Some Properties of a Nonlinear Model of a System for Synchronizing Digital Transmission Networks*

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(Manuscript received April 22, 1969)

J. R. Pierce has recently proposed a system for synchronizing an arbitrary number of geographically separated oscillators, and, under the assumption of zero transmission delays between stations, has shown that a certain linear model of the system is stable in the sense that all of the station frequencies approach a common final value as $t \rightarrow \infty$.

The purpose of this paper is to report on some results concerning the dynamic behavior of a nonlinear version of an important special case of Pierce's model. The nonlinear model takes into account transmission delays.

It is proved under certain very general conditions that the nonlinear model possesses the stability property required of a synchronization system. More explicitly, it is proved that the model is stable for all nonnegative values of the delays. The results show that the model possesses some additional fundamental properties of engineering interest, and they provide an analytical basis for using a computer for further studies. In particular, a complete solution to the problem of determining the final frequency of the system and the final value of the content of an arbitrary buffer is presented, in the sense that it is shown that these quantities can be determined by solving a certain set of nonlinear equations which is proved to possess a unique solution.

I. INTRODUCTION

The purpose of this paper is to report on some results concerning properties of the solution $f_1(t)$, $f_2(t)$, $\cdots$, $f_n(t)$ of the set of equations

* This paper was presented as an invited contribution at both the Symposium on Mathematical Aspects of Electrical Networks (sponsored by the American Mathematical Society, New York City, April 1969) and the Joint Conference on Mathematical and Computer Aids to Design (sponsored by the ACM, SIAM, and IEEE; Anaheim, California; October 1969).

$$f_i(t) = \varphi_i\left\{\sum_{j\neq i} \varphi_{ij}\left\{\int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)]\, d\tau + b_{ij}(0)\right\}\right\} + c_i$$

$$i = 1, 2, \cdots, n$$

$$t \geqq 0 \tag{1}$$

in which n is an arbitrary positive integer such that $n \geqq 2$, the $\varphi_i(\cdot)$ and the $\varphi_{ij}(\cdot)$ are monotone functions that map the real interval $(-\infty, \infty)$ into itself, the τ_{ij} are nonnegative constants, and the c_i and the $b_{ij}(0)$ are real constants.

The set of equations (1) governs the behavior of a nonlinear model of the key part of a system for synchronizing digital transmission networks. Our main result is that synchronization is possible under very general conditions concerning the nonlinearities and the time delays τ_{ij}. In addition, an analytical basis for computing the final frequency of the system is presented; this involves proving that a certain set of nonlinear equations possesses a unique solution. Other results are presented concerning, for example, buffer requirements* and certain monotonicity properties of the frequency functions $f_i(\cdot)$.

1.1 *Pierce's Model*

When $\tau_{ij} = 0$ for all $i \neq j$, when $\varphi_i(x) = x$ for all i and all real x, and when $\varphi_{ij}(x) = a_{ij}x$ for all real x and all $i \neq j$, in which a_{ij} is a real constant for all $i \neq j$, we have

$$f_i(t) = \sum_{j\neq i} a_{ij}\left\{\int_0^t [f_j(\tau) - f_i(\tau)]\, d\tau + b_{ij}(0)\right\} + c_i$$

$$i = 1, 2, \cdots, n \qquad t \geqq 0. \tag{2}$$

Equations (2) are the equations of a linear model of the principal part of a system for synchronizing digital transmission networks recently proposed by J. R. Pierce.[1] His system employs oscillators of adjustable frequency and buffers which accept pulses at an incoming rate and which produce corresponding output pulses at the local clock rate.

In Pierce's model the content b_{ij} of the buffer at station i which accepts pulses from station j is assumed to satisfy the equation†

$$\dot{b}_{ij}(t) = f_j(t) - f_i(t), \qquad t \geqq 0 \tag{3}$$

in which $f_j(t)$ and $f_i(t)$ are the frequencies at time t at stations j and i,

* An explanation of the function of the device called a buffer is given in Section 1.1.
† As usual, a dot over a mathematical symbol denotes the derivative with respect to time.

respectively, and the overall system of coupled oscillators is assumed to satisfy equations (2) with $a_{ij} = a_{ji} \geqq 0$ for all $i \neq j$. Under the natural assumption that there is some path from each station to every other station, Pierce has shown, by directing attention to a passive RL network analog of equations (2), that the model is stable in the sense that each frequency f_i approaches the same final value as $t \to \infty$.*

1.2 *The Nonlinear Model*

Our interest in the properties of the solution of equations (1) arises as a consequence of Pierce's work as follows. First, we wish to take into account the time delay τ_{ij} associated with transmission to an arbitrary station i from an arbitrary station $j \neq i$. Thus we replace $f_j(t)$ by $f_j(t - \tau_{ij})$ in (3) and (2). The content $b_{ij}(t)$ of the ijth buffer is then

$$\int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)]\, d\tau + b_{ij}(0) \tag{4}$$

for all $t \geqq 0$.

Our mathematical model of a buffer does not reflect the fact that the capacity of a real buffer is bounded; a real buffer is a device that can store at most some fixed finite number of pulses. Therefore it makes sense to study how a linear model of a synchronization system employing buffers, such as the one governed by (2), can be modified to reduce the possibility of occurrence of buffer overload (that is, the possibility that the capacity of the buffers will be exceeded). It is therefore reasonable to replace the expression (4) for the buffer content by some monotone nonlinear function $\varphi_{ij}(\cdot)$ of (4), with the idea in mind that $\varphi_{ij}(\cdot)$ is a function with moderate slope near the origin and very large slope corresponding to values of (4) that are in the neighborhoods of buffer overload. Similarly, in order to ease the requirements on the extent to which the frequencies of the adjustable oscillators must be variable, and in order to reduce the tendency of very large excursions in the frequencies f_i during a transient phase, it is reasonable to replace the sum

$$\sum_{j \neq i} \varphi_{ij}[b_{ij}(t)] \tag{5}$$

formed at the ith station by some monotone nonlinear function $\varphi_i(\cdot)$

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* In Ref. 1 Pierce actually deals with a more general linear model than we have described here, but treats in most detail the important case described above. In connection with the more general model, Pierce has exploited the network analogy further in order to obtain an expression for the final frequency, and to make assertions concerning the behavior of the system when certain elements are nonlinear. For additional material dealing with various aspects of the problem of synchronizing geographically separated oscillators, see, for example, Refs. 2–7. In particular, Ref. 4 contains a short history of the problem.

of (5), in which $\varphi_i(\cdot)$ has moderate slope near the origin and very small slope far from the origin.

These considerations lead at once to the study of the properties of the set of equations (1). Of course the crucial question is: "Does the system governed by (1) possess the basic stability property required of a synchronization system?" Our main result concerning (1) is that, no matter what the values of the time delays τ_{ij}, under some conditions which are quite trivial from the engineering viewpoint (and rather weak from the mathematical viewpoint), it does.

II. SUMMARY OF RESULTS, AND SOME APPLICATIONS

2.1 *The Main Result Concerning (1)*

In order to describe the result, we first introduce some definitions and assumptions.

Definition 1: Let M denote an arbitrary $n \times n$ matrix with elements m_{ij}. Let the *graph of M* denote the graph containing n vertices (that is, n nodes), a directed edge (that is, a directed line segment or arc) from node j to node i for every pair i, j with $i \neq j$ and $m_{ij} \neq 0$, and no other directed edges.

Definition 2: Let M denote an arbitrary $n \times n$ matrix. Then we shall say that the graph of M is a *communicating graph* if and only if there is some path (not necessarily a direct path) from each node to every other node.

We assume throughout the paper that:

(*i*) τ_{ij} denotes an arbitrary nonnegative constant for all $i \neq j$.

