Jump Criteria of Nonlinear Control Systems and the Validity of Statistical Linearization Approximation

By SANG H. KYONG

(Manuscript received March 26, 1969)

We study the conditions for the unique response in a class of nonlinear control systems subject to random inputs using statistical linearization approximation. As in the case of sinusoidal inputs, we show that jump phenomena may occur if the inverse vector locus of the linear part passes through certain regions on the complex plane, where the regions are defined by the characteristics of nonlinear part. Such jump phenomena regions for several typical nonlinearities are given; we also show that, among a restricted class of nonlinearities, the saturation and dead zone produce the largest jump phenomena regions.

A new result concerning the validity of statistical linearization approximation of nonlinear control systems is also presented. We show that the condition for the uniqueness of response to a given input in a nonlinear feedback system obtained through statistical linearization approximation is compatible with a related rigorous result, thus providing additional confidence in the applicability of statistical linearization.

I. INTRODUCTION

It is well known that jump resonance can occur in nonlinear control systems with attendant worsening of the control performance. In the case of periodic input signals, the rigorous conditions for the unique response,* or equivalently, for the absence of jump resonance, are available.[1] In addition, various authors have studied the conditions for the absence of jump resonance using the describing function method (see Refs. 2 and 3); the describing function method criteria

* Although the present terminology is widely used, a more precise term will be "unique solution to the equations arising from the steady state situation for a given input realization."

for jump resonance have been found for many common nonlinearities. For systems with random inputs, the exact condition for the unique response is not known, although a rigorous condition for the convergence of a successive approximation is available.[4]

A useful approximate technique for studying the performance of nonlinear feedback systems subject to random inputs is Booton's method of statistical linearization.[5] Although the method of statistical linearization has been widely used, the conditions for its validity are not fully known.

The first part of this paper concerns the determination of the criteria for unique response, in a class of nonlinear control systems subject to random inputs, using statistical linearization approximation. We present the statistical linearization criteria for unique response for several common nonlinearities. We also show that an idealized saturation and an idealized deadzone yield the limit jump phenomena regions among a restricted class of nonlinearities.

In view of the uncertainty concerning the conditions for the validity of statistical linearization approximation, it is of interest to compare the results of statistical linearization analysis with those of a rigorous analysis. The second part of this paper presents a result that provides new evidence on the validity of statistical linearization approximation. More specifically, the conditions for the unique response obtained in the first part on the basis of statistical linearization are compared with a related result of Holtzman,[4] which is a rigorous sufficient condition for the convergence of a successive approximation starting with the statistical linearization approximation. We show that the two conditions are "compatible"; that is, the satisfaction of the rigorous condition for the convergence of the successive approximation guarantees the satisfaction of the conditions for unique response based on statistical linearization. However, since Holtzman's rigorous condition guarantees only the convergence of a specific successive approximation but not necessarily a unique reponse, while the conditions derived from statistical linearization are for the globally unique response, the precise interpretation of the present comparison is largely open to debate. The present comparison lacks the finality of a similar comparison concerning the method of describing function in the case of periodic inputs.[6] Still, the comparison appears to provide some additional confidence in the validity of statistical linearization approximation.

Section II defines the class of nonlinear control systems to be studied and derives the conditions for the unique response based on the statistical linearization analysis. Section III presents such conditions for the unique response for several typical nonlinearities. Section IV shows that if the conditions for the unique response are met by saturation and deadzone nonlinearities, then a large class of other nonlinearities will also meet the conditions. Section V shows that the statistical linearization conditions for the unique response are compatible with a related rigorous condition.

II. CONDITIONS FOR THE UNIQUENESS OF RESPONSE

Consider the nonlinear feedback system of Fig. 1. The nonlinear characteristic $f(\cdot)$ is assumed to be single-valued, odd, and piecewise continuously differentiable, and to satisfy

$$0 \leqq a \leqq f'(m) \leqq b \tag{1}$$

for all real m, where a and b are real. Concerning the linear element, it is assumed that:

(*i*) $G(j\omega)$ is the Fourier transform of a real function g satisfying

$$\int_{-\infty}^{\infty} | g(t) | \, dt < \infty, \tag{2}$$

(*ii*)
$$1 + \tfrac{1}{2}(a + b)G(j\omega) \neq 0 \tag{3}$$

for all $\omega \in (-\infty, \infty)$, and

(*iii*)
$$1 + K_{eq}G(j\omega) \neq 0 \tag{4}$$

for all $\omega \in (-\infty, \infty)$, where K_{eq} is the equivalent gain of the nonlinear characteristic $f(\cdot)$ obtained by statistical linearization; that is,

$$K_{eq} = \frac{E[mf(m)]}{E[m^2]}. \tag{5}$$

In equation (5), $E[\cdot]$ denotes expectation taken over the probability distribution of m. The input r to the feedback system is assumed to be a zero-mean, stationary gaussian random function with the power spectral density given by $\sigma_r^2\phi_r(\omega)$.

