\lambda}$ in equation (31). Third, the coefficients of negative powers are to be such that, with an appropriate ϵ, $|\exp [\text{if } (\varphi)]| = 1$ when φ is real, so that $f(\varphi)$ is real and the error extrema are equal per equation (27). Sometimes it turns out that there are no negative powers beyond $-(2n + 1)$. Then the left side of equation (31) can be adjusted to match the right side. In many other problems, approximately equal error extrema can be obtained by simply ignoring terms in negative powers beyond $-(2n + 1)$.

Now consider the class of functions $F(\cdot)$ such that, in equation (31)

$$\sum_{\lambda=0}^{\infty} C_{n+1+\lambda} e^{i\lambda\varphi} = \frac{B(e^{i\varphi})}{A(e^{i\varphi})} \tag{32}$$

in which $A(\cdot)$ is a polynomial of degree m and $B(\cdot)$ is a polynomial of degree μ. If the series converges, as assumed, the zeros of $A(z)$ will lie outside the unit circle.

Under conditions which we shall examine further, the appropriate exp $[if(\varphi)]$ is now as follows:

$$e^{if(\varphi)} = \frac{A(e^{-i\varphi})X(e^{i\varphi})}{A(e^{i\varphi})X(e^{-i\varphi})} + \delta(e^{i\varphi}) \tag{33}$$

in which δ is small (at real φ) and $X(\cdot)$ is a polynomial determined by two further conditions. First, the zeros of $X(z)$ must lie outside the unit circle. Second ϵ and $X(\cdot)$ must be such that

$$\frac{\epsilon}{2}\frac{A(e^{-i\varphi})X(e^{i\varphi})}{A(e^{i\varphi})X(e^{-i\varphi})} + \frac{B(e^{i\varphi})}{A(e^{i\varphi})} = \frac{e^{-i\varphi}N(e^{-i\varphi})}{X(e^{-i\varphi})} \tag{34}$$

in which $N(\cdot)$ is a polynomial. Let us examine the implications first and the existence of such an ϵ and $X(\cdot)$ thereafter.

When φ is real exp $(i\varphi)$ and exp $(-i\varphi)$ are conjugates, and so are identical polynomials in these two variables. This makes $f(\varphi)$ real in equation (33), except for small corrections due to δ. When the zeros of $A(z)$ and $X(z)$ lie outside the unit circle, as required, the unit circle in the z plane maps into contours in the polynomial planes which do not enclose 0. This makes $f(\varphi)$ periodic in φ.

The condition on the zeros of $X(z)$ also permits the right side of equation (34) to be expanded:

$$\frac{e^{-i\varphi}N(e^{-i\varphi})}{X(e^{-i\varphi})} = \sum_{\sigma=1}^{\infty} \hat{G}_\sigma e^{-i\sigma\varphi}. \tag{35}$$

Using equations (32), (33), (34) and (35) in equation (31) now gives

$$\delta(e^{-i\varphi}) + \sum_{\sigma=1}^{\infty} \hat{G}_\sigma e^{-i\sigma\varphi} = \sum_{\sigma=1}^{2n+1} G_\sigma e^{-i\sigma\varphi}; \tag{36}$$

$$\begin{aligned} P_\sigma &= \hat{G}_\sigma + C_{n+1-\sigma}, \qquad \sigma \leqq n+1; \\ &= \hat{G}_\sigma, \qquad n+1 < \sigma < 2n+1; \end{aligned}$$

$$\delta(e^{i\varphi}) = -\sum_{2n+2}^{\infty} \hat{G}_\sigma e^{-i\sigma\varphi}.$$

This δ is small provided the zeros z_j of $X(z)$ are such that $z_j^{-(2n+2)}$ is small. When δ is small, the actual error extrema will differ from ϵ, but by no more than $\pm\,|\delta|\,\epsilon$.

Equation (34) requires

$$\frac{\epsilon}{2} A(e^{-i\varphi})X(e^{i\varphi}) + B(e^{i\varphi})X(e^{-i\varphi}) = A(e^{i\varphi})e^{-i\varphi}N(e^{-i\varphi}). \tag{37}$$

The appropriate degree η of polynomial $X(\cdot)$ turns out to be one less than the number of poles of $zB(z)/A(z)$ (including any poles at $z = \infty$).

When the degree of $A(\cdot)$ is greater than the degree of $B(\cdot)$ and the zeros of $A(\cdot)$ are distinct, a set of $\eta + 1$ homogeneous equations in the coefficients q_i of $X(\cdot)$ can be derived by evaluating equation (37) at the zeros of $A(\cdot)$. Then

$$\left(M_A + \frac{\epsilon}{2} M_B\right)Q = 0 \tag{38}$$

in which Q is the column matrix of the coefficients q_i of X and M_A and M_B are square matrices of order $\eta + 1$. Under other conditions an equation of the same form can be obtained in other ways.

Equation (38) requires $\epsilon/2$ to be one of $\eta + 1$ eigenvalues for which the matrix coefficient of Q is singular. Each eigenvalue determines a polynomial $X(\cdot)$ [including an arbitrary scale factor which cancels out in equation (33)]. For our purposes, we must choose an ϵ which is real and such that the zeros of $X(z)$ lie outside the unit circle. This raises a question of the existence of a suitable ϵ and $X(\cdot)$.

When degree $\eta = 0$, $X(\cdot)$ is a constant, equation (38) is a real linear equation in ϵ, and there is no zero of $X(\cdot)$. When $\eta = 1$, $X(\cdot)$ is linear, ϵ is a root of a quadratic equation. It is not hard to show that the two roots are real [under our assumptions regarding zeros of $A(\cdot)$] and that one (the larger) yields a zero z_1 of $X(z)$ such that $|z_1| \geq 1$. Equality occurs only in singular cases such that $n + 2$ zero error points are possible [even though there are only $n + 1$ disposable coefficients of $\mathbf{P}(x)$] and can be so placed that there are $n + 3$ equal error extrema, instead of only $n + 2$. This may be seen by assuming that the zero of a linear $X(z)$ is ± 1, and then noting that, in equation (33),

$$\frac{X(e^{i\varphi})}{X(e^{-i\varphi})} = \frac{1 \pm e^{i\varphi}}{1 \pm e^{-i\varphi}} = \pm e^{i\varphi}. \tag{39}$$

Conditions for the existence of a suitable ϵ have not been established for $\eta > 2$. They are probably at least closely related to the (unknown) general conditions under which the minimax approximation has the equal ripple form of Fig. 1.