(*ii*) For each i, $\varphi_i(\cdot)$ denotes a real-valued continuously differentiable function defined on $(-\infty, \infty)$ such that

$$\underline{k}_i \leqq \varphi_i'(x) \leqq \bar{k}_i \tag{6}$$

for all x, with $\underline{k}_i$ and $\bar{k}_i$ positive constants.

(*iii*) For each $i \neq j$, $\varphi_{ij}(\cdot)$ denotes a continuously differentiable real-valued function defined on $(-\infty, \infty)$ such that either $\varphi_{ij}(x) = 0$ for all x, or

$$\underline{k}_{ij} \leqq \varphi_{ij}'(x) \leqq \bar{k}_{ij} \tag{7}$$

for all x, with $\underline{k}_{ij}$ and $\bar{k}_{ij}$ positive constants.*

* At the price of some additional complication, we could have replaced assumptions (*ii*) and (*iii*) with assumptions concerning the behavior of the $\varphi_i(\cdot)$ and the $\varphi_{ij}(\cdot)$ on finite intervals. See Section 2.2.

(*iv*) The matrix M defined by

$$(M)_{ii} = 0 \qquad \text{for all} \quad i$$

$$(M)_{ij} = \varphi'_{ij}(0) \quad \text{for all} \quad i \neq j$$

is the matrix of a communicating graph.

(*v*) Each $f_i(\cdot)$ is defined and differentiable on $[-\bar{\tau}, \infty)$ in which $\bar{\tau} = \max_{i \neq j}\{\tau_{ij}\}$.

Assumption (*iv*) possesses a simple physical interpretation. It is a natural connectivity assumption of the type needed if synchronization is to be possible in the sense that all of the station frequencies approach a common final value as $t \to \infty$.

Our basic set of equations is

$$f_i(t) = \varphi_i\left\{\sum_{j \neq i} \varphi_{ij}\left\{\int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)]\, d\tau + b_{ij}(0)\right\}\right\} + c_i \tag{8}$$

for all i and all $t \geqq 0$. By differentiating both sides of these equations with respect to t, we have

$$\dot{f}_i(t) = \varphi'_i[\xi_i(t)] \sum_{j \neq i} \varphi'_{ij}[\xi_{ij}(t)][f_j(t - \tau_{ij}) - f_i(t)], \qquad t \geqq 0 \tag{9}$$

for all i, in which of course

$$\xi_i(t) = \sum_{j \neq i} \varphi_{ij}\left\{\int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)]\, d\tau + b_{ij}(0)\right\}$$

and

$$\xi_{ij}(t) = \int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)]\, d\tau + b_{ij}(0).$$

Let $h_{ij}(t) = \varphi'_i[\xi_i(t)]\varphi'_{ij}[\xi_{ij}(t)]$ for all $t \geqq 0$ and all $j \neq i$. Then

$$\dot{f}_i(t) = \sum_{j \neq i} h_{ij}(t)[f_j(t - \tau_{ij}) - f_i(t)], \qquad t \geqq 0 \tag{10}$$

for all i. According to Theorem 1 (Section III) the coefficients $h_{ij}(\cdot)$ of (10) are such that there exists a real constant ρ with the property that for all i, $f_i(t) - \rho \to 0$ as $t \to \infty$. This means that the system is stable in the sense that all of the station frequencies approach a common final value. Note that this result does not involve assumptions concerning the values of the nonnegative delays τ_{ij}, that it is valid for monotone nonlinearities of a very general type, and that it does not involve symmetry assumptions such as $\varphi_{ij}(\cdot) = \varphi_{ji}(\cdot)$ for all $i \neq j$.

2.2. *A Monotonicity Property of the* $f_i(\cdot)$

The first of the two lemmas used in the proof of Theorem 1 asserts that the solution $f_1(\cdot), f_2(\cdot), \cdots, f_n(\cdot)$ of (10) possesses an interesting monotonicity property. Let T be an arbitrary nonnegative value of time t, and let the upper envelope and lower envelope $\bar{f}(t)$ and $\underline{f}(t)$, respectively, of the $f_i(t)$ be defined for each $t \geqq -\bar{\tau}$ by $\bar{f}(t) = \max_i f_i(t)$, $\underline{f}(t) = \min_i f_i(t)$. Let $\bar{f}_{\bar{\tau}}(T)$ and $\underline{f}_{\bar{\tau}}(T)$, respectively, denote the largest and smallest value of $\bar{f}(t)$ and $\underline{f}(t)$ for t belonging to the interval $[-\bar{\tau} + T, T]$. Then, according to the lemma just referred to, $\bar{f}(t) \leqq \bar{f}_{\bar{\tau}}(T)$ and $\underline{f}(t) \geqq \underline{f}_{\bar{\tau}}(T)$ for all $t \geqq T$. In particular, since the $f_i(t)$ approach a common final value, we see that the interval envelope functions $\bar{f}_{\bar{\tau}}(T)$ and $\underline{f}_{\bar{\tau}}(T)$ approach each other as $T \to \infty$.

Our assumptions (*ii*) and (*iii*) on the $\varphi_i(\cdot)$ and the $\varphi_{ij}(\cdot)$ concern the behavior of those functions for all, and in particular arbitrarily large, arguments. The upper and lower bounds just described show that it would have sufficed to have made similar assumptions on the behavior of the $\varphi_i(\cdot)$ on any finite interval $[-a, a]$ such that for all i

$$\varphi_i(x) \in [\underline{f}_{\bar{\tau}}(0) - \max_i c_i \ , \ \bar{f}_{\bar{\tau}}(0) - \min_i c_i]$$

for all $x \in [-a, a]$. On the basis of bounds of the type described in Section 2.4, similar statements can be made concerning the pertinent range of arguments of the $\varphi_{ij}(\cdot)$.

2.3. *Final-Frequency Determination*

We now turn our attention to the matter of determining the final frequency of the model governed by (1).

Let

$$p_i(t) = \int_0^t f_i(\tau)\, d\tau \tag{11}$$

for all $t \geqq 0$ and all i. Then, since for all $t \geqq 0$

$$\int_0^t f_j(\tau - \tau_{ij})\, d\tau = \int_0^{(t-\tau_{ij})} f_j(\tau)\, d\tau + \int_{-\tau_{ij}}^0 f_j(\tau)\, d\tau,$$

we have, using (1),

$$\dot{p}_i(t) = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[p_j(t - \tau_{ij}) - p_i(t) + \lambda_{ij}]\} + c_i \tag{12}$$

for all i and all $t \geqq 0$, in which

$$\lambda_{ij} = b_{ij}(0) + \int_{-\tau_{ij}}^0 f_j(\tau)\, d\tau. \tag{13}$$

According to Theorem 2 (Section III), there exists a unique real constant ρ and some real n-vector q such that

$$\rho = \varphi_i\{\sum_{j\neq i} \varphi_{ij}[-\rho\tau_{ij} + q_j - q_i + \lambda_{ij}]\} + c_i \quad \text{for all} \quad i. \tag{14}$$

With ρ and q such that (14) is satisfied, let

$$p_i(t) = \rho t + q_i + r_i(t), \qquad t \geqq -\bar{\tau} \tag{15}$$

for all i, in which the q_i are the components of q, and the $r_i(t)$ are some functions of t. Then, using (12),

$$\rho + \dot{r}_i(t) = \varphi_i\{\sum_{j\neq i} \varphi_{ij}[-\rho\tau_{ij} + q_j - q_i + \lambda_{ij} + r_j(t - \tau_{ij}) - r_i(t)]\} + c_i \tag{16}$$

for all i and all $t \geqq 0$. But, using (14) and (16),

$$\dot{r}_i(t) = \varphi_i\{\sum_{j\neq i} \varphi_{ij}[r_j(t - \tau_{ij}) - r_i(t) + s_{ij}]\} - \varphi_i\{\sum_{j\neq i} \varphi_{ij}[s_{ij}]\} \tag{17}$$

for all i and all $t \geqq 0$, in which $s_{ij} = -\rho\tau_{ij} + q_j - q_i + \lambda_{ij}$.