We further assume that m can be represented by a zero-mean gaussian probability distribution. That m can be zero-mean follows

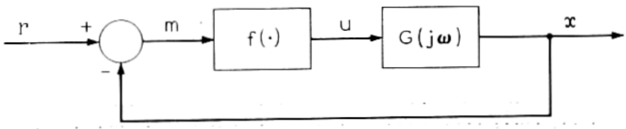


Fig. 1 — Basic feedback system.

from $f(\cdot)$ being odd. This assumption is consistent with the usual one made in connection with a statistical linearization analysis of nonlinear feedback systems.[5]

If the nonlinearity $f(\cdot)$ is replaced by the equivalent gain K_{eq}, then the variance of m can easily be determined from

$$\sigma_m^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma_r^2 \phi_r(\omega)}{\mid 1 + K_{eq} G(j\omega) \mid^2} \, d\omega. \tag{6}$$

From equation (6), it is seen that

$$\frac{d}{d\sigma_m} (\sigma_m^2 \mid 1 + K_{eq} G(j\omega) \mid^2) > 0 \tag{7}$$

for all $\omega \in (-\infty, \infty)$ is sufficient to guarantee*

$$\frac{d\sigma_r}{d\sigma_m} > 0 \qquad \text{for all } \sigma_r \; . \tag{8}$$

Condition (8) implies that σ_m is a monotonically increasing function of σ_r, which in turn implies that there is a unique value of σ_m given by equation (6) for a given σ_r. This is the context in which the term "uniqueness" is used in this paper. Suppose that $(d\sigma_r/d\sigma_m) < 0$ in a certain interval of the values of σ_r. Then, the curve of σ_m versus σ_r will have the shape given by either Fig. 2a or b. Figure 2a indicates nonunique σ_m, and hence nonunique responses, or the presence of jump phenomena in the nonlinear feedback system of Fig. 1. Thus, the condition given by equation (7) is sufficient for the absence of jump phenomena in the system of Fig. 1.

Rewriting equation (7) with $H(j\omega) = G^{-1}(j\omega)$, Re $H(j\omega) = \xi_\omega$, and Im $H(j\omega) = \eta_\omega$, one obtains

$$\left(\xi_\omega + K_{eq} + \frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m}\right)^2 + \eta_\omega^2 > \left(\frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m}\right)^2. \tag{9}$$

Thus, inequality (7) is equivalent to the condition that the locus of $H(j\omega) = G^{-1}(j\omega)$, when plotted on the complex plane for $\omega \in (-\infty, \infty)$, remains outside of the circle centered at

$$\left(-\left[K_{eq} + \frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m}\right], 0\right) \tag{10}$$

and with radius

* Inequality (7) may be considered to be necessary as well as sufficient for condition (8), in the sense that if the inequality is reversed in inequality (7), then there is at least one $\phi_r(\omega)$; for example, $\phi_r(\omega) = \delta(\omega - \omega')$, such that condition (8) is violated

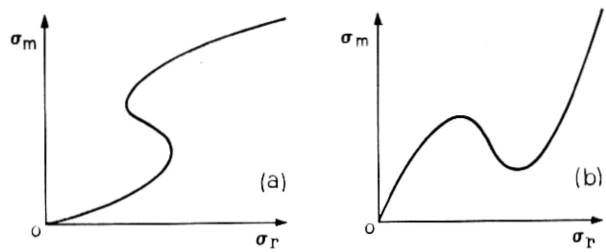


Fig. 2 — Curves of σ_m versus σ_r.

$$\rho = \left| \frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m} \right|. \tag{11}$$

The union of all such circles for all nonnegative real values of σ_m gives a region on the $H(j\omega)$-plane such that the sufficient condition (on the basis of statistical linearization) for unique response or for the absence of jump phenomena is that the locus of $H(j\omega) = G^{-1}(j\omega)$ remains outside of that region as ω is varied on $(-\infty, \infty)$.

As in Ref. 3, the circles defined by equations (10) and (11) will be called the iso-σ_m circles, and the union of all iso-σ_m circles for positive σ_m will be referred to as the jump phenomena region. Both the iso-σ_m circles and the jump phenomena region are determined by the characteristics of nonlinearity only.

III. JUMP PHENOMENA REGIONS FOR TYPICAL NONLINEARITIES

Centers and radii of iso-σ_m circles for several typical nonlinearities are tabulated in Table I along with their normalized characteristics. Figure 3 shows the jump phenomena regions of these nonlinearities.