The procedures described here are appropriate only when a suitable ϵ does in fact exist. However, degrees $\eta = 0$ and 1, for which existence has been established, are sufficient for many practical problems. For approximately equal error extrema, equation (32) itself need be only an approximation, and polynomials $A(\cdot)$ and $B(\cdot)$ for which $\eta = 0$ or 1 are likely to give a good enough approximation. This is particularly true when degree n of $\mathbf{P}(x)$ is sufficiently large so that coefficients

C_σ , $\sigma > n$, in equation (26) approach a simple asymptotic behavior. Percentage variations between error extrema need not have to be very small even though the absolute errors must be very small. For example, a 10 percent variation between very small extrema may be acceptable, compared with large variations obtained by truncation of the infinite Chebyshev polynomial series.

Table I indicates the degree m of $A(\cdot)$ and μ of $B(\cdot)$ for which $\eta = 0$ or 1. The column headed $m + \mu + 1$ indicates the number of disposable parameters in the rational fraction A/B which can be adjusted to approximate the sum in equation (32).

The procedures for $\eta = 0$ and 1 are particularly well suited for rapid explorations of available error magnitudes as functions of initial design parameters, such as degree of the disposable polynomial, extent of the approximation interval, and parameters in the approximated function. When only the error magnitude is needed, it is not necessary to calculate the coefficients of polynomial $P(\cdot)$, which requires the series expansion (35). When $\eta = 0$, the error magnitude is (approximately) the single ϵ determined by equation (38). Then simple closed form formulas can frequently be obtained (and will be included in 4 of the 5 examples in Section V). When $\eta = 1$, ϵ is one of the two roots of the quadratic equation required by equation (38). (To meet the condition on the zero of $X(\cdot)$, the larger ϵ must be chosen.)

When ϵ has been determined, it can be compared with the error ϵ_T obtained by simply truncating the Chebyshev polynomial expansion of $\mathbf{F}(x)$. In terms of equations (26) and (32)

$$\epsilon_T \cong \frac{B(e^{i\varphi})}{A(e^{i\varphi})}\, e^{i(n+1)\varphi} + \frac{B(e^{-i\varphi})}{A(e^{-i\varphi})}\, e^{-i(n+1)\varphi}. \tag{40}$$

Comparing the maximum ϵ_T (at real φ) with ϵ indicates the improve-

TABLE I—Value of m and μ for which η is 0 or 1.

Degree m of $A(\cdot)$	Degree μ of $B(\cdot)$	Degree η of $X(\cdot)$	$m + \mu + 1$
0	0	0	1
1	0	0	2
2	0	1	3
0	1	1	2
1	1	1	3
2	1	1	4

ment to be obtained by the minimax refinement of the truncated series. Frequently, max ϵ_T occurs at $\varphi = 0$ or π, and then

$$\max \epsilon_T = 2 \frac{B(\pm 1)}{A(\pm 1)}. \tag{41}$$

4.2 *Weighted Minimax Approximations*

Let

$$\mathbf{P}(x) = \mathbf{F}(x) + \frac{1}{\mathbf{W}(x)} \mathbf{e}(x) \tag{42}$$

in which $\mathbf{P}(x)$ is again a disposable polynomial of degree n, $\mathbf{F}(x)$ is again a given function to be approximated in the interval $x_a \leqq x \leqq x_b$, and the new function $\mathbf{W}(x)$ is a given weight factor. For what $\mathbf{P}(x)$ is $\mathbf{e}(x)$ smallest in the minimax sense? We will again assume that the minimax $\mathbf{e}(x)$ has the equal ripple form and will seek only approximations to equal ripples. We will also assume that $\mathbf{W}(x)$ is bounded and positive definite in the approximation interval. (*A* point where $\mathbf{W}(x) = 0$ or ∞ would probably spoil the equal ripple character of the minimax approximation.)

Map from x to φ as before and define $P(\cdot)$, $F(\cdot)$ and $E(\cdot)$ again by equations (24), (26), and (28). Express $\mathbf{W}(x)$ also in terms of exponentials, but as a product instead of a sum. More specifically, let

$$\mathbf{H}(x) = -\log \mathbf{W}(x) = H(e^{i\varphi}) + H(e^{-i\varphi}) \tag{43}$$

and assume that $\mathbf{W}(x)$ is sufficiently smooth, as well as bounded and positive definite, so that $H(z)$ is regular at $|z| \leqq 1$. Then

$$\frac{1}{\mathbf{W}(x)} = D(e^{i\varphi})D(e^{-i\varphi}), \qquad D(e^{i\varphi}) = e^{H(e^{i\varphi})} \tag{44}$$

with $\log D(z)$ regular when $|z| \leqq 1$. This $D(\cdot)$ is a generalization of the $D(\cdot)$ of Subsection 3.1 and of $[S(\cdot)]^{\frac{1}{2}}$ of Subsection 3.2. Frequently it can be found by direct factorization of a function of $e^{i\varphi}$ as in Section III.

Equations like (29) and (30) can now be obtained as before. The only difference is that $E(\cdot)$ must now be multiplied by the product of functions of φ in equation (44). Then equation (30) becomes

$$P(e^{i\varphi}) = F(e^{i\varphi}) + D(e^{i\varphi})D(e^{-i\varphi})E(e^{i\varphi}) \tag{45}$$

and equation (31) becomes

$$\sum_{\lambda=1}^{2n+1} G_\lambda e^{-i\lambda\varphi} = \frac{\epsilon}{2} D(e^{i\varphi})D(e^{-i\varphi})e^{if(\varphi)} + \sum_{\lambda=0}^{\infty} C_{n+1+\lambda}e^{i\lambda\varphi}; \tag{46}$$

$$G_\lambda = P_{n+1-\lambda} - C_{n+1-\lambda}, \qquad \lambda \leqq n+1;$$

$$= P_{n+1-\lambda}, \qquad n+1 < \lambda \leqq 2n+1.$$

Retain the rational fraction $B(\cdot)/A(\cdot)$ of equation (32), but change equation (33) to

$$e^{if(\varphi)} = \frac{D(e^{-i\varphi})A(e^{-i\varphi})X(e^{i\varphi})}{D(e^{i\varphi})A(e^{i\varphi})X(e^{-i\varphi})} + \hat{\delta}(e^{i\varphi}) \tag{47}$$

so that

$$D(e^{i\varphi})D(e^{-i\varphi})e^{if(\varphi)} = \frac{D^2(e^{-i\varphi})A(e^{-i\varphi})X(e^{i\varphi})}{A(e^{i\varphi})X(e^{-i\varphi})} + \delta(e^{i\varphi}) \tag{48}$$

in which $\hat{\delta}$ and δ are small. Then change equation (34) to

$$\frac{\epsilon}{2}\,\frac{D^2(e^{-i\varphi})A(e^{-i\varphi})X(e^{i\varphi})}{A(e^{i\varphi})X(e^{-i\varphi})} + \frac{B(e^{i\varphi})}{A(e^{i\varphi})} = \frac{e^{-i(\varphi)}N(e^{-i\varphi})}{X(e^{-i\varphi})}. \tag{49}$$

Using equations (32), (47) and (35) in equation (46) now gives equation (36) again. From equation (49), equation (37) must be changed to

$$\frac{\epsilon}{2} D^2(e^{-i\varphi})A(e^{-i\varphi})X(e^{i\varphi}) + B(e^{i\varphi})X(e^{-i\varphi}) = A(e^{i\varphi})e^{-i\varphi}N(e^{-i\varphi}). \tag{50}$$

Equation (50) can be used to find ϵ, and the $X(\cdot)$ and $N(\cdot)$ needed for equations (35) and (36).