For each i and each $t \in [0, \infty)$, we have, by the mean-value theorem,

$$\varphi_i\{\sum_{j\neq i} \varphi_{ij}[r_j(t - \tau_{ij}) - r_i(t) + s_{ij}]\} - \varphi_i\{\sum_{j\neq i} \varphi_{ij}[s_{ij}]\} = \varphi_i'[u_i(t)]\{\sum_{j\neq i} \varphi_{ij}[r_j(t - \tau_{ij}) - r_i(t) + s_{ij}] - \sum_{j\neq i} \varphi_{ij}[s_{ij}]\}$$

for some $u_i(t)$ such that $u_i(t)$ lies within the closed interval with end-points $\sum_{j\neq i} \varphi_{ij}[s_{ij}]$ and $\sum_{j\neq i} \varphi_{ij}[r_j(t - \tau_{ij}) - r_i(t) + s_{ij}]$. Similarly for each $j \neq i$ and each $t \in [0, \infty)$,

$$\varphi_{ij}[r_j(t - \tau_{ij}) - r_i(t) + s_{ij}] - \varphi_{ij}[s_{ij}] = \varphi_{ij}'[w_{ij}(t)][r_j(t - \tau_{ij}) - r_i(t)]$$

for a suitably chosen $w_{ij}(t)$. Therefore (17) can be written as

$$\dot{r}_i(t) = \sum_{j\neq i} c_{ij}(t)[r_j(t - \tau_{ij}) - r_i(t)] \tag{18}$$

for all i and all $t \geqq 0$, where $c_{ij}(t) = \varphi_i'[u_i(t)]\varphi_{ij}'[w_{ij}(t)]$. But, by Theorem 1, the coefficients $c_{ij}(\cdot)$ of (18) are such that there exists a constant σ with the property that for all i, $r_i(t) \to \sigma$ as $t \to \infty$. It follows [see (18)] that for all i, $\dot{r}_i(t) \to 0$ as $t \to \infty$. Since

$$\int_0^t f_i(\tau)\, d\tau = \rho t + q_i + r_i(t), \qquad t \geqq 0$$

for all i, it is clear that ρ is the final value of the $f_i(\cdot)$.

According to Theorem 2: there exists exactly one real n-vector q such that, with $U^{tr} = (1, 1, \cdots, 1)$,

$$U^{tr}q = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\tau_{ij}U^{tr}q + q_j - q_i + \lambda_{ij}]\} + c_i$$

for all i, and $\rho = U^{tr}q$.

There are some simple special cases in which we can exhibit an explicit expression for ρ. Suppose, for example, that $\tau_{ij} = 0$ for all $i \neq j$, that $b_{ij}(0) = -b_{ji}(0)$ for all $i \neq j$, and that $\varphi_{ij}(x) = -\varphi_{ji}(-x)$ for all $i \neq j$ and all real x. Then, using (14), we have for all i

$$\varphi_i^{-1}(\rho - c_i) = \sum_{j \neq i} \varphi_{ij}[q_j - q_i + b_{ij}(0)]$$

in which $\varphi_i^{-1}(\cdot)$ is the inverse of $\varphi_i(\cdot)$, and

$$\sum_i \varphi_i^{-1}(\rho - c_i) = \sum_i \sum_{j \neq i} \varphi_{ij}[q_j - q_i - b_{ij}(0)] = 0.$$

Therefore, $n\rho = \sum_i c_i$ if $\varphi_i(x) = x$ for all real x and all i, or if $n = 2$ and $\varphi_1(x) = \varphi_2(x) = -\varphi_2(-x)$ for all real x.

Finally, as a relevant application of the material of Section 2.2, we have when $\tau_{ij} = 0$ for all $i \neq j$

$$\min_i \left(c_i + \varphi_i\{\sum_{j \neq i} \varphi_{ij}[b_{ij}(0)]\}\right) \leqq \rho \leqq \max_i \left(c_i + \varphi_i\{\sum_{j \neq i} \varphi_{ij}[b_{ij}(0)]\}\right)$$

since $\bar{f}(t) \leqq \max_i f_i(0)$ and $\underline{f}(t) \geqq \min_i f_i(0)$ for all $t \geqq 0$, and, by (1),

$$f_i(0) = c_i + \varphi_i\{\sum_{j \neq i} \varphi_{ij}[b_{ij}(0)]\}$$

for all i.

2.4. *Bounds on Buffer Content*

In order to analytically formulate specifications to be met by real buffers such that buffer overload does not occur in a real synchronization system of the type under study, it is natural to consider the problem of obtaining useful upper bounds on the contents of the mathematical buffers of our model. We do not treat this entire problem in detail in this paper. However, we show here that under some strong assumptions, it is possible to exploit the material of Sections 2.2 and 2.3 to obtain a simple *uniform* bound on buffer content. In addition, in terms of the constant ρ and the vector q introduced in Section 2.3, we present a complete solution to the problem of evaluating the *final value* of the content of an arbitrary buffer.

According to Theorem 2, the vector q that satisfies (14) is unique to within an additive n-vector of the form αU, in which α is a real constant and U is the transpose of $(1, 1, \cdots, 1)$. In particular, the quantity $\Delta_q = (\max_i q_i - \min_i q_i)$ associated with any solution pair ρ, q of (14) is unique. In this section it is shown that when

$$\tau_{ij} = b_{ij}(0) = 0 \quad \text{for all} \quad i \neq j, \tag{19}$$

then the magnitude of the content

$$\int_0^t [f_j(\tau) - f_i(\tau)] \, d\tau \tag{20}$$

of an arbitrary buffer is bounded for all $t \geqq 0$ by $2\Delta_q$.

Let (19) be satisfied. As in Section 2.3, let

$$p_i(t) = \int_0^t f_i(\tau) \, d\tau, \qquad t \geqq 0$$

for all i. Then with $p_i(t) = \rho t + q_i + r_i(t)$, $t \geqq 0$ for all i, in which ρ and q satisfy (14), we find as in Section 2.3 that for suitably chosen functions $u_i(\cdot)$ and $w_{ij}(\cdot)$,

$$\dot{r}_i(t) = \sum_{j \neq i} c_{ij}(t)[r_j(t) - r_i(t)], \qquad t \geqq 0 \tag{21}$$

for all i, in which $c_{ij}(t) = \varphi_i'[u_i(t)]\varphi_{ij}'[w_{ij}(t)]$. Since (21) is an equation of the same type as (10) (more precisely, see Lemma 1 of the proof of Theorem 1), it follows that for all $t \geqq 0$, $r_i(t) \leqq \max_i r_i(0)$ and $r_i(t) \geqq \min_i r_i(0)$. But $r_i(0) = -q_i$ for all i. Thus, for any j and i with $j \neq i$

$$\begin{aligned} p_j(t) - p_i(t) &= q_j - q_i + r_j(t) - r_i(t), \qquad t \geqq 0 \\ &\leqq 2\Delta_q, \qquad t \geqq 0 \end{aligned}$$

and, similarly, $p_j(t) - p_i(t) \geqq -2\Delta_q$, $t \geqq 0$.

Concerning the problem of evaluating Δ_q, there are some cases in which it is possible to obtain simple and useful upper bounds. In one simple case we can obtain an explicit expression for Δ_q. For example, suppose that (19) is satisfied and that $n = 2$. Suppose also that $\varphi_1(x) = \varphi_2(x) = -\varphi_2(-x)$ for all x, and that $\varphi_{12}(x) = \varphi_{21}(x) = -\varphi_{21}(-x)$ for all x. Then $\rho = \varphi_1[\varphi_{12}(q_2 - q_1)] + c_1$, $\rho = \varphi_2[\varphi_{21}(q_1 - q_2)] + c_2$, and, using the fact that $\varphi_2(\cdot)$ and $\varphi_{21}(\cdot)$ are odd, $2\varphi_1[\varphi_{12}(q_2 - q_1)] = c_2 - c_1$. Therefore, in this case $\Delta_q = |q_2 - q_1| = |\varphi_{12}^{-1}\{\varphi_1^{-1}[\frac{1}{2}(c_2 - c_1)]\}|$.