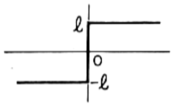
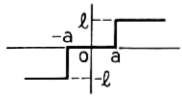
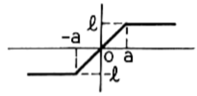
IV. LIMIT JUMP PHENOMENA REGION

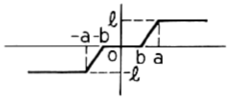
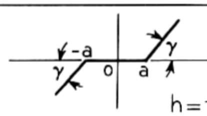
Fukuma and Matsubara have shown that, using the describing function method for the system of Fig. 1 subject to sinusoidal inputs, the jump resonance regions for idealized saturation and idealized deadzone include the jump resonance regions for all other nonlinearities satisfying

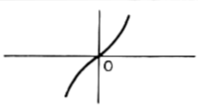
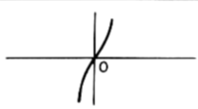
$$0 \leqq f'(m) \leqq 1, \tag{12}$$

in addition to being single-valued and odd.[3] The idealized saturation is given by

TABLE I—CHARACTERISTICS OF NONLINEARITES

	(a) RELAY	(b) RELAY WITH DEADZONE	(c) SATURATION
NONLINEAR CHARACTERISTIC			
K	$\left(\frac{2}{\pi}\right)^{1/2}\frac{\ell}{\sigma_m}$	$\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\sigma_m}\,\mathrm{EXP}\left(\frac{-a^2}{2\sigma_m^2}\right)$	$\left(\frac{2}{\pi}\right)^{1/2}\frac{\ell}{a}\int_0^{a/\sigma_m}\mathrm{EXP}\left(\frac{-t^2}{2}\right)dt$
ρ	$\frac{1}{(2\pi)^{1/2}}\frac{\ell}{\sigma_m}$	$-\frac{1}{(2\pi)^{1/2}}\frac{a^2\ell}{\sigma_m^3}\,\mathrm{EXP}\left(\frac{-a^2}{2\sigma_m^2}\right)$	$\frac{1}{(2\pi)^{1/2}}\frac{\ell}{\sigma_m}\,\mathrm{EXP}\left(\frac{-a^2}{2\sigma_m^2}\right)$
NORMALIZATION†	$\ell = 1$	$\ell = 1,\ a = 1$	$\ell = 1,\ a = 1$

	(d) SATURATION WITH DEADZONE	(e) DEADZONE
NONLINEAR CHARACTERISTIC		 h = TAN γ
K	$\left(\frac{2}{\pi}\right)^{1/2}\frac{\ell}{(a-b)}\left[\int_0^{a/\sigma_m}\mathrm{EXP}\left(\frac{-t^2}{2}\right)dt-\int_0^{b/\sigma_m}\mathrm{EXP}\left(\frac{-t^2}{2}\right)dt\right]$	$h\left[1-\left(\frac{2}{\pi}\right)^{1/2}\int_0^{a/\sigma_m}\mathrm{EXP}\left(\frac{-t^2}{2}\right)dt\right]$
ρ	$\frac{1}{(2\pi)^{1/2}}\frac{\ell}{(a-b)\sigma_m}\left[a\,\mathrm{EXP}\left(\frac{-a^2}{2\sigma_m^2}\right)-b\,\mathrm{EXP}\left(\frac{-b^2}{2\sigma_m^2}\right)\right]$	$-\frac{1}{(2\pi)^{1/2}}\frac{ah}{\sigma_m}\,\mathrm{EXP}\left(\frac{-a^2}{2\sigma_m^2}\right)$
NORMALIZATION†	$\ell = 1,\ a = 2,\ b = 1$	$a = 1,\ \gamma = \frac{\pi}{4}$

	(f) $f(m) = Nm^2\,\mathrm{sgn}\,(m)$	(g) $f(m) = Nm^3$
NONLINEAR CHARACTERISTIC		
K	$2\left(\frac{2}{\pi}\right)^{1/2}N\sigma_m$	$3N\sigma_m^2$
ρ	$-\left(\frac{2}{\pi}\right)^{1/2}N\sigma_m$	$-3N\sigma_m^2$
NORMALIZATION†	$N = 1$	$N = 1$

*IN ALL CASES CENTER IS GIVEN BY $-\lambda + j0$, RADIUS IS GIVEN BY $|\rho|$, WHERE $\lambda = K - \rho$.
† NORMALIZATION OF THE PARAMETER VALUES OF NONLINEAR CHARACTERISTIC USED IN FIGURE 3.