4.3 *More General Approximating Functions*

Let

$$\Psi[\mathbf{P}(x), x] = \mathbf{G}(x) + \boldsymbol{\varepsilon}(x) \tag{51}$$

in which $\Psi[\mathbf{P}(x), x]$ is a given function of x and a disposable polynomial $\mathbf{P}(x)$, $\mathbf{G}(x)$ is a given function to be approximated by $\Psi[\mathbf{P}(x), x]$ in the interval $x_a \leqq x \leqq x_b$, and $\boldsymbol{\varepsilon}(x)$ is the error in the approximation. For what $\mathbf{P}(x)$ is $\boldsymbol{\varepsilon}(x)$ smallest in the minimax sense? Under certain further assumptions regarding $\Psi(\cdot\ ,\ \cdot)$ this approximation can be transformed into a weighted minimax polynomial approximation.

Assume an inverse Ψ^{-1} of $\Psi[\mathbf{P}(x), x]$, with respect to $\mathbf{P}(x)$, exists over the approximation interval. Then equation (51) can be replaced by

$$\mathbf{P}(x) = \Psi^{-1}\{[\mathbf{G}(x) + \boldsymbol{\varepsilon}(x)], x\}. \tag{52}$$

Assume $\Psi^{-1}(\cdot\ ,\ \cdot)$ is sufficiently smooth and $\boldsymbol{\varepsilon}(x)$ sufficiently small to justify the following approximation (in the interval $x_a \leqq x \leqq x_b$):

$$\mathbf{P}(x) = \Psi^{-1}[\mathbf{G}(x), x] + \frac{\partial \Psi^{-1}[\mathbf{G}(x), x]}{\partial \mathbf{G}(x)} \boldsymbol{\varepsilon}(x). \tag{53}$$

This is in the general form (42) with

$$\begin{aligned} \mathbf{F}(x) &= \Psi^{-1}[\mathbf{G}(x), x] \\ \frac{1}{\mathbf{W}(x)} &= \frac{\partial \Psi^{-1}[\mathbf{G}(x), x]}{\partial \mathbf{G}(x)}. \end{aligned} \tag{54}$$

Thus Subsection 4.2 can now be applied provided the $\mathbf{F}(\cdot)$ and $\mathbf{W}(\cdot)$ determined by equation (54) meet the appropriate conditions. Recall that we required $\mathbf{W}(x)$ to be bounded and positive definite over the interval of approximation. However, reversing the sign of $\mathbf{W}(x)$ merely reverses the sign of $\boldsymbol{\varepsilon}(x)$. Hence, in equation (54), we need only require that the partial derivative must be bounded and either positive definite or negative definite and sufficiently smooth for log $D(z)$ to be regular when $|z| < 1$.

As a first example of the inversion of equation (51), let

$$\mathbf{W}(x)\mathbf{P}(x) = \mathbf{G}(x) + \boldsymbol{\varepsilon}(x) \tag{55}$$

where $\mathbf{W}(x)$ and $\mathbf{G}(x)$ are given functions of x. Then

$$\mathbf{P}(x) = \frac{\mathbf{G}(x) + \boldsymbol{\varepsilon}(x)}{\mathbf{W}(x)} = \frac{\mathbf{G}(x)}{\mathbf{W}(x)} + \frac{1}{\mathbf{W}(x)} \boldsymbol{\varepsilon}(x). \tag{56}$$

As a second example, let

$$[\mathbf{A}(x) + \mathbf{B}(x)\mathbf{P}(x)]^{\frac{1}{2}} = \mathbf{G}(x) + \boldsymbol{\varepsilon}(x) \tag{57}$$

where $\mathbf{A}(x)$, $\mathbf{B}(x)$, and $\mathbf{G}(x)$ are given functions of x, with $\mathbf{B}(x)$ and $\mathbf{G}(x)$ positive definite over the approximation interval. Solving for $\mathbf{P}(x)$ gives

$$\mathbf{P}(x) = \frac{\mathbf{G}^2(x) - \mathbf{A}(x)}{\mathbf{B}(x)} + 2\frac{\mathbf{G}(x)}{\mathbf{B}(x)} \boldsymbol{\varepsilon}(x) + \frac{1}{\mathbf{B}(x)} \boldsymbol{\varepsilon}^2(x). \tag{58}$$

If the term in $\boldsymbol{\varepsilon}^2(x)$ is omitted

$$[\mathbf{A}(x) + \mathbf{B}(x)\mathbf{P}(x)]^{\frac{1}{2}} \cong \mathbf{G}(x) + \boldsymbol{\varepsilon}(x) - \frac{\boldsymbol{\varepsilon}^2(x)}{2\mathbf{G}(x)} \tag{59}$$

in which terms in $\boldsymbol{\varepsilon}^\sigma(x)$ have been neglected for $\sigma > 2$. An equal ripple $\boldsymbol{\varepsilon}(x)$ in equation (57) yields an approximately equal ripple error if the last term is somewhat smaller than the extrema of $\boldsymbol{\varepsilon}(x)$.

4.4 *Relation to Phase Modulation*

The function of φ defined by equation (28) is similar to functions of time t used in communication theory to describe phase modulated signals. If φ is replaced by t in equation (28),

$$E(e^{it}) = \frac{\epsilon}{2} e^{i[(n+1)t+if(t)]}. \tag{60}$$

This is the exponential representation of a phase modulated signal in which the carrier (radian) frequency is $n + 1$ and the baseband signal $f(t)$ is periodic with one period every $n + 1$ periods of the carrier. The signal may be regarded as the carrier plus sequences of upper and lower sidebands. The upper sidebands are determined by the coefficients $C_{n+1+\lambda}$ in equation (31). The lower sidebands are determined by the requirement of a purely phase modulated signal. Finally, if the sequence of lower sidebands extends as far as the negative carrier frequency $-(n + 1)$ we truncate it at $-n$.