We now consider the matter of (proving the existence of and) evaluating the final value $\lim_{t\to\infty} b_{ij}(t)$ of the content of an arbitrary buffer. With ρ, q, the $r_i(\cdot)$, and the $p_i(\cdot)$ as defined in Section 2.3, we have for $t \geqq 0$ and any $i \neq j$

$$b_{ij}(t) = \int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)]\, d\tau + b_{ij}(0)$$

$$= p_j(t - \tau_{ij}) - p_i(t) + b_{ij}(0) + \int_{-\tau_{ij}}^{0} f_j(\tau)\, d\tau$$

$$= -\rho\tau_{ij} + q_j - q_i + r_j(t - \tau_{ij}) - r_i(t) + b_{ij}(0) + \int_{-\tau_{ij}}^{0} f_j(\tau)\, d\tau.$$

Since $r_j(t - \tau_{ij}) - r_i(t) \to 0$ as $t \to \infty$, we have the result

$$\lim_{t \to \infty} b_{ij}(t) = -\rho\tau_{ij} + q_j - q_i + b_{ij}(0) + \int_{-\tau_{ij}}^{0} f_j(\tau)\, d\tau. \tag{22}$$

Finally, if (19) is satisfied, then, using (22),

$$\max_{j \neq i} \mid \lim_{t \to \infty} b_{ij}(t) \mid = \max_{j \neq i} \mid q_j - q_i \mid = \Delta_q ,$$

which shows that our *uniform bound* $2\Delta_q$ is not unreasonable.

2.5 *Discussion*

The results presented in this paper are concerned with a reasonably realistic strongly-nonlinear model of an important type of synchronization system. They answer several key questions concerning the dynamic behavior of the system, and provide an analytical basis for using a computer for further studies in so far as we have proved, for example, that a solution pair ρ, q of the set of equations (14) exists, that this pair is unique in the sense indicated, and that it can be determined by computing the unique solution q of a related set of equations.

On the other hand, although we have proved that under very general conditions our nonlinear model possess the basic properties of a synchronization system, in this paper we have not considered the next natural problem, that of determining the extent to which the system performance can be improved as a result of the presence of the nonlinearities. There are several other important practical problems that are not considered here, such as the problem of predicting the effects of variable transmission delays (due to temperature changes). There is a clear need for much more work in this area, especially in connection with the problem of comparing the performance of alternative synchronization systems.

III. THEOREMS 1 AND 2

Throughout Sections III and IV:

(*i*) n denotes an arbitrary fixed positive integer such that $n \geqq 2$;

the statement "for all i" means for all $i = 1, 2, \cdots, n$, and "for all $j \neq i$" means for all $j \in \{1, 2, \cdots, n\}$ except $j = i$.

(ii) With v an arbitrary n-vector, v^{tr} denotes the transpose of v. The zero n-vector is denoted by θ.

(iii) If x denotes a differentiable function of t, then $\dot{x}$ indicates the derivative of x with respect to t.

(iv) All functions and constants considered are real valued.

The following two theorems are proved in Section IV.

Theorem 1: Suppose that the following conditions are satisfied:

(i) For each $i \neq j$, $a_{ij}(\cdot)$ denotes a nonnegative bounded measurable function defined for all $t \in [0, \infty)$.

(ii) With $\underline{a}$ and $\bar{a}$ positive constants such that $\underline{a} \leqq \bar{a}$, for each $i \neq j$, $a_{ij}(\cdot)$ satisfies either $a_{ij}(t) = 0$ for all $t \in [0, \infty)$ or $\underline{a} \leqq a_{ij}(t) \leqq \bar{a}$ for all $t \in [0, \infty)$.

*(iii) For $t \in [0, \infty)$, the $n \times n$ matrix A, with $(A)_{ij} = a_{ij}(t)$ for all $i \neq j$ and $(A)_{ii} = 0$ for all i, is the matrix of a communicating graph.**

(iv) For each $i \neq j$, τ_{ij} denotes a nonnegative constant and $\bar{\tau} = \max_{i \neq j} \tau_{ij}$.

(v) For each i, $x_i(\cdot)$ denotes a differentiable function defined on $[-\bar{\tau}, \infty)$ such that

$$\dot{x}_i(t) = \sum_{j \neq i} a_{ij}(t)[x_j(t - \tau_{ij}) - x_i(t)], \qquad t \geqq 0$$

for all i.

Then there exists a constant ρ such that $x(t) - \rho U \to \theta$ as $t \to \infty$, in which $U = (1, 1, \cdots, 1)^{tr}$.

Theorem 2: Suppose that assumptions (i) through (iv) in Section 2.1 are satisfied. Let U denote the n-vector $(1, 1, \cdots, 1)^{tr}$. Then (a) there exists a unique n-vector q such that

$$U^{tr}q = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\tau_{ij}U^{tr}q + q_j - q_i + \lambda_{ij}]\} + c_i \quad \text{for all} \quad i,$$

in which the λ_{ij} and the c_i are constants, and (b) concerning the solution ρ, q of

$$\rho = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho\tau_{ij} + q_j - q_i + \lambda_{ij}]\} + c_i \quad \text{for all} \quad i,$$

the value of ρ is unique, and q is unique to within an additive n-vector αU, in which α is an arbitrary real constant.

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* See definitions 1 and 2 in Section 2.1.

IV. PROOF OF THEOREMS 1 AND 2

In this section:

(*i*) 1_n *denotes the identity matrix of order* n.
(*ii*) *The transpose of any matrix* M *is denoted by* M^{tr}.
(*iii*) *If* v *is an* n*-vector, then* $\| v \|$ *denotes* $(v^{tr}v)^{\frac{1}{2}}$.
(*iv*) *If* F *denotes an* n*-vector-valued function, then* $(F)_i$ *denotes the* i^{th} *component of* F.

4.1. *Proof of Theorem 1*

We first prove the following lemma.

Lemma 1: *Suppose that (i), (iv), and (v) of Theorem 1 are satisfied. For all* $t \in [-\bar{\tau}, \infty)$, *let* $\bar{x}(t)$ *and* $\underline{x}(t)$ *denote* $\max_q x_q(t)$ *and* $\min_q x_q(t)$, *respectively. Let* T *be a nonnegative constant. Then, for all* $t \geqq T$, $\bar{x}(t) \leqq \sup_{[-\bar{\tau}+T, T]} \bar{x}(t)$ *and* $\underline{x}(t) \geqq \inf_{[-\bar{\tau}+T, T]} \underline{x}(t)$.

Proof: (upper bound) We have for all i

$$\dot{x}_i(t) = \sum_{j \neq i} a_{ij}(t)[x_j(t - \tau_{ij}) - x_i(t)], \qquad t \geqq 0. \tag{23}$$

Thus

$$\dot{x}_i(t) + x_i(t) \sum_{j \neq i} a_{ij}(t) = \sum_{j \neq i} a_{ij}(t) x_j(t - \tau_{ij}), \qquad t \geqq 0$$

and

$$x_i(t) = x_i(T) \exp\left[-\int_T^t \sum_{j \neq i} a_{ij}(t)\, dt\right]$$
$$+ \int_T^t \exp\left[-\int_\tau^t \sum_{j \neq i} a_{ij}(t)\, dt\right] \sum_{j \neq i} a_{ij}(\tau) x_j(\tau - \tau_{ij})\, d\tau, \quad t \geqq T \tag{24}$$

for all i.