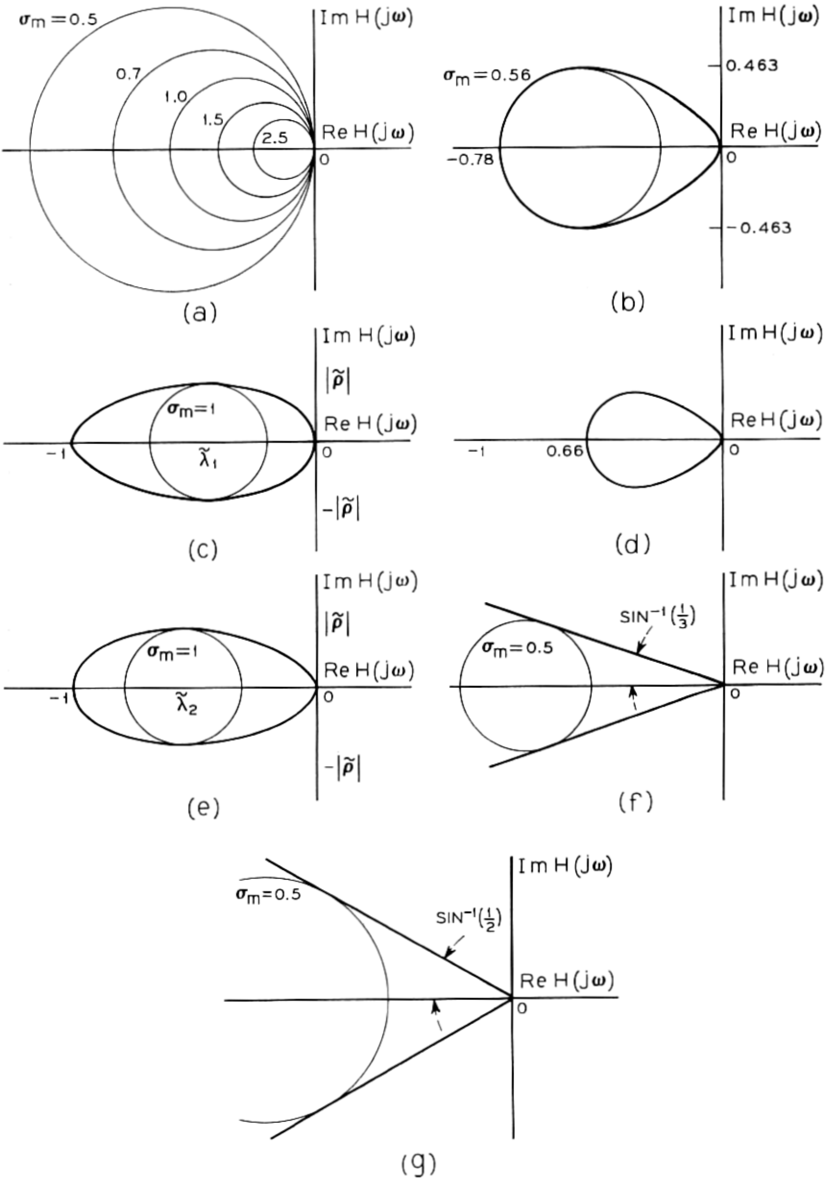


Fig. 3 — Jump phenomena regions: (a) relay, (b) relay with deadzone, (c) saturation, (d) saturation with deadzone, (e) deadzone, (f) $f(m) = m^2 \operatorname{sgn}(m)$, and (g) $f(m) = m^3$.

$$f(m) = \begin{cases} -\alpha; & m < -\alpha, \\ m; & -\alpha \leqq m \leqq +\alpha, \\ +\alpha; & +\alpha < m, \end{cases} \tag{13}$$

and the idealized deadzone is given by

$$f(m) = \begin{cases} m + \beta; & m < -\beta, \\ 0 \quad ; & -\beta \leqq m \leqq +\beta, \\ m - \beta; & +\beta < m, \end{cases} \tag{14}$$

where α and β are positive real constants. Such limit jump resonance regions are determined by finding the nonlinearity satisfying inequality (12) which maximizes the radius of the circle given the coordinates of the center of the circle.

In this section we show that the idealized saturation and idealized deadzone give a limit jump phenomena region also in the case of random inputs, if $f(\cdot)$ is restricted to those satisfying inequality (12).

