Mobile Radio Diversity Reception

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This paper examines a particular kind of diversity system, under conditions of multipath fading, when there is interference from either random noise or from an unwanted station. The transmitter sends a pilot wave along with the modulated signal. The receiver's mixer stage heterodynes the signal with the pilot (instead of with a locally generated tone). Doppler phase distortion, which affects the signal and pilot in nearly the same way, cancels out during mixing. The diversity system with N antennas adds the outputs from N such mixers. This kind of diversity tends to add the N signal outputs in phase, while random noise components as well as certain other interferences add powerwise. In the presence of an interfering station, diversity smooths out amplitude fluctuations. It thereby reduces the probability that the interference will override the desired station.

I. INTRODUCTION

D. O. Reudink, in an unpublished work, has suggested a diversity system especially suited for mobile radio. In his system the transmitter sends a pilot wave along with the modulated signal. The receiver's mixer stage beats the signal against the received pilot (instead of against a locally generated tone). Doppler distortion, which affects the signal and pilot in nearly the same way, cancels out during mixing. The diversity system with N antennas adds the outputs of N such mixers and demodulates the sum by means of an ordinary AM or FM detector.

The receiver obtains a signal-to-noise advantage by adding signal components from the N mixers in phase while adding most interference terms powerwise. To obtain this advantage under multipath propagation conditions, the receiver's IF (that is, the difference f between the signal and pilot frequencies) must be chosen small enough. It suffices to make f so small that the propagation times along the different paths all agree to within a small fraction of $1/f$ (see Section

2.2). The analysis presented in this paper is only valid for the situations in which signal components add in phase.

The effectiveness of these receivers is most clearly seen by examining the signal and noise levels at their outputs. Here the noise in question may be either random noise or an unwanted beat from an interfering station. Several kinds of signal-to-noise ratios can be defined because the signal and noise levels fluctuate as the receiver moves. The ratio snr of output signal power to output noise power depends on the receiver's position. Here snr is regarded as a random variable and its probability distribution function is derived. A simpler ratio, called SNR, is obtained by dividing the mean output signal power by the mean output noise power. SNR is simply a fixed number but it gives less information about receiver failure than the distribution of snr does.

The probability distribution of snr is derived for cases in which the signal experiences rayleigh fading. The rayleigh fading model is known to agree well with experiment within small areas, say ten wavelengths across, although it cannot account for largescale effects like shadowing by buildings and hills.[1] SNR is derived without assuming rayleigh fading.

Table I gives excerpts from more complete tables which follow. It compares receivers under rayleigh fading conditions by giving transmitter powers needed to keep snr above 3 dB or 10 dB with probability 0.99. The transmitter powers are given in decibels above a common level which need not be specified at this point. Of course the required powers depend on the interference power and on the propagation losses, but these terms are the same in all cases; they contribute a constant number of decibels to all the tabulated values. Only differences in decibel values need be considered when comparing receivers.

The table considers four kinds of interference and gives the signal power needed to keep snr at the given level for each separately. Random interference is supposed to be gaussian noise. In diversity receivers an interfering station produces three noise signals having different properties. These are called $2PS'$, $2P'S$, $2P'S'$, the letters denoting the components which beat to produce the noise. Thus $2P'S$ is a beat between interfering *P*ilot and desired *S*ignal. For comparison, the conventional receiver has only one kind of output noise. Notice that the relative strengths of the three noises in the diversity receiver, and hence the character of the combined noise, depends both on N and on the signal level. Even a two-antenna diversity system has a noise

Table I—Relative Transmitter Powers (dB) Required for 0.01 Probability of snr $\geqq$ 3 dB or 10 dB

snr (dB)	Interference		Diversity Receivers $N = 1$	2	4	8	Conventional Receiver
3							
	random		26.0	14.3	6.6	1.4	20.0
	station	$2PS'$	23.0	12.5	6.3	1.9	23.0
		$2P'S$	23.0	12.5	6.3	1.9	
		$2P'S'$	21.5	13.5	9.3	6.8	
10							
	random		36.0	24.3	16.6	11.4	30.0
	station	$2PS'$	30.0	19.5	13.3	8.9	33.0
		$2P'S$	30.0	19.5	13.3	8.9	
		$2P'S'$	25.0	17.0	12.8	10.3	

advantage over the conventional system and has immunity to doppler distortion too.