It is convenient to introduce the function $I(\cdot, \cdot, \cdot)$ defined by

$$I(u, v, k) = \exp\left[-\int_u^v \sum_{j \neq k} a_{kj}(t)\, dt\right]$$

for all real $u \leqq v$ and all positive integer $k \leqq n$. Thus, for example, (24) is equivalent to

$$x_i(t) = x_i(T) I(T, t, i) + \int_T^t I(\tau, t, i) \sum_{j \neq i} a_{ij}(\tau) x_j(\tau - \tau_{ij})\, d\tau,$$
$$t \geqq T. \tag{25}$$

Let t_0 denote an arbitrary positive constant. There exist an index k and a $t_1 \in [T, T + t_0]$ such that

$$x_k(t_1) = \sup_{[T, T+t_0]} \bar{x}(t).$$

Clearly

$$x_k(t_1) = x_k(T)I(T, t_1, k) + \int_T^{t_1} I(\tau, t_1, k) \sum_{j \neq k} a_{kj} x_j(\tau - \tau_{kj})\, d\tau.$$

Therefore, since the a_{kj} are nonnegative,

$$x_k(t_1) \leqq x_k(T)I(T, t_1, k) + \int_T^{t_1} I(\tau, t_1, k) \sum_{j \neq k} a_{kj}(\tau)\, d\tau \cdot \max_{j \neq k} \sup_{[T-\tau_{kj}, t_1-\tau_{kj}]} x_j(t)$$

$$\leqq x_k(T)I(T, t_1, k) + \int_T^{t_1} I(\tau, t_1, k) \sum_{j \neq k} a_{kj}(\tau)\, d\tau \sup_{[T-\bar{\tau}, t_1]} \bar{x}(t).$$

But

$$\int_T^{t_1} I(\tau, t_1, k) \sum_{j \neq k} a_{kj}(\tau)\, d\tau = 1 - I(T, t_1, k).$$

Thus

$$x_k(t_1) \leqq x_k(T)I(T, t_1, k) + [1 - I(T, t_1, k)] \sup_{[T-\bar{\tau}, t_1]} \bar{x}(t).$$

Either

$$\sup_{[T-\bar{\tau}, T]} \bar{x}(t) \leqq \sup_{[T, t_1]} \bar{x}(t) \tag{26}$$

or

$$\sup_{[T-\bar{\tau}, T]} \bar{x}(t) > \sup_{[T, t_1]} \bar{x}(t). \tag{27}$$

If (26) holds, then

$$x_k(t_1) \leqq x_k(T)I(T, t_1, k) + [1 - I(T, t_1, k)]x_k(t_1)$$

[since $x_k(t_1) = \sup_{[T, t_1]} \bar{x}(t)$], and hence

$$x_k(t_1) \leqq x_k(T),$$

which implies that $x_k(t_1) \leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t)$. If (27) holds, then [since $x_k(T) \leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t)$]

$$x_k(t_1) \leqq I(T, t_1, k) \sup_{[T-\bar{\tau}, T]} \bar{x}(t) + [1 - I(T, t_1, k)] \sup_{[T-\bar{\tau}, T]} \bar{x}(t)$$
$$\leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t).$$

We have shown that

$$\sup_{[T, T+t_0]} \bar{x}(t) \leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t). \tag{28}$$

But t_0 is an arbitrary positive number. Therefore

$$\sup_{t \geqq T} \bar{x}(t) \leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t).$$

(lower bound) Our proof of the inequality

$$\inf_{t \geqq T} \underline{x}(t) \geqq \inf_{[T-\bar{\tau}, T]} \underline{x}(t) \tag{29}$$

parallels the derivation of the upper bound, and is outlined below.

There exists an index l and a $t_2 \in [T, T + t_0]$ such that

$$x_l(t_2) = \inf_{[T, T+t_0]} \underline{x}(t).$$

Thus

$$x_l(t_2) \geqq x_l(T)I(T, t_2, l) + [1 - I(T, t_2, l)] \inf_{[T-\bar{\tau}, t_2]} \underline{x}(t). \tag{30}$$

Either

$$\inf_{[T-\bar{\tau}, T]} \underline{x}(t) \geqq \inf_{[T, t_2]} \underline{x}(t)$$

or

$$\inf_{[T-\bar{\tau}, T]} \underline{x}(t) < \inf_{[T, t_2]} \underline{x}(t).$$

In either case, we find using (30), that (29) is satisfied. □

We note that it is a consequence of Lemma 1 that the components of $x(\cdot)$ are bounded on $[0, \infty)$, and, since $x(\cdot)$ and $\dot{x}(\cdot)$ are related by (23), that the components of $\dot{x}(\cdot)$ are bounded on $[0, \infty)$.

Assume that

$$\sup_{[u-\bar{\tau}, u]} \bar{x}(t) - \inf_{[u-\bar{\tau}, u]} \underline{x}(t)$$

[$\bar{x}(\cdot)$ and $\underline{x}(\cdot)$ are defined in the statement of Lemma 1] *does not* approach zero as $u \to \infty$. We shall show that this assumption implies that the components of $x(\cdot)$ are *not bounded* on $[0, \infty)$, a contradiction.

Since, by assumption, $\sup_{[u-\bar{\tau}, u]} \bar{x}(t) - \inf_{[u-\bar{\tau}, u]} \underline{x}(t)$ does not approach zero as $u \to \infty$, there exist a positive constant ϵ and a set $\{u_q\}_0^\infty$

with $u_q \in [0, \infty)$ and $\sup_q u_q = \infty$ such that

$$\sup_{[u_q-\bar{\tau}, u_q]} \bar{x}(t) - \inf_{[u_q-\bar{\tau}, u_q]} \underline{x}(t) \geqq 2\epsilon$$

for all q. For each q let $t_q \in [u_q - \bar{\tau}, u_q]$ and $t_q' \in [u_q - \bar{\tau}, u_q]$ be such that

$$\underline{x}(t_q) = \inf_{[u_q-\bar{\tau}, u_q]} \underline{x}(t)$$

$$\bar{x}(t_q') = \sup_{[u_q-\bar{\tau}, u_q]} \bar{x}(t).$$

Of course $\sup_q t_q = \infty$ and $| t_q - t_q' | \leqq \bar{\tau}$. Thus there exists a set $\{\lambda_q\}_0^\infty$ of real constants such that $| \lambda_q | \leqq \bar{\tau}$ for all q, with the property that $\bar{x}(t_q + \lambda_q) - \underline{x}(t_q) \geqq 2\epsilon$ for all q. It follows from the definition of $\underline{x}(\cdot)$ and $\bar{x}(\cdot)$ that for each q there exist indices $l(q)$ and $s(q)$ such that $x_{l(q)}(t_q + \lambda_q) - x_{s(q)}(t_q) \geqq 2\epsilon$.

Finally, since the components of $\dot{x}(\cdot)$ are bounded on $[0, \infty)$, there exists a positive constant δ such that *for all* q $x_{l(q)}(t + \lambda_q) - x_{s(q)}(t) \geqq \epsilon$ for all $t \in [t_q - \frac{1}{2}\delta, t_q + \frac{1}{2}\delta]$.

At this point we need the following lemma.

Lemma 2: *If the hypotheses of Theorem 1 are satisfied, if T is a nonnegative constant, and if there exist three positive constants t_q, ϵ, and δ and indices $l(q)$ and $s(q)$ such that $t_q - \frac{1}{2}\delta > T + \bar{\tau}$ and $x_{l(q)}(t + \lambda_q) - x_{s(q)}(t) \geqq \epsilon$ for all $t \in [t_q - \frac{1}{2}\delta, t_q + \frac{1}{2}\delta]$, with λ_q a constant and $| \lambda_q | \leqq \bar{\tau}$, then there exist positive constants ξ and Δ such that, with $\bar{x}(t)$ as defined in the statement of Lemma 1,*

$$\sup_{t \geqq \xi} \bar{x}(t) \leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t) - \Delta$$

and Δ depends only on $\underline{a}$, $\bar{a}$, $\bar{\tau}$, ϵ, and δ.