Notice that

$$\frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m} = \frac{d}{dv} E[mf(m)] - K_{eq} ,$$

where $v = \sigma_m^2$. From a theorem given in Ref. 7, $(d/dv)E[mf(m)] = E[f'(m)] + \frac{1}{2}E[mf''(m)]$, where prime denotes differentiation with respect to the argument. Integrating the first term on the right by parts,

$$E[f'(m)] = K_{eq} . \tag{15}$$

These relations reduce to

$$\frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m} = \tfrac{1}{2}E[mf''(m)].$$

If $f(\cdot)$ is such that $f''(m)$ is piecewise continuous, then the right side of the above equation may be integrated by parts to give

$$\frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m} = -\frac{1}{2} \int_{-\infty}^{\infty} f'(m)[p(m) + mp'(m)]\, dm. \tag{16}$$

For the gaussian probability density function for $p(m)$,

$$mp'(m) = -\frac{m^2}{\sigma_m^2} p(m).$$

Therefore, equation (16) may be rewritten as

$$\frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m} = \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{m^2}{\sigma_m^2} - 1 \right) f'(m) p(m) \, dm. \tag{17}$$

If $f'(m)$ is only piecewise continuous (as in the case of saturation and deadzone given by equations (13) and (14), respectively) then $f''(m)$ is not piecewise continuous, and the integration by parts used above to obtain equation (16) may not be valid in the ordinary sense. However, if the meaning of the integration

$$E[mf''(m)] = \int_{-\infty}^{\infty} mp(m) f''(m) \, dm$$

is extended, and is considered as an operation of a distribution $f''(m)$ on an infinitely smooth function $mp(m)$, then a generalized integration by parts can be used.[8] The use of integration by parts, in the generalized sense, does not change the result in the present case, and equation (17) remains valid.

Now, combining equation (17) with equation (15),

$$-\left(K_{eq} + \frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m} \right) = -\frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{m^2}{\sigma_m^2} + 1 \right) f'(m) p(m) \, dm. \tag{18}$$

Suppose that the coordinate of the center of the circle is fixed, that is,

$$-\left(K_{eq} + \frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m} \right) = -\lambda, \tag{19}$$

where λ is a constant. Clearly from equation (18), $0 \leqq \lambda \leqq 1$ for $f'(m)$ satisfying inequality (12). From equations (18) and (19),

$$\int_{-\infty}^{\infty} (m^2 + \sigma_m^2) f'(m) p(m) \, dm = 2\lambda\sigma_m^2 . \tag{20}$$

The nonlinearity that gives the limit jump phenomena region is found by determining $f'(m)$ such that it maximizes

$$\left| \frac{\sigma_m}{2} \frac{dK_{eq}}{d\sigma_m} \right| = \frac{1}{2\sigma_m^2} \left| \int_{-\infty}^{\infty} (m^2 - \sigma_m^2) f'(m) p(m) \, dm \right| \tag{21}$$

subject to the constraints given by equations (12) and (20).

By using Pontryagin's maximum principle the appendix shows that the solution of above optimization problem is given by an idealized saturation and an idealized deadzone, or the nonlinearities of the form of equations (13) and (14), respectively. In other words, the idealized saturation and idealized deadzone yield the limit jump phenomena regions among all nonlinearities which are single-valued and odd, and satisfy $0 \leqq f'(m) \leqq 1$, in the case of gaussian random input, as well as in the case of sinusoidal input.

Suppose that the unit of the signals r, m, and so on, is normalized such that σ_m is taken as the unit. Then the appendix also shows that the jump phenomena circles giving the maximum radius are centered at $(-\tilde{\lambda}_1, 0)$ for the idealized saturation with $\alpha = 1$ in equation (13) and at $(-\tilde{\lambda}_2, 0)$ for the idealized deadzone with $\beta = 1$ in equation (14), where

$$\tilde{\lambda}_1 = \frac{1}{2(2\pi)^{\frac{1}{2}}} \left[\int_0^1 e^{-\gamma/2} \gamma^{1/2} \, d\gamma + \int_0^1 e^{-\gamma/2} \gamma^{-1/2} \, d\gamma \right], \tag{22}$$

$$\tilde{\lambda}_2 = \frac{1}{2(2\pi)^{\frac{1}{2}}} \left[\int_1^\infty e^{-\gamma/2} \gamma^{1/2} \, d\gamma + \int_1^\infty e^{-\gamma/2} \gamma^{-1/2} \, d\gamma \right]. \tag{23}$$

In both cases, the magnitude of the maximum radius is given by

$$\tilde{\rho} = \frac{1}{2(2\pi)^{\frac{1}{2}}} \left[\int_0^1 e^{-\gamma/2} \gamma^{-1/2} \, d\gamma - \int_0^1 e^{-\gamma/2} \gamma^{1/2} \, d\gamma \right]. \tag{24}$$

The values of the integrals in equations (22) through (24) are tabulated in Ref. 9; it is found that

$$\tilde{\lambda}_1 = 0.44072, \qquad \tilde{\lambda}_2 = 0.55928, \qquad \tilde{\rho} = 0.24197.$$

V. COMPATIBILITY OF CONDITIONS

In this section, we compare inequality (7), which is an approximate condition for the uniqueness of response or the absence of jump phenomenon based on statistical linearization, with a related rigorous condition to obtain further evidence concerning the validity of statistical linearization approximation. Section II showed that inequality (7) implies the condition that the locus of $H(j\omega) = G^{-1}(j\omega)$ must remain outside of the circle defined by equation (10) and (11) on the complex plane as ω is varied on $(-\infty, \infty)$.