II. THE DIVERSITY RECEIVER

The transmitter sends a pilot tone $A \cos 2\pi Ft$ along with the modulated signal $AB \cos[2\pi(F + f)t + \theta]$. Here f is an intermediate frequency, small compared with F but large enough so that the signal spectrum does not overlap the pilot. B and θ are an amplitude and a phase, either one of which may be varied slowly to represent the modulating signal. The receiver (see the block diagram, Fig. 1), contains elements SQ which square received antenna voltages. Each square contains a component at frequency f which results from a beat between the pilot and the modulated signal. This component contains the modulation, AM or FM, of the original transmission. The N squares are added and the sum is filtered to remove other components at frequencies far from f. The filtered sum is an IF signal to be demodulated in the usual way.

2.1 *Single Path In Phase Addition*

In effect the transmitted pilot tone replaces the local oscillator tone which a conventional receiver generates internally. The advantage is that any doppler distortion affects the pilot as well as the modulated signal. As a result, the circuit of Fig. 1 tends to add IF components in phase if f is small. This may be seen as follows.

Figure 2 shows N antennas receiving a signal which arrives from

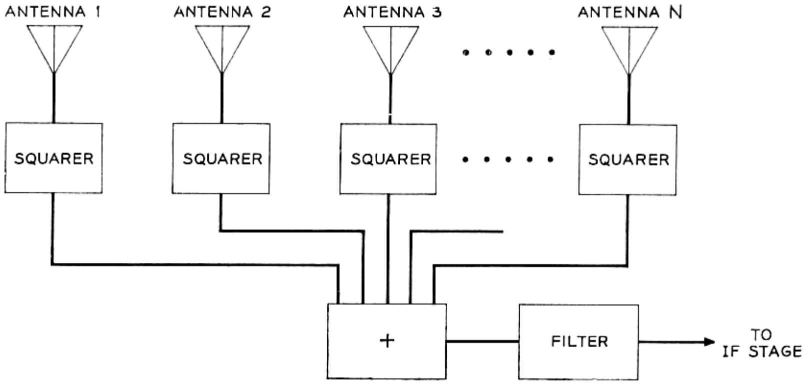

Fig. 1 — Diversity receiver.

the direction indicated by the arrow. Suppose for the moment that this is the only incident signal (no multipath effects). Now consider two typical antennas, say 1 and 2. Let the difference between the lengths of the paths from 1 and 2 to the transmitter be called s.

If the voltage in antenna 1 is

$$A \cos (2\pi F t + \varphi) + AB \cos [2\pi(F + f)t + \psi], \tag{1}$$

then the voltage in antenna 2 is

$$A \cos [2\pi F(t - s/c) + \varphi] + AB \cos [2\pi(F + f)(t - s/c) + \psi], \tag{2}$$

where c is the velocity of light. After squaring, the IF components are $\frac{1}{2}A^2B \cos (2\pi ft + \psi - \varphi)$ from antenna 1, and $\frac{1}{2}A^2B \cos (2\pi ft + \psi - \varphi - 2\pi fs/c)$ from antenna 2.

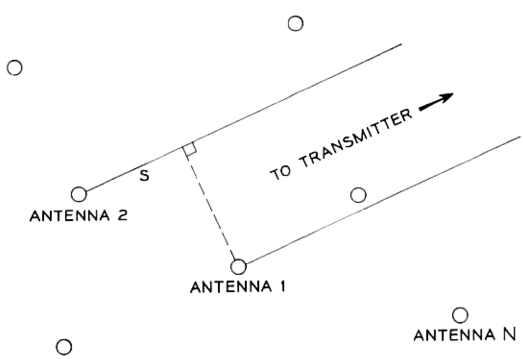

Fig. 2 — Reception by N antennas.