Weighted minimax approximations can be interpreted similarly, in terms of simultaneous phase and amplitude modulation.

4.5 *Alternative Procedures*

It is obvious that the procedures described above can be varied in many different ways. A very few of the possible variations are noted below.

Preliminary manipulations may be needed to obtain a formulation in which the disposable part is a polynomial. Also, the pertinent Fourier series may be sums of sines instead of cosines. Both these situations will be illustrated by Example 5, in Section V.

If $\partial\Psi^{-1}[\mathbf{G}(x), x]/\partial\mathbf{G}(x)$ is expressed as a product of functions of x, $D(e^{i\varphi})$ can be formulated as a product of corresponding factors. A factor of the form $(1 - x/x_\sigma)^\rho$ contributes a factor of the form $[M_\sigma(1 - \gamma_\sigma e^{i\varphi})]^\rho$, as in Section III. More generally, there may be advantages to replacing the $D(\cdot)$ of equation (44) by $\hat{D}(\cdot)$ defined

$$\mathbf{W}^{\pm 2}(x) = \hat{D}(e^{i\varphi})\hat{D}(e^{-i\varphi}). \tag{61}$$

Then the $D^2(\cdot)$ in equations (48), (49) and (50) is replaced by $\hat{D}^{\pm 1}(\cdot)$.

It may sometimes be convenient to express $\mathbf{P}(x)$ and $\mathbf{F}(x)$ as products of factors in exp $(\pm i\varphi)$ instead of sums, say

$$\mathbf{P}(x) = P(e^{i\varphi})P(e^{-i\varphi}), \qquad \mathbf{F}(x) = F(e^{i\varphi})F(e^{-i\varphi}) \tag{62}$$

in which $P(\cdot)$ is a polynomial of degree n with no negative powered terms. If a term in ϵ^2 is neglected, one can now replace equation (29) by

$$P(e^{i\varphi})P(e^{-i\varphi}) = \left[F(e^{i\varphi}) + \frac{E(e^{i\varphi})}{F(e^{-i\varphi})}\right]\left[F(e^{-i\varphi}) + \frac{E(e^{-i\varphi})}{F(e^{i\varphi})}\right]. \tag{63}$$

Equating factors separately replaces equation (30) by

$$P(e^{i\varphi}) = F(e^{i\varphi}) + \frac{E(e^{i\varphi})}{F(e^{-i\varphi})}. \tag{64}$$

Subsequent modifications of our previous procedures are now easily worked out.

It would be possible to replace equation (33) by other functional forms for exp $[if(\varphi)]$. The moduli must approximate unity at real φ and expansions in positive and negative powers of exp $(i\varphi)$ must exist. Disposable parameters are to be adjusted so as to approximate the required coefficients of positive powers. However, except for very special functional forms [such as equation (33)], the adjustment is likely to be a quite complicated task.

V. EXAMPLES

This section further clarifies the general procedures by means of five examples.

5.1 *Example 1*

Let

$$\frac{\mathbf{P}(x)}{(1 - x/x_0)^{\frac{1}{2}}} = 1 + \boldsymbol{\varepsilon}(x), \qquad -1 \leqq x \leqq +1 \tag{65}$$

in which x_0 is a given constant, $|x_0| > 1$, and the degree n of the disposable polynomial $\mathbf{P}(x)$ is large. What is the approximate amplitude $|\epsilon|$ of the equal error extrema of the minimax approximation?

This is a special case of equations (55) and (56), which can be solved as a special case of equation (42) for which

$$\mathbf{F}(x) = \frac{1}{\mathbf{W}(x)} = (1 - x/x_0)^{\frac{1}{2}}. \tag{66}$$

To apply Section IV, define $F(\cdot)$ and $D(\cdot)$ by

$$\begin{aligned} (1 - x/x_0)^{\frac{1}{2}} &= F(e^{i\varphi}) + F(e^{-i\varphi}) = D(e^{i\varphi})D(e^{-i\varphi}), \\ F(e^{i\varphi}) &= \sum_{\sigma=0}^{\infty} C_\sigma e^{i\sigma\varphi}, \end{aligned} \tag{67}$$

Log $D(z)$ regular, $|z| \leqq 1$.

We have already factored a linear function of x in terms of exp $(\pm i\varphi)$, in Section III. A similar factorization now gives

$$D(e^{i\varphi}) = \frac{(1 - \gamma e^{i\varphi})^{\frac{1}{2}}}{(1 + \gamma^2)^{\frac{1}{4}}}, \qquad |\gamma| < 1 \tag{68}$$

and then the coefficients of C_σ correspond to an expansion of

$$\left[\frac{(1 - \gamma e^{i\varphi})(1 - \gamma e^{-i\varphi})}{1 + \gamma^2}\right]^{\frac{1}{2}} = \sum_{\sigma=0}^{\infty} C_\sigma e^{i\sigma\varphi} + \sum_{\sigma=0}^{\infty} C_\sigma e^{-i\sigma\varphi}. \tag{69}$$

To determine ϵ we only need the coefficients C_σ for $\sigma > n$, which we have assumed to be large.

The following expansion of $[1 - \gamma \exp(i\varphi)]$, valid for $|\gamma| \leqq 1$, is well known

$$\begin{aligned} (1 - \gamma e^{i\varphi})^{\frac{1}{2}} &= \sum_{\sigma=0}^{\infty} K_\sigma e^{i\sigma\varphi}; \\ K_0 &= -1, \qquad K_1 = -\gamma/2; \\ K_\sigma &= \frac{-(2\sigma - 3)!}{4^{\sigma-1}(\sigma - 2)!\,\sigma!}\gamma^\sigma, \qquad \sigma \geqq 2. \end{aligned} \tag{70}$$

When n is large and $\lambda \ll n$,

$$\frac{K_{n+\lambda+1}}{K_{n+\lambda}} = \frac{2(n + \lambda) - 1}{2(n + \lambda) + 2}\gamma \cong k\gamma \tag{71}$$

in which

$$k = \frac{2n + 1}{2n + 4} = 1 - \frac{3}{2n + 4} \cong 1. \tag{72}$$

As a result, when n is large

$$(1 - \gamma e^{i\varphi})^{\frac{1}{2}} \cong \sum_{\sigma=0}^{n} K_\sigma e^{i\sigma\varphi} + \frac{K_{n+1}e^{i(n+1)\varphi}}{1 - k\gamma e^{i\varphi}}. \tag{73}$$