On the other hand, the rigorous sufficient condition for the convergence of a successive approximation is given in Ref. 4 as

$$\sup_{\omega \in (-\infty, \infty)} \left| \frac{G(j\omega)}{1 + \frac{1}{2}(a + b)G(j\omega)} \right| \tfrac{1}{2}(b - a) < 1. \tag{25}$$

Inequality (25) implies that the locus of $H(j\omega) = G^{-1}(j\omega)$ on the $H(j\omega)$-plane, as ω is varied on $(-\infty, \infty)$, must not enter the circle centered at

$$(-\tfrac{1}{2}[a + b], 0) \tag{26}$$

with radius

$$\rho = \tfrac{1}{2}(b - a). \tag{27}$$

The circle defined by equations (10) and (11) intersects the real axis of the complex plane at $-K_{eq}$ and $-[K_{eq} + \sigma_m(dK_{eq}/d\sigma_m)]$ with its center lying on the real axis. Similarly, the circle defined by equations (26) and (27) intersects the real axis at $-a$ and $-b$ with its center also lying on the real axis. Thus it suffices to show

$$a \leqq K_{eq} + \sigma_m \frac{dK_{eq}}{d\sigma_m} \leqq b, \tag{28}$$

and

$$a \leqq K_{eq} \leqq b, \tag{29}$$

for all positive σ_m.

But inequality (29) follows immediately from equations (15) and (1). Combining equations (17) and (18),

$$K_{eq} + \sigma_m \frac{dK_{eq}}{d\sigma_m} = \frac{1}{\sigma_m^2} \int_{-\infty}^{\infty} m^2 f'(m) p(m)\, dm. \tag{30}$$

From equations (1) and (30), $\alpha \leqq K_{eq} + \sigma_m(dK_{eq}/d\sigma_m) \leqq \beta$, which is the inequality (28).

Thus, inequalities (28) and (29) are satisfied for all $\sigma_m > 0$, which implies the two conditions are compatible; that is the satisfaction of condition (25) implies the satisfaction of condition (7) for all $\sigma_m > 0$, and conversely, the violation of condition (7) for some $\sigma_m > 0$ implies the violation of condition (25).

VI. CONCLUDING REMARKS

The conditions for the unique response in a randomly excited nonlinear control system were studied using a statistical linearization approximation. The jump phenomena regions of several common nonlinearities were given. It was shown that, among nonlinearities satisfying $0 \leqq f'(m) \leqq 1$, the idealized saturation and idealized deadzone yield the limit jump phenomena regions.

It was also shown that, concerning the uniqueness of the response in nonlinear feedback systems subject to random input, the criterion obtained by the statistical linearization approximation is not contradicted by a related, although not equivalent, rigorous result. However, as mentioned in the introduction, the interpretation of this result is largely open to debate since (*i*) the comparison made is between an approximate and a rigorous sufficient condition, and (*ii*) the two sufficient conditions are not concerned with exactly the same requirement. More specifically, the approximate criterion obtained in Section II is for a globally unique response, while the rigorous result of Holtzman, with which the comparison is made, is for the convergence of a specific successive approximation.

It is shown in Ref. 10 that, in a system closely related to that with which the present paper is concerned, there is a unique response up to an equivalence to an input r satisfying

$$\limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\, r(t)\,|^2\, dt < \infty ,$$

if the condition identical to condition (25) is satisfied. This result strongly suggests that condition (25) may be sufficient not only for the convergence of a specific successive approximation as shown in Ref. 4, but also for a globally unique response (up to an equivalence). If this is true, then the meaning of the result of comparison made in the present paper is correspondingly strengthened.

It is interesting to compare the limit jump phenomena regions of the present approximate analysis (Fig. 3c and e) with the circle of rigorous analysis, and to notice that the limit jump phenomena regions occupy substantial portions of the interior of the circle of exact analysis. Also notice that in view of inequalities (25) and (29), the statistical linearization analysis of the system of Fig. 1 always has a solution under the conditions dicussed in Section II.

VII. ACKNOWLEDGMENTS

The author wishes to thank Mr. J. M. Holtzman for many helpful discussions.