These two components differ in phase by $2\pi fs/c$ radians. To keep this angle small, s must be a small fraction of c/f, the wavelength at IF. For instance if the IF is $f \leqq 1$ MHz and if no two antennas are more than ten feet apart, then s is less than 0.01 wavelength and the N beat components are in phase to within 3.6°.

2.2 *Multipath Inphase Addition*

Under multipath conditions cross beats occur between pilots and modulated signals received via different paths. This section derives a more stringent sufficient condition for inphase addition. Now the lengths of all major propagation paths from transmitter to receiving antennas must agree within a small fraction of the IF wavelength. For example, if the IF were 100 kHz, the wavelength in question would be 3000 meters. Path differences of hundreds of feet would still permit nearly inphase addition. Path differences of this size might occur if only nearby buildings serve as reflectors. The data which W. R. Young took in New York City shows that some longer path dierences can be expected there.[1]

The voltages in antennas 1 and 2 of Fig. 2 are now sums of voltages received over different paths. The kth path contributes terms like (1) and (2) but with parameters A_k, ϕ_k, ψ_k, and s_k which depend on k. Suppose the kth path has length L_k. Then ϕ_k is a sum of phase shifts at reflections plus a propagation term $-2\pi FL_k/c$. Likewise ψ_k is a sum of the same phase shifts at reflections, a propagation term $-2\pi(F + f)L_k/c$, and the modulation angle θ. Then $\psi_k = \phi_k + \theta - 2\pi fL_k/c$. At antenna 2 the kth pilot is $P_k = A_k \cos(2\pi Ft + \phi_k - 2\pi Fs_k/c)$ and the kth modulated signal is $S_k = A_kB \cos[2\pi(F + f)t + \theta + \phi_k - 2\pi fL_k/c - 2\pi(F + f)s_k/c]$. At antenna 1 the kth path produces voltages of the same form but with $s_k = 0$.

When the antenna 2 voltage is squared, cross beats between the jth and kth paths occur. The IF part of P_kS_j is

$$P_kS_j : \tfrac{1}{2}A_kA_jB \cos[2\pi ft + \theta + \phi_j - \phi_k - 2\pi fL_j/c - 2\pi(F + f)s_j/c + 2\pi Fs_k/c].$$

There is also a P_jS_k beat, and the sum of the two beats contains the IF component

$$P_kS_j + P_jS_k : A_kA_jB \cos[2\pi ft + \theta - \pi f(L_k + L_j + s_k + s_j)/c] \cdot \cos[\phi_j - \phi_k - \pi f(L_j - L_k + s_j - s_k)/c - 2\pi F(s_j - s_k)/c].$$

The same expression gives the IF component of $P_kS_j + P_jS_k$ at antenna 1 when s_j and s_k are replaced by zero. In this expression the first cosine contains the time dependence while the second cosine is purely an amplitude factor.

Now suppose, as in Section 2.1, that $s_1, s_2, \cdots$, are all so small that the terms $\pi f s_k/c$ are small angles. Then the first cosine in the $P_kS_j + P_jS_k$ contribution is nearly the same at antenna 2 as it is at antenna 1. However the second cosine contains the large angle $2\pi F(s_j - s_k)/c$ at antenna 2 only. Indeed one can construct numerical examples to show that further assumptions are needed to make the total IF outputs of the two squarers be inphase. It will suffice to assume that the path lengths $L_1, L_2, \cdots$, are nearly equal, differing from one another by only a small fraction of c/f. Under this extra condition, the first cosine factor is approximately $\cos(2\pi ft + \theta - 2\pi L_1/c)$ for all k, j and at both antennas. For a given k, j the second cosine factor can still have opposite signs at the two antennas. However, when all beats are combined, the amplitude at antenna 2 is approximately

$$\begin{aligned}\tfrac{1}{2} \sum_{k,j} A_kA_jB \cos[\phi_j - \phi_k - 2\pi F(s_j - s_k)/c] &= \tfrac{1}{2}B \operatorname{Re} \sum_{k,j} A_kA_j \exp i[\phi_j - \phi_k - 2\pi F(s_j - s_k)/c] \\ &= \tfrac{1}{2}B \operatorname{Re} \left| \sum_j A_j \exp i[\phi_j - 2\pi Fs_j/c] \right|