Proof:

As in the proof of Lemma 1, it is convenient to introduce the function $I(\cdot, \cdot, \cdot)$ defined by

$$I(u, v, k) = \exp\left[-\int_u^v \sum_{j \neq k} a_{kj}(\tau)\, d\tau\right]$$

for all real $u \leqq v$ and all positive integer $k \leqq n$. The relation between T, $\bar{\tau}$, t_q, and δ is indicated in Fig. 1.

From (23)

$$x_i(t) = x_i(T)I(T, t, i) + \int_T^t I(\tau, t, i) \sum_{j \neq i} a_{ij}(\tau) x_j(\tau - \tau_{ij})\, d\tau$$

for all $t \geqq T$ and all i. By Lemma 1, $\bar{x}(t) \leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t)$ for all $t \geqq T$.

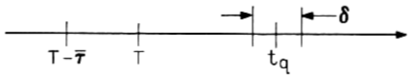

Fig. 1 — Relation between T, $\bar{\tau}$, t_q, δ.

Therefore

$$\sup_{[T-\bar{\tau},T]} \bar{x}(t) - x_{s(q)}(t) \geqq \epsilon \tag{31}$$

for all $t \in [t_q - \frac{1}{2}\delta, t_q + \frac{1}{2}\delta]$.

Let k_1 be an index such that $a_{k_1 s(q)}(t) \neq 0$ for all $t \geqq 0$. Then for $t \geqq t_q + \frac{1}{2}\delta + \bar{\tau}$

$$\begin{aligned} x_{k_1}(t) &= x_{k_1}(T)I(T, t, k_1) + \int_T^t I(\tau, t, k_1) \sum_{j \neq k_1} a_{k_1 j}(\tau) x_j(\tau - \tau_{k_1 j})\, d\tau \\ &\leqq I(T, t, k_1) \sup_{[T-\bar{\tau},T]} \bar{x}(t) \\ &\quad + \int_T^t I(\tau, t, k_1) \sum_{\substack{j \neq k_1 \\ j \neq s(q)}} a_{k_1 j}(\tau) x_j(\tau - \tau_{k_1 j})\, d\tau \\ &\quad + \int_T^{t_q - \frac{1}{2}\delta + \tau_{k_1 s(q)}} I(\tau, t, k_1) a_{k_1 s(q)}(\tau) x_{s(q)}(\tau - \tau_{k_1 s(q)})\, d\tau \\ &\quad + \int_{t_q - \frac{1}{2}\delta + \tau_{k_1 s(q)}}^{t_q + \frac{1}{2}\delta + \tau_{k_1 s(q)}} I(\tau, t, k_1) a_{k_1 s(q)}(\tau) x_{s(q)}(\tau - \tau_{k_1 s(q)})\, d\tau \\ &\quad + \int_{t_q + \frac{1}{2}\delta + \tau_{k_1 s(q)}}^{t} I(\tau, t, k_1) a_{k_1 s(q)}(\tau) x_{s(q)}(\tau - \tau_{k_1 s(q)})\, d\tau. \end{aligned}$$

By Lemma 1, for each j,

$$x_j(\tau - \tau_{k_1 j}) \leqq \sup_{[T-\bar{\tau},T]} \bar{x}(t) \tag{32}$$

for all $\tau \geqq T + \tau_{k_1 j}$. But (32) is obviously satisfied also for $\tau \in [T, T + \tau_{k_1 j}]$. That is, (32) holds for all $\tau \geqq T$ and all j. Thus, using (31),

$$\begin{aligned} x_{k_1}(t) &\leqq I(T, t, k_1) \sup_{[T-\bar{\tau},T]} \bar{x}(t) \\ &\qquad + \int_T^t I(\tau, t, k_1) \sum_{j \neq k_1} a_{k_1 j}(\tau)\, d\tau \sup_{[T-\bar{\tau},T]} \bar{x}(t) \\ &\qquad - \epsilon \int_{t_q - \frac{1}{2}\delta + \tau_{k_1 s(q)}}^{t_q + \frac{1}{2}\delta + \tau_{k_1 s(q)}} I(\tau, t, k_1) a_{k_1 s(q)}(\tau)\, d\tau \\ &\leqq \sup_{[T-\bar{\tau},T]} \bar{x}(t) - \epsilon \int_{t_q - \frac{1}{2}\delta + \tau_{k_1 s(q)}}^{t_q + \frac{1}{2}\delta + \tau_{k_1 s(q)}} I(\tau, t, k_1) a_{k_1 s(q)}\, d\tau \end{aligned}$$

for all $t \geqq t_q + \frac{1}{2}\delta + \bar{\tau}$, since

$$\int_T^t I(\tau, t, k_1) \sum_{j \neq k_1} a_{k_1 j}(\tau)\, d\tau = 1 - I(T, t, k_1).$$

But, for all k_1,

$$\int_{t_q - \frac{1}{2}\delta + \tau_{k_1 s(q)}}^{t_q + \frac{1}{2}\delta + \tau_{k_1 s(q)}} I(\tau, t, k_1) a_{k_1 s(q)}(\tau)\, d\tau \geqq \underline{a} \int_{t_q - \frac{1}{2}\delta + \tau_{k_1 s(q)}}^{t_q + \frac{1}{2}\delta + \tau_{k_1 s(q)}} e^{-(n-1)\bar{a}(t-\tau)}\, d\tau$$

$$\geqq K \exp\{-\beta[t - t_q - \tau_{k_1 s(q)} - \tfrac{1}{2}\delta]\}$$

in which $\beta = (n-1)\bar{a}$ and $K = \underline{a}\{1 - \exp[-(n-1)\bar{a}\delta]\}[(n-1)\bar{a}]^{-1}$.

Therefore

$$x_{k_1}(t) \leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t) - \epsilon K \exp\{-\beta[t - t_q - \tau_{k_1 s(q)} - \tfrac{1}{2}\delta]\}$$

for all $t \geqq t_q + \frac{1}{2}\delta + \bar{\tau}$. In particular,

$$\sup_{[T-\bar{\tau}, T]} \bar{x}(t) - x_{k_1}(t) \geqq \epsilon K e^{-\beta(\delta + \bar{\tau})}$$

for all $t \in [t_q + \frac{1}{2}\delta + \bar{\tau}, t_q + \frac{3}{2}\delta + \bar{\tau}]$. Similarly, if the index k_2 is such that $a_{k_2 k_1}(t) \neq 0$ for all $t \geqq 0$, we have for all $t \geqq t_q + \frac{3}{2}\delta + 2\bar{\tau}$

$$\begin{aligned} x_{k_2}(t) &= x_{k_2}(T) I(T, t, k_2) \\ &\quad + \int_T^t I(\tau, t, k_2) \sum_{j \neq k_2} a_{k_2 j}(\tau) x_j(\tau - \tau_{k_2 j})\, d\tau \\ &\leqq \sup_{[T-\bar{\tau}, T]} \bar{x}(t) - \epsilon K^2 e^{-\beta(\delta+\bar{\tau})} \\ &\quad \cdot \exp\{-\beta[t - t_q - \tau_{k_2 k_1} - \tfrac{3}{2}\delta - \bar{\tau}]\}. \end{aligned} \tag{33}$$

In particular, for $t \in [t_q + \frac{3}{2}\delta + 2\bar{\tau}, t_q + \frac{5}{2}\delta + 2\bar{\tau}]$

$$\sup_{[T-\bar{\tau}, T]} \bar{x}(t) - x_{k_2}(t) \geqq \epsilon K^2 e^{-2\beta(\delta+\bar{\tau})}.$$