Now note that

$$\left[\frac{1 - \gamma e^{-i\varphi}}{1 + \gamma^2}\right]^{\frac{1}{2}}\left[\frac{K_{n+1}}{1 - k\gamma e^{i\varphi}}\right] = \sum_{\sigma=1}^{\infty} L_\sigma e^{-i\varphi} + \frac{(1 - k\gamma^2)^{\frac{1}{2}}K_{n+1}}{(1 + \gamma^2)^{\frac{1}{2}}(1 - k\gamma e^{i\varphi})}. \tag{74}$$

If equation (74) is used to evaluate C_σ in equation (67), only the last term contributes to C_σ when $\sigma > n$. Then

$$\sum_{\lambda=0}^{\infty} C_{n+1+\lambda} e^{i\lambda\varphi} \cong \frac{(1 - k\gamma^2)^{\frac{1}{2}} K_{n+1}}{(1 + \gamma^2)^{\frac{1}{2}}(1 - k\gamma e^{i\varphi})} \tag{75}$$

which is a special case of equation (32) with $A(\cdot)$ a linear polynomial and $B(\cdot)$ a constant. The corresponding $X(\cdot)$ in equation (47) is a constant and cancels out. Then equations (68) and (75) applied to equation (50) give

$$\epsilon = \frac{-2K_{n+1}}{(1 - k^2\gamma^2)(1 - k\gamma^2)^{\frac{1}{2}}}, \tag{76}$$

$$N(e^{-i\varphi}) = G_1 + G_2 e^{-i\varphi}.$$

The constants G_1 and G_2 contribute to the two highest degree terms in the polynomial $P(\cdot)$. They need not be computed unless the specific polynomial is needed as well as the amplitude $|\epsilon|$ of the approximation errors.

The linear $A(\cdot)$ and constant $B(\cdot)$ determined by equation (75) can be used in equation (40) to approximate the truncation error for the polynomial approximation defined by equation (66). The corresponding error in equation (65) can be found by dividing by $(1 - x/x_0)^{1/2}$. This gives

$$r = \frac{\max |\epsilon_T|}{|\epsilon|} = \frac{(1 - k\gamma^2)(1 + k|\gamma|)}{(1 - |\gamma|)}. \tag{77}$$

If $k = 1$, $r = (1 + |\gamma|)^2 < 4$. Actually $k < 1$, but further analysis indicates that r will not be significantly >4 when γ^n is small.

When $|x_0| \to \infty$, $\mathbf{W}(x) \to 1$, $C_{n+1+\lambda}/C_{n+1} \to 0$, and ϵ_T is dominated by a single Chebyshev polynomial (which has equal extrema). Consistent with this our $\gamma \to 0$, then G_1, $G_2 \to 0$ in equation (76) and $r \to 1$ in equation (77).

In equation (74), K_{n+1} can be determined by the formula for K_σ in equation (70). However, the following simpler approximate formula may be more useful:

$$K_{n+1} \cong \frac{-\gamma^{n+1}}{2(\pi)^{\frac{1}{2}}(n + 1)(n + 1/4)^{\frac{1}{2}}}. \tag{78}$$

The error amounts to about 0.3 percent at $n = 2$ and about 0.04 percent at $n = 6$. The derivation is related to, but requires more than substitution of Stirling's approximation for the factorials in equation (70).

Fig. 3 illustrates computed errors $\boldsymbol{\varepsilon}(x)$ and $\boldsymbol{\varepsilon}_T(x)$ corresponding re-

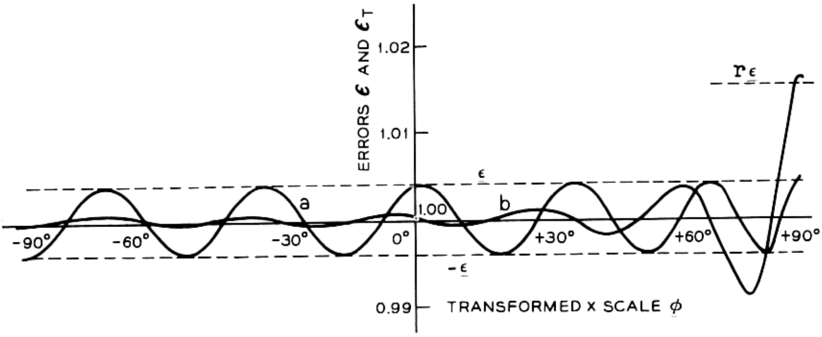

Fig. 3 — Illustrating Example 1: (a) $\varepsilon(x)$ and (b) $\varepsilon_T(x)$.

spectively to our approximately equal ripple solution and truncation of the Chebyshev polynomial series. The constants $\pm\epsilon$ and $r\epsilon$ determined by equations (74) and (75) are included for comparison. The computations started with

$$x_0 = 1.025, \qquad n = 10$$

for which, as computed by equations (76), (77) and (78),

$$\gamma = 0.8, \qquad k = 7/8,$$

$$K_{n+1} \cong 0.0006881, \qquad \epsilon \cong 0.0040678, \qquad r \cong 3.74.$$

5.2 *Example 2*

Let

$$\mathbf{P}(x) = (1 - x/x_0)^{1/2} + \boldsymbol{\varepsilon}(x), \qquad -1 \leqq x \leqq +1 \tag{79}$$

in which the degree n of $\mathbf{P}(x)$ is again large and x_0 is again a given constant, $|x_0| > 1$.

Since the function $\mathbf{F}(x)$ is the same as in Example 1, equation (75) is again valid. Now, however, $\mathbf{W}(x) = 1$ and hence Section 4.1 (on unweighted polynomial approximations) is appropriate. Applying equation (75) to equation (37) gives

$$\epsilon = \frac{(1 - k\gamma^2)^{\frac{1}{2}} K_{n+1}}{(1 + \gamma^2)^{\frac{1}{2}}(1 - k^2\gamma^2)} \tag{80}$$

$$N(e^{-i\varphi}) = G_1$$

in which $N(\cdot)$ contributes only to the highest degree term in $P(\cdot)$. The error ratio r turns out to be

$$r = \frac{\max |\epsilon_T|}{|\epsilon|} = 1 + k|\gamma| < 2. \tag{81}$$

5.3 *Example 3**

Let

$$(1 - x/x_0)^{1/2}\mathbf{P}(x) = 1 + \varepsilon(x), \qquad -1 \leqq x \leqq 1 \tag{82}$$

in which degree n of $\mathbf{P}(x)$ is again large and x_0 is given, $|x_0| > 1$.