APPENDIX

Optimization Problem

The following optimization problem is stated in Section IV: Maximize $|\,\rho\,|$, where

$$\rho = \frac{1}{2\sigma_m^2}\int_{-\infty}^{\infty} (m^2 - \sigma_m^2) f'(m) p(m)\, dm, \tag{31}$$

by choosing $f'(m)$, $-\infty < m < \infty$, satisfying the condition

$$0 \leqq f'(m) \leqq 1, \tag{32}$$

subject to the constraint

$$\int_{-\infty}^{\infty} (m^2 + \sigma^2) f'(m) p(m) \, dm = 2\lambda\sigma_m^2 , \tag{33}$$

where λ is a given constant such that $0 \leqq \lambda \leqq 1$. This problem may be solved by making use of Pontryagin's maximum principle.

Since both $(m^2 - \sigma_m^2)p(m)$ and $(m^2 + \sigma^2)p(m)$ are even functions of m, it suffices to find $f'(m)$ for $m \geqq 0$, and to let $f'(-m) = f'(m)$. Thus, the problem may be reformulated in the following way. Let

$$\dot{x}_1(m) = (m^2 - \sigma_m^2)p(m)f'(m), \tag{34}$$

$$\dot{x}_2(m) = (m^2 + \sigma_m^2)p(m)f'(m), \tag{35}$$

where $x_1(0) = x_2(0) = 0$. We want to minimize or maximize $x_1(\infty)$ subject to $x_2(\infty) = \lambda\sigma_m^2$. Pontryagin's maximum principle may be used to the above reformulation. The Hamiltonian function is

$$H = g_1(m)(m^2 - \sigma_m^2)p(m)f'(m) + g_2(m)(m^2 + \sigma_m^2)p(m)f'(m), \tag{36}$$

where $g_1(m)$ and $g_2(m)$ are the adjoint variables. Clearly, $\dot{g}_1(m) = \dot{g}_2(m) = 0$.

Suppose first that $x_1(\infty)$ is to be minimized. Then g_1 may be set as $g_1 = -1$; and maximizing the resulting H with respect to $f'(m)$ satisfying inequality (32), one obtains,

$$f'(m) = \tfrac{1}{2} + \tfrac{1}{2} \operatorname{sgn} [-(m^2 - \sigma_m^2) + g_2(m^2 + \sigma_m^2)]. \tag{37}$$

It is easy to determine that

$$-1 \leqq g_2 \leqq +1 \tag{38}$$

to satisfy the constraint $x_2(\infty) = \lambda\sigma_m^2$. For the values of g_2 satisfying inequality (38), equation (37) and $f'(m) = f'(-m)$ give

$$f'(m) = \begin{cases} 1; & | m | \leq \left(\dfrac{1 + g_2}{1 - g_2}\right)^{\frac{1}{2}} \sigma_m , \\[2ex] 0; & | m | > \left(\dfrac{1 + g_2}{1 - g_2}\right)^{\frac{1}{2}} \sigma_m , \end{cases} \tag{39}$$

as the one that minimizes $x_1(\infty)$. The actual value of g_2 is determined from equation (33), or

$$\int_0^{\alpha} (m^2 + \sigma_m^2)p(m) \, dm = \lambda\sigma_m^2 , \tag{40}$$

where $\alpha = (1 + g_2/1 - g_2)^{\frac{1}{2}}\sigma_m$.

Proceeding similarly, the function $f'(m)$ which maximizes $x_1(\infty)$ is

$$f'(m) = \begin{cases} 0; & |m| \leqq \left(\dfrac{1 - g_2}{1 + g_2}\right)^{\frac{1}{2}} \sigma_m , \\ 1; & |m| > \left(\dfrac{1 - g_2}{1 + g_2}\right)^{\frac{1}{2}} \sigma_m , \end{cases} \tag{41}$$

where $-1 \leqq g_2 \leqq 1$. The actual value of g_2 is found from equation (33), or

$$\int_\beta^\infty (m^2 + \sigma_m^2) p(m)\, dm = \lambda \sigma_m^2 , \tag{42}$$

where $\beta = (1 - g_2/1 + g_2)^{\frac{1}{2}} \sigma_m$.

The functions $f'(m)$ of equations (39) and (41) correspond to idealized saturation and idealized deadzone, respectively. Thus, among nonlinearities giving $\rho < 0$, $f'(m)$ of equation (39) yields the limit jump phenomena region, and among the ones giving $\rho > 0$, $f'(m)$ of equation (41) yields the same.