Since the graph of A is a communicating graph, we may continue in this manner to obtain an upper bound of the type (33) for *all* of the $x_k(\cdot)$. More explicitly, for each $k_1 \in \{1, 2, \cdots, n\}$, let $\{k_1, k_2, \cdots, k_p\}$ denote a finite set of positive integers, with the integer p dependent on k_1, such that $\{k_1, k_2, \cdots, k_p\} \supset \{1, 2, \cdots, n\}$ and

$$a_{k_2 k_1} a_{k_3 k_2} \cdots a_{k_p k(p-1)} \neq 0, \qquad t \geqq 0.$$

Then, with $B = \sup_{[T-\bar{\tau}, T]} \bar{x}(t)$, $u = e^{-\beta(\delta+\bar{\tau})}$, and

$$T_r = t_q + \tfrac{1}{2}\delta + \bar{\tau} + (r-1)(\delta + \bar{\tau}) \quad \text{for all} \quad r = 1, 2, \cdots, p,$$

we have

$$x_{k_1}(t) \leqq B - \epsilon K e^{-\beta\bar{\tau}} e^{-\beta(t-T_1)}, \qquad t \geqq T_1$$
$$x_{k_2}(t) \leqq B - \epsilon K^2 u e^{-\beta\bar{\tau}} e^{-\beta(t-T_2)}, \qquad t \geqq T_2$$
$$\vdots$$
$$x_{k_p}(t) \leqq B - \epsilon K^p u^{p-1} e^{-\beta\bar{\tau}} e^{-\beta(t-T_p)}, \qquad t \geqq T_p .$$

Now let $t = T_p + \eta$ with $\eta \in [0, \bar{\tau}]$. Then

$$x_{k_1}(t) \leqq B - \epsilon K e^{-\beta\bar{\tau}} e^{-\beta(p-1)(\delta+\bar{\tau})} e^{-\beta\eta}$$
$$x_{k_2}(t) \leqq B - \epsilon K^2 u e^{-\beta\bar{\tau}} e^{-\beta(p-2)(\delta+\bar{\tau})} e^{-\beta\eta}$$
$$\vdots$$
$$x_{k_p}(t) \leqq B - \epsilon K^p u^{p-1} e^{-\beta\bar{\tau}} e^{-\beta\eta}.$$

Thus, for all $t \in [T_p, T_p + \bar{\tau}]$,

$$x_{k_r}(t) \leqq B - \Delta_{k_1}$$

for all $r = 1, 2, \cdots, p$, in which

$$\Delta_{k_1} = \min_r \{\epsilon K^r u^{r-1} e^{-2\beta\bar{\tau}} e^{-\beta(p-r)(\delta+\bar{\tau})}\}.$$

Let $\Delta = \min_{k_1} \Delta_{k_1}$, and observe that Δ depends only on $\underline{a}$, $\bar{a}$, $\bar{\tau}$, ϵ, and δ. By Lemma 1,

$$\bar{x}(t) \leqq B - \Delta$$

for all $t \geqq T_p + \bar{\tau}$. □

Since as indicated earlier, there are an infinite number of δ-intervals with centers t_q such that $\sup \{t_q\} = \infty$, and such that there exist indices $l(q)$ and $s(q)$ with the property that

$$x_{l(q)}(t + \lambda_q) - x_{s(q)}(t) \geqq \epsilon \tag{34}$$

for all $t \in [t_q - \frac{1}{2}\delta, t_q + \frac{1}{2}\delta]$, with the constants λ_q such that $|\lambda_q| \leqq \bar{\tau}$ we see that Lemma 2 and the assumption that

$$\sup_{[u-\bar{\tau},u]} \bar{x}(t) - \inf_{[u-\bar{\tau},u]} \underline{x}(t) \tag{35}$$

does not approach zero as $u \to \infty$ imply that $\bar{x}(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $\bar{x}(\cdot)$ is bounded on $[0, \infty)$. Therefore (35) approaches zero as $u \to \infty$. But, by Lemma 1, $\sup_{[u-\bar{\tau},u]} \bar{x}(t)$ is monotone nonincreasing in u and bounded from below. Thus there is a constant $\bar{L}$ such that

$$\sup_{[u-\bar{\tau},u]} \bar{x}(t) \rightarrow \bar{L}$$

as $u \rightarrow \infty$. Similarly, by Lemma 1, $\inf_{[u-\bar{\tau},u]} \underline{x}(t)$ is monotone nondecreasing in u and bounded from above. Thus there is a constant $\underline{L}$ such that

$$\inf_{[u-\bar{\tau},u]} \underline{x}(t) \rightarrow \underline{L}$$

as $u \rightarrow \infty$. But we have proved that $\underline{L} = \bar{L}$. Therefore $\bar{x}(t)$ and $\underline{x}(t)$ both approach L as $t \rightarrow \infty$, which means that there is a constant ρ such that

$$x(t) - \rho U \rightarrow \theta \quad \text{as} \quad t \rightarrow \infty. \quad \square$$

4.2. *Proof of Theorem 2*

In part (*a*) of this proof we employ a theorem of R. S. Palais* according to which: if $F(\cdot)$ is a continuously-differentiable mapping of real Euclidean n-space E^n into itself with values $F(q)$ for $q \in E^n$ such that

(*i*) $\det J_q \neq 0$ for all $q \in E^n$, in which J_q is the Jacobian matrix of $F(\cdot)$ with respect to q, and
(*ii*) $\lim_{\|q\| \rightarrow \infty} \| F(q) \| = \infty$,

then $F(\cdot)$ is an invertible mapping of E^n onto itself and $F(\cdot)^{-1}$ is continually differentiable on E^n.

We have

$$U^{tr}q = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\tau_{ij}U^{tr}q_i + q_j - q_i + \lambda_{ij}]\} + c_i \quad \text{for all } i.$$

Let $F(\cdot)$ denote the mapping of E^n into itself defined by the condition that for all i and all $q \in E^n$:

$$[F(q)]_i = U^{tr}q - \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\tau_{ij}U^{tr}q + q_j - q_i + \lambda_{ij}]\}.$$

Our objective is to show that $F(\cdot)$ satisfies conditions (*i*) and (*ii*) of Palais' theorem.

We have, with F_i denoting $[F(\cdot)]_i$,

$$\frac{\partial F_i}{\partial q_i} = 1 + \varphi_i' \sum_{j \neq i} (1 + \tau_{ij})\varphi_{ij}' \quad \text{for all } i$$

* See Ref. 8 and the appendix of Ref. 9.

and

$$\frac{\partial F_i}{\partial q_k} = 1 + \varphi_i' \sum_{j \neq i} \tau_{ij}\varphi_{ij}' - \varphi_i'\varphi_{ik}' \quad \text{for all} \quad k \neq i$$

in which

$$\varphi_i' = \varphi_i'\{\sum_{j \neq i} \varphi_{ij}[-\tau_{ij}U^{tr}q + q_j - q_i + \lambda_{ij}]\}$$

and

$$\varphi_{ij}' = \varphi_{ij}'[-\tau_{ij}U^{tr}q + q_j - q_i + \lambda_{ij}].$$

Let $\beta_{ij} = \varphi_i'\varphi_{ij}'$ for all $i \neq j$, let V be the n-vector defined by

$$V^{tr} = (1 + \sum_{j \neq 1} \beta_{1j}\tau_{1j}, 1 + \sum_{j \neq 2} \beta_{2j}\tau_{2j}, \cdots, 1 + \sum_{j \neq n} \beta_{nj}\tau_{nj}),$$

and let B denote the $n \times n$ matrix defined by

$$(B)_{ii} = \sum_{j \neq i} \beta_{ij} \quad \text{for all } i, \qquad (B)_{ij} = -\beta_{ij} \quad \text{for all} \quad i \neq j.$$

Then $J_q = B + VU^{tr}$.