In the equivalent weighted polynomial approximation

$$\mathbf{F}(x) = \frac{1}{\mathbf{W}(x)} = (1 - x/x_0)^{-\frac{1}{2}}. \tag{83}$$

Proceeding as in Example 1, one now gets

$$\begin{aligned} D(e^{i\varphi}) &= \frac{(1+\gamma^2)^{\frac{1}{4}}}{(1-\gamma e^{i\varphi})^{\frac{1}{2}}}, \\ \sum_{\sigma=0}^{\infty} C_{n+1+\lambda} e^{i\lambda\varphi} &\cong \frac{(1+\gamma^2)^{\frac{1}{4}} K_{n+1}}{(1-k\gamma^2)^{\frac{1}{2}}(1-k\gamma e^{i\varphi})}, \\ K_{n+1} &= \frac{(2n+1)!\,\gamma^{n+1}}{2^{n+1} n!\,(n+1)!} \cong \frac{\gamma^{n+1}}{[\pi(n+5/4)]^{\frac{1}{2}}}, \\ k &= \frac{2n+3}{2n+4} = 1 - \frac{1}{2n+4}. \end{aligned} \tag{84}$$

Then equation (49) gives

$$\begin{aligned} \epsilon &= \frac{2K_{n+1}(1-k\gamma^2)^{\frac{1}{2}}}{1-k^2\gamma^2}, \\ e^{-i\varphi}N(e^{-i\varphi}) &= \frac{N_1 e^{-i\varphi}}{1-\gamma e^{-i\varphi}}, \\ N_1 &= \frac{\epsilon}{2}\,\frac{\gamma(1-k)(1+\gamma^2)^{\frac{1}{4}}}{1-k\gamma^2}. \end{aligned} \tag{85}$$

Equation (35) is now

$$\frac{N_1 e^{-i\varphi}}{1-\gamma e^{-i\varphi}} = \sum_{\sigma=1}^{\infty} \hat{G}_\sigma e^{-i\sigma\varphi}. \tag{86}$$

The first $2n + 1$ terms in this series contribute to $P(\cdot)$ per equation

* The author has encountered this problem in connection with two different circuit theory studies, which will be described in other papers.

(36). The remainder can be summed, to get

$$\delta(e^{i\varphi}) = \frac{N_1\gamma^{2n+1}e^{-i(2n+2)\varphi}}{1-\gamma e^{-i\varphi}}. \tag{87}$$

Evaluating the error corresponding to simple truncation now gives

$$r = \frac{\max|\epsilon_T|}{|\epsilon|} \cong \frac{(1-|\gamma|)(1-k^2\gamma^2)}{(1-k|\gamma|)(1-k\gamma^2)}. \tag{88}$$

When n is large, $k \cong 1$ and r is so close to unity that the minimax refinement of simple truncation is not likely to be justified. However, our analysis has been useful in disclosing this fact, without the detailed computation of any minimax approximations.

5.4 *Example 4*

Previous work, which we shall discuss in Section VI, concerns the following problem: Let

$$\mathbf{P}_n(x) = \mathbf{P}_{n+\nu}(x) + \varepsilon(x), \qquad -1 \leqq x \leqq +1 \tag{89}$$

in which $\mathbf{P}_{n+\nu}(x)$ is a given polynomial of degree $n+\nu$ and $\mathbf{P}_n(x)$ is a disposable polynomial of degree n. For what $\mathbf{P}_n(x)$ does $\varepsilon(x)$ have the equal ripple form?

Equations (32), (33) and (37) now simplify to

$$\begin{aligned} \sum_{\lambda=0}^{\infty} C_{n+1+\lambda}(e^{i\lambda\varphi}) &= \sum_{\sigma=0}^{\nu-1} C_{n+1+\lambda}(e^{i\lambda\varphi}) = B(e^{i\lambda\varphi}), \\ e^{if(\varphi)} &= \frac{X(e^{i\varphi})}{X(e^{-i\varphi})} + \delta(e^{i\varphi}), \end{aligned} \tag{90}$$

$$\frac{\epsilon}{2}X(e^{i\varphi}) + B(e^{i\varphi})X(e^{-i\varphi}) = e^{-i\varphi}N(e^{-i\varphi})$$

in which $B(\cdot)$ is a polynomial of degree $\nu - 1$, with coefficients $C_{n+1+\lambda}$ and $X(\cdot)$ is a polynomial of degree $\nu - 1$, to be found therefrom. The coefficients of $B(\cdot)$ can be found by expanding the left side of the last equation and equating to zero the coefficients of pgsitive powers of exp $(i\varphi)$. The result can be expressed as the following specialization of equation (38):

$$\left(C + \frac{\epsilon}{2}I\right)Q = 0 \tag{91}$$

in which Q is again a column matrix whose elements are the ν coefficients of $X(\cdot)$ and I is the identity matrix of order ν. The matrix C has the special form (assuming the elements q_σ of Q to be ordered per $X(z) = \sum q_\sigma z^\sigma$)

$$C = \begin{vmatrix} C_{n+1} & C_{n+2} & \cdots & C_{n+\nu-1} & C_{n+\nu} \\ C_{n+2} & C_{n+3} & \cdots & C_{n+\nu} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ C_{n+\nu} & 0 & \cdots & 0 & 0 \end{vmatrix}, \tag{92}$$

When $\nu = 1$, X is a constant and the solution is elementary. When $\nu = 2$, $X(z)$ is linear. Let z_1 be its zero. Then equations (91) and (92) require

$$\epsilon^2 + 2C_{n+1}\epsilon - 4C_{n+2}^2 = 0,$$

$$z_1 = \frac{\epsilon}{2C_{n+2}}. \tag{93}$$

The roots of the quadratic equation in ϵ are real. When $C_{n+1} \neq 0$, the larger $|\epsilon| > 2|C_{n+2}|$ and $|z_1| > 1$, as required. Then δ in equation (36) turns out to be a power series in $\exp(-i\varphi)$ which can be summed to get

$$\delta(e^{i\varphi}) = \frac{-\gamma^{2n+2}C_{2n+2}e^{-i(2n+2)\varphi}}{1 - \gamma e^{-i\varphi}}, \tag{94}$$

$$\gamma = 1/z_1 .$$

When $C_{n+1} = 0$, $\epsilon = \pm 2C_{n+2}$ and $z_1 = \pm 1$. But then the error due to simple truncation of the Chebyshev polynomial expansion of $\mathbf{P}_{n+2}(x)$ is proportional to a single Chebyshev polynomial of degree $n + 2$, which has equal ripples with $n + 3$ extrema instead of $n + 2$.

5.5 *Example 5**

As a last example consider the following nonalgebraic approximation:

$$\bar{\Gamma}(\theta) = \sum_{\sigma=1}^{n} \mathbf{A}_\sigma \sin \sigma\theta = \theta + \boldsymbol{\varepsilon}(\theta) \tag{95}$$

$$-\pi < -\theta_c \leqq \theta \leqq \theta_c < \pi.$$

For what coefficients A_σ does the error $\boldsymbol{\varepsilon}(\theta)$ have the equal ripple form and what is the amplitude ϵ of the ripples?