Having determined the functions $f'(m)$ that maximize $|\rho|$, it is also of interest to determine the actual values of maximum $|\rho|$ and the location of the center of the corresponding circles on complex plane. In case of idealized saturation, the maximum of $|\rho|$ corresponds to the minimum of ρ, and

$$\tilde{\rho} = \frac{1}{\sigma_m^2} \int_0^\alpha (m^2 - \sigma_m^2) p(m)\, dm, \tag{43}$$

where α is given following equation (40). We want to find the value of λ, $0 \leqq \lambda \leqq 1$, such that $\tilde{\rho}$ above is further minimized, and to find that minimum value of $\tilde{\rho}$. Differentiating equation (43) with respect to λ,

$$\frac{d\tilde{\rho}}{d\lambda} = \frac{1}{\sigma_m^2} (\alpha^2 - \sigma_m^2) p(\alpha) \frac{d\alpha}{d\lambda}. \tag{44}$$

But, from equation (40),

$$p(\alpha) \frac{d\alpha}{d\lambda} = \frac{\sigma_m^2}{\sigma_m^2 + \alpha^2}. \tag{45}$$

Thus, equation (44) becomes

$$\frac{d\tilde{\rho}}{d\lambda} = \frac{\alpha^2 - \sigma_m^2}{\alpha^2 + \sigma_m^2}. \tag{46}$$

For minimum $\tilde{\rho}$, $\alpha = \sigma_m$ or $g_1 = 0$. Thus,

$$\tilde{\rho}_{\min} = \frac{1}{\sigma_m^2} \int_0^{\sigma_m} (m^2 - \sigma_m^2) p(m)\, dm, \tag{47}$$

and the corresponding value of λ is given by

$$\tilde{\lambda}_1 = \frac{1}{\sigma_m^2} \int_0^{\sigma_m} (m^2 + \sigma_m^2) p(m)\, dm. \tag{48}$$

In order to obtain the results which are independent of the particular signals used, suppose that the idealized saturation being considered is further normalized such that

$$f(m) = \begin{cases} -1; & m < -1, \\ m; & -1 \leqq m \leqq 1, \\ 1; & 1 < m. \end{cases} \tag{49}$$

The units of the signals are also normalized such that σ_m is taken as the unit. With these normalizations, the integrals of equations (47) and (48) may be evaluated using the tables in Ref. 9 to obtain $\tilde{\lambda}_1 = 0.44072$, $\tilde{\rho}_{\min} = 0.24197$.

In a similar manner, for the normalized idealized deadzone given by

$$f(m) = \begin{cases} m + 1; & m < -1, \\ 0\ ; & -1 \leqq m \leqq 1, \\ m - 1; & 1 < m, \end{cases} \tag{50}$$

it is found that $\tilde{\lambda}_2 = 0.55928$, $\tilde{\rho}_{\max} = 0.24197$.

REFERENCES

1. Sandberg, I. W., "On the Response of Nonlinear Control Systems to Periodic Input Signals," B.S.T.J., *43*, No. 3 (May 1964), pp. 911–926.
2. Hatanaka, H., "The Frequency Responses and Jump Resonance Phenomena of Nonlinear Feedback Control Systems," Trans. Amer. Soc. Mechanical Engineers, J. of Basic Eng., *85*, No. 2 (June 1963), pp. 236–242.
3. Fukuma, A., and Matsubara, M., "Jump Resonance Criteria of Nonlinear Control System," Trans. IEEE Automatic Control, *AC-11*, No. 4 (October 1966), pp. 699–706.
4. Holtzman, J. M., "Analysis of Statistical Linearization of Nonlinear Control Systems," SIAM J. Control, *6*, No. 2 (May 1968), pp. 235–243.
5. Booton, R. C., Jr., "The Analysis of Nonlinear Control Systems with Random Input," *Proc. Symp. on Nonlinear Circuit Analysis,* New York: Polytechnic Institute of Brooklyn, 1953, pp. 369–391.
6. Holtzman, J. M., *Nonlinear System Theory,* Prentice-Hall: Englewood Cliffs, N. J., (in press).
7. Papoulis, A., *Probability, Random Variables and Stochastic Processes,* New York: McGraw-Hill, 1965, pp. 160–161.
8. Zemanian, A. H., *Distribution Theory and Transform Analysis,* New York: McGraw-Hill, 1965, pp. 67–70.
9. Pearson, E. S., and Hartley, H. O., *Biometrika Tables for Statisticians,* Vol. 1, 3rd Ed., Cambridge, England: University Press, 1966, pp. 9, 128–135.
10. Beneš, V. E., "A Nonlinear Integral Equation in the Marcinkiewicz Space $\mathfrak{M}_2$," J. Math. Phys., *44*, No. 1 (January 1965), pp. 24–35.