Suppose now that $\det J_q = 0$ for some n-vector q. For that q, there would exist an n-vector $x \neq \theta$ such that $J_q^{tr}x = (B^{tr} + UV^{tr})x = \theta$. Since the column space of B^{tr} is orthogonal to U, we must have $B^{tr}x = \theta$ and $V^{tr}x = 0$. But B is of rank $(n - 1)$ and the cofactors of B are nonnegative.*

Thus $B^{tr}x = \theta$ implies that $x = \xi y$, in which y is any column of the matrix of cofactors of B and ξ is some real nonzero constant.† But we must have $V^{tr}x = \xi V^{tr}y = 0$, which is a contradiction, since at least one element of y and all of the elements of V are positive. Therefore $F(\cdot)$ meets condition (*i*) of Palais' theorem.

We now show that $F(\cdot)$ satisfies condition (*ii*) of the theorem of Palais. It is a simple matter to verify that for all i

$$F_i = U^{tr}q - r_i \sum_{j \neq i} r_{ij}[-\tau_{ij}U^{tr}q + q_j - q_i] - \varphi_i[\sum_{j \neq i} \varphi_{ij}(\lambda_{ij})]$$

in which

$$r_i = \frac{\varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\tau_{ij}U^{tr}q + q_j - q_i + \lambda_{ij}]\} - \varphi_i\{\sum_{j \neq i} \varphi_{ij}[\lambda_{ij}]\}}{\sum_{j \neq i} \varphi_{ij}[-\tau_{ij}U^{tr}q + q_j - q_i + \lambda_{ij}] - \sum_{j \neq i} \varphi_{ij}[\lambda_{ij}]}$$

* See Ref. 10 for a proof that B is of rank $(n - 1)$ and that the cofactors of all of the $(B)_{ii}$ elements of B are positive. Since $BU = \theta$, each of the columns of the transposed matrix of cofactors of B is proportional to the vector U (see the footnote that follows). Therefore *all* of the cofactors of B are positive.

† This follows from the well-known proposition that $M^{tr}C = 1_n \det M$, in which M is any square matrix and C is the matrix of cofactors of M.

and

$$r_{ij} = \frac{\varphi_{ij}[-\tau_{ij}U^{tr}q + q_j - q_i + \lambda_{ij}] - \varphi_{ij}[\lambda_{ij}]}{-\tau_{ij}U^{tr}q + q_j - q_i},$$

with the understanding that r_i is unity when the corresponding numerator is zero, r_{ij} is zero for all q and all $i \neq j$ for which φ_{ij} is identically zero, and r_{ij} is unity for all $i \neq j$ for which φ_{ij} is not identically zero and for all q for which the corresponding numerator is zero. Therefore, for all $q \varepsilon E^n$, $F(q) = Mq + s$, in which the $n \times n$ matrix M is obtained from J_q by replacing φ_i' by r_i and φ_{ij}' by r_{ij} for all i and all $i \neq j$, respectively, and the ith component of s is $-\varphi_i[\sum_{j \neq i} \varphi_{ij}(\lambda_{ij})]$ for all i. In particular, det $M \neq 0$ for all q. Therefore det $(M^{tr}M) > 0$ for all q. Since all of the r_i as well as all of the nonidentically zero r_{ij} are bounded above and below by positive constants uniformly for $q \varepsilon E^n$, there exists a positive constant ϵ such that det $(M^{tr}M) \geqq \epsilon$ for all $q \varepsilon E^n$.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the eigenvalues of $M^{tr}M$. Then $\lambda_1\lambda_2 \cdots \lambda_n \geqq \epsilon$ for all $q \varepsilon E^n$. Assume that $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$. Since all of the r_i and all of the r_{ij} are bounded from above uniformly for $q \varepsilon E^n$, there exists a positive constant λ such that $\lambda_n \leqq \lambda$ for all $q \varepsilon E^n$. Thus, for all $q \varepsilon E^n$ we have $\lambda_1 \geqq \epsilon\lambda^{-(n-1)}$. Therefore,

$$\begin{aligned} \| F(q) \| = \| Mq + s \| &\geqq \| Mq \| - \| s \| \\ &\geqq \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{2}(n-1)} \| q \| - \| s \| \end{aligned}$$

for all $q \varepsilon E^n$, from which it is clear that $\| F(q) \| \to \infty$ as $\| q \| \to \infty$ This completes the proof of part (*a*) of our theorem.

Next we show that there is *at most* one ρ with the property that there exists a $q \varepsilon E^n$ such that for all i

$$\rho = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho\tau_{ij} + q_j - q_i + \lambda_{ij}]\} + c_i .$$

Let $\rho^{(a)}$ and $\rho^{(b)}$ be two constants, and $q^{(a)}$ and $q^{(b)}$ two n-vectors, such that for all i

$$\rho^{(a)} = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho^{(a)}\tau_{ij} + q_j^{(a)} - q_i^{(a)} + \lambda_{ij}]\} + c_i$$

$$\rho^{(b)} = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho^{(b)}\tau_{ij} + q_j^{(b)} - q_i^{(b)} + \lambda_{ij}]\} + c_i .$$

Then with $q^{(c)} = q^{(b)} + \alpha U$, in which the constant α is chosen so that $\rho^{(a)} - \rho^{(b)} = U^{tr}(q^{(a)} - q^{(c)})$, we have for all i

$$\rho^{(c)} = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho^{(c)}\tau_{ij} + q_j^{(c)} - q_i^{(c)} + \lambda_{ij}]\} + c_i$$

where $\rho^{(c)} = \rho^{(b)}$. Thus

$$\rho^{(a)} - \rho^{(c)} = \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho^{(a)}\tau_{ij} + q_j^{(a)} - q_i^{(a)} + \lambda_{ij}]\} - \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho^{(c)}\tau_{ij} + q_j^{(c)} - q_i^{(c)} + \lambda_{ij}]\}$$

and

$$\rho^{(a)} - \rho^{(c)} = U^{tr}(q^{(a)} - q^{(c)}).$$

Therefore we can define nonnegative ratios p_i and p_{ij} similar to the r_i and r_{ij} above, such that

$$U^{tr}(q^{(a)} - q^{(c)}) = p_i \sum_{j \neq i} p_{ij}[-U^{tr}(q^{(a)} - q^{(c)})\tau_{ij} + (q_j^{(a)} - q_j^{(c)}) - (q_i^{(a)} - q_i^{(c)})] \quad \text{for all } i,$$

and such that these equations are equivalent to $M'(q^{(a)} - q^{(c)}) = \theta$ in which the $n \times n$ matrix M' is obtained from J_q by replacing φ_i' by p_i and φ_{ij}' by p_{ij} for all i and all $i \neq j$, respectively, so that det $M' \neq 0$. But this implies that $q^{(a)} = q^{(c)}$ and hence that $\rho^{(a)} = \rho^{(b)}$.

We shall how prove that q is specified to within an additive vector of the form αU in which α is a real constant.

Suppose that, with $q^{(a)}$ and $q^{(b)}$ two n-vectors,

$$\rho - \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho\tau_{ij} + q_j^{(a)} - q_i^{(a)} + \lambda_{ij}]\} = \rho - \varphi_i\{\sum_{j \neq i} \varphi_{ij}[-\rho\tau_{ij} + q_j^{(b)} - q_i^{(b)} + \lambda_{ij}]\}$$

for all i. Then, with the p_i and p_{ij} as introduced above,

$$p_i \sum_{j \neq i} p_{ij}[(q_j^{(a)} - q_j^{(b)}) - (q_i^{(a)} - q_i^{(b)})] = 0$$

for all i. Thus, since $p_i \neq 0$ for all i, and the $n \times n$ matrix P defined by

$$(P)_{ii} = \sum_{j \neq i} p_{ij}\,, \quad \text{for all } i$$

$$(P)_{ij} = -p_{ij}\,, \quad \text{for all} \quad i \neq j$$

is of rank* $(n - 1)$, and $PU = \theta$, we have $q^{(a)} - q^{(b)} = \alpha U$ for some real constant α. □

* See Ref. 10.

V. ACKNOWLEDGMENT

The writer is indebted to A. N. Willson, Jr., for carefully reading the draft.

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