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* This problem is of interest in, for example, the approximation of differentiation with a tapped delay line. A more detailed treatment is planned for a future paper.

A sequence of transformations changes equation (95) into a weighted polynomial approximation. First, equation (95) is equivalent to

$$\bar{\Gamma}(\theta) = \sin\theta \sum_{\rho=0}^{n-1} \mathbf{B}_\rho \cos(\rho\theta), \tag{96}$$

$$2\mathbf{A}_\rho = \mathbf{B}_{\rho-1} - \mathbf{B}_{\rho+1}\,.$$

Second, relate θ to a new variable φ by

$$\sin\frac{\theta}{2} = q\sin\varphi, \qquad |\,\theta\,| < \pi; \tag{97}$$

$$q = \sin\frac{\theta_c}{2} < 1.$$

Real φ maps into $-\theta_c \leqq \theta \leqq \theta_c$. Also

$$\begin{aligned} \cos\theta &= 1 - q^2 + q^2\cos 2\varphi, \\ \sin\theta &= 2q(1 - q^2\sin^2\varphi)^{\frac{1}{2}}\sin\varphi. \end{aligned} \tag{98}$$

In these terms, equation (96) becomes

$$\bar{\Gamma}(\theta) = 2q(1 - q^2\sin^2\varphi)^{\frac{1}{2}}\sin\varphi \sum_{\rho=0}^{n-1} B_\rho \cos(2\rho\varphi) \tag{99}$$

in which the set of coefficients B_ρ is linearly related to the set $\mathbf{B}_\rho$ of equation (96). In equivalent exponential terms

$$\hat{\Gamma}(\theta) = \frac{q}{i}(1 - q^2\sin^2\varphi)^{\frac{1}{2}}[P(e^{i\varphi}) - P(e^{-i\varphi})] \tag{100}$$

in which $P(z)$ is a polynomial of degree $2n - 1$ in odd powers of z only.

Use equation (100) in equation (95) and solve for the factor in [ ]. The result can be expressed in terms of exponentials:

$$\begin{aligned} P(e^{i\varphi}) - P(e^{-i\varphi}) &= F(e^{i\varphi}) - F(e^{-i\varphi}) \\ &\quad + D(e^{i\varphi})D(e^{-i\varphi})[E(e^{i\varphi}) - E(e^{-i\varphi})] \end{aligned} \tag{101}$$

in which $F(\cdot)$, $D(\cdot)$, and $E(\cdot)$ are related to previous functions by

$$\frac{\theta}{q}(1 - q^2\sin^2\varphi)^{-\frac{1}{2}} = \sum_{\sigma=1}^{\infty} 2C_{2\sigma-1}\sin(2\sigma - 1)\varphi; \tag{102}$$

$$F(e^{i\varphi}) = \sum_{\sigma=1}^{\infty} C_{2\sigma-1}e^{i(2\sigma-1)\varphi};$$

$$\frac{1}{q}(1 - q^2\sin^2\varphi)^{-\frac{1}{2}} = D(e^{i\varphi})D(e^{-i\varphi});$$

$$D(e^{i\varphi}) = \left[\frac{q(1 + \gamma e^{i2\varphi})}{1 + \gamma}\right]^{\frac{1}{2}}, \qquad |\gamma| < 1;$$

$$\mathbf{E}(\theta) = \frac{1}{i}[E(e^{i\varphi}) - E(e^{-i\varphi})];$$

$$E(e^{i\varphi}) = \frac{\epsilon}{2} e^{i[(2n+1)\varphi + f(\varphi)]}.$$

It can be shown that the coefficients $C_{2\sigma-1}$ obey a difference equation of order 2. The asymptotic behavior of the difference equation shows that

$$\text{as } \sigma \to \infty, \qquad C_{2\sigma+1} \to -\left(\frac{2\sigma - 1}{2\sigma + 1}\right)^{\frac{1}{2}} \gamma C_{2\sigma-1} . \tag{103}$$

As a result, for a sufficiently large n

$$\sum_{\sigma=n+1}^{\infty} C_{2\sigma-1} e^{i(2\sigma-1)\varphi} \cong \frac{C_{2n+1} e^{i(2n+1)\varphi}}{1 + k\gamma e^{i2\varphi}},$$
$$k = \left(\frac{2n+1}{2n+3}\right)^{\frac{1}{2}} \cong \frac{4n+1}{4n+3}. \tag{104}$$

Proceeding almost exactly as in Example 3, using equation (104) and the $D(\cdot)$ of equation (102), one can now obtain an approximation to the minimax error ϵ, to the error $\epsilon_T(\varphi)$ due to simply truncating the expansion of $F(e^{i\varphi})$, and to the ratio r of max $|\epsilon_T|$ and $|\epsilon|$. As in Example 3, it turns out that the minimax approximation is only a little better than the approximation by truncation, at least when n is large.

Figure 4 compares a computed $\epsilon_T(\varphi)$ with the approximate ϵ and ratio r of max $|\epsilon_T|$ to $|\epsilon|$, using

$$\theta_c = 170^\circ \qquad n = 15,$$

for which

$$\gamma = 0.8397 \qquad k = 0.96825,$$
$$\epsilon \cong 5.690^\circ \qquad r \cong 1.0840.$$

VI. COMPARISON WITH OTHER WORK

This section compares the present paper with previous publications in various related fields. It is not intended, however, to be an exhaustive survey of all related publications.

The transformation from x to φ followed by distortion of the φ scale

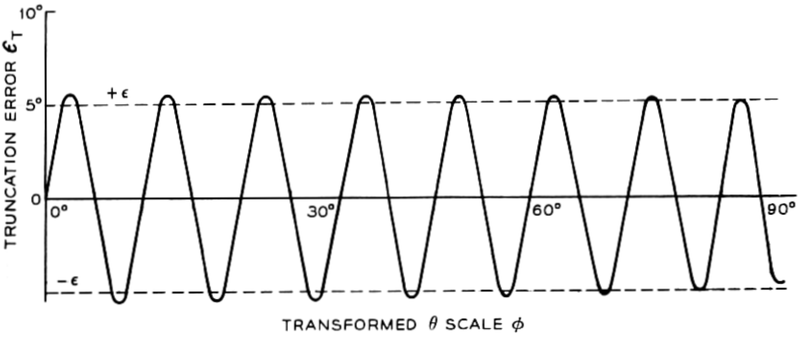

Fig. 4 — Illustrating Example 5.

to obtain an error function of the form $\epsilon \cos [(n + 1)\varphi + f(\varphi)]$ has been used before. Pertinent references are papers by Clenshaw[2] and Stiefel,[3,4] who call our $f(\varphi)$ the "phase function". Of these, Clenshaw's paper is quite close to ours, and in fact our work might be regarded as a generalization of his.

Clenshaw devotes much of his paper to the approximation of a polynomial of degree $n + \nu$ with a polynomial of degree n, which is our Example 4. Clenshaw, (Ref. 2, pp. 30, 31) solves the problem for $\nu = 2$ in a quite similar way, except that he expresses his Fourier series in terms of cosine functions instead of power series in $e^{\pm i\varphi}$. He exhibits an approximate solution which can be shown to be almost, but not quite equivalent to ours. We would have obtained an exact equivalent if we had restricted our polynomial $P(\cdot)$ to positive powers only, corresponding to $\sigma = 0$ to n in equation (25) instead of $-n$ to n. In equation (36), this restricts the disposable G_λ's to $\lambda = 1$ to $n + 1$ and increases the number of terms in $\delta(\cdot)$ to $\lambda = n + 2$ to ∞. The result is a somewhat poorer, but frequently adequate, approximation to an equal ripple error. Clenshaw also notes how his approximation can be improved, but does not fill in the details. It can be shown that the improved approximation would be an exact equivalent of ours. However, we have found that our formulations in terms of $e^{\pm i\varphi}$, instead of $\cos \varphi$, are simpler, and also more revealing concerning, for example, the nature of the approximations.

Clenshaw (Ref. 2, pp. 31–36) also considers $\nu > 2$, and obtains approximate solutions for $\nu = 3, 4$ in terms of roots of cubic and quartic equations. However, he retains the use of $\cos \varphi$, instead of $e^{\pm i\varphi}$. As a result, he does not include a formulation for a general ν in terms of an eigenvalue and eigen vector of a matrix, like our equation (91).

Clenshaw (Ref. 2, p. 29) notes that the equal ripple approximation to $(x_0 - x)^{-1}$ has been solved exactly and cites Hornecker[6] and Rivlin[7]. Any solution to this problem is easily applied to more general problems in which the given function yields the same sort of remainder when the Chebyshev polynomial series is truncated. Examples are our Example 2 and our general formulation for unweighted polynomial approximations with weight factor $W = 1$, $m = 1$, $\mu = 0$ in the remainder function (32).

Our procedures are more general in the following ways: First, remainder functions can have the more general form (32). For practical purposes, degrees m and μ should not be large. However, they need not be restricted to the special cases $m = 0$, $\mu \leqq 4$ and $m = 1$, $\mu = 0$. Second, minimax *weighted* errors can be obtained [by using suitable weight factors $\mathbf{W}(x)$ in equations (42), (44), and so on]. Third, unweighted minimax approximations can be obtained with approximating functions of which the disposable polynomial is only a part (by solving an equivalent polynomial approximation with a weighted minimax error, as in our Examples 1, 3 and 5). Finally, relatively simple formulations have been obtained by using exponentials instead of cosine functions.

Stiefel's papers[3,4] have much less relevance to our work. They use the error formulation $\epsilon \cos [(n + 1)\varphi + f(\varphi)]$ but obtain solutions by numerical iteration. Reference 3 also includes a general integral equation, which determines the required coefficients implicitly but is not easily solved.

Our use of rational fractions to approximate remainder functions, as in equation (32), and so on, is at least reminscent of the so-called ϵ algorithm. The ϵ algorithm also uses rational function approximations to remainders but for a different purpose—to increase the rate of convergence when functions are evaluated from their power series. It is quite different from the use of rational functions in the formulation of minimax polynomial approximations. References for the ϵ algorithm are Shanks[8] and Wynn.[9]

Our procedures require evaluating certain of the coefficients in the Chebyshev polynomial expansion of a given $\mathbf{F}(x)$ (or in the equivalent Fourier series expansion in terms of φ). Various established numerical methods are available for this.[1] The best choice depends on the form in which $\mathbf{F}(x)$ is specified (for example in closed analytic form, as a power series in x, or numerically at a set of discrete points). When $\mathbf{F}(x)$ satisfies a differential equation with polynomial coefficients, the coefficients in the Chebyshev polynomial series are related by a difference equation of finite order and can be computed recursively. Our Example 5 is a special case. The general relation is described by Clenshaw[10] who also includes numerical tabulations of coefficients for some common functions.

The author's 1952 paper on network synthesis in terms of Chebyshev polynomials is only remotely related to the present work.[11]

VII. CONCLUSIONS

Techniques like those described in Section IV and illustrated in Section V can be applied to many approximation problems in which the disposable part of the approximating function is a polynomial and approximately equal weighted or unweighted error extrema are desired. However, to be useful they must compete with other possible techniques, especially established numerical methods whereby equal-ripple approximations are obtained by iterative improvement of a sequence of unequal-error approximations. This section notes some circumstances under which procedures like those described here may perhaps be preferable.

First, the techniques described here are more likely to be competitive when the degree n of the disposable polynomial is large. When n is large iterative numerical methods are more likely to entail excessive amounts of computing. On the other hand, certain aspects of the more analytic techniques described here are likely to become easier as n becomes larger. These concern particularly the use of a simple rational fraction to approximate a remainder function, as in equation (32).

Second, the techniques described here are particularly suitable for exploring relationships between error amplitude $|\epsilon|$, the limits x_a, x_b of the approximation interval, the degree n of the disposable polynomial, and other parameters in the approximating function (such as x_0 in examples 1, 2, and 3). In explorations of this sort the computation of the actual coefficients of the disposable polynomial $\mathbf{P}(x)$ can usually be omitted. When n is large this can mean omitting most of the computations required for a complete determination of the approximating function. Frequently, computations which end with $|\epsilon|$ remain very simple even though n becomes arbitrarily large.

Third, sometimes, as in our Examples 1, 2, 3 and 5, our techniques give quite simple estimates of the advantage of an equal-ripple approximation over simple truncation of an infinite series of Chebyshev polynomials. Such a comparison may be useful, for example, in deciding what sort of approximation should be computed in detail.

More generally, an attractive combination may be an initial exploration in terms of the techniques described here, followed by the detailed computation of one or more preferred cases by established iterative numerical methods.

We have assumed here that the parameters disposable for purposes of approximation are all the coefficients in a *polynomial.* Preliminary investigation indicates that similar methods may be feasible for disposable rational functions, or ratios of polynomials, provided the polynomials in the denominators are of quite modest degree. This will be the subject of a later paper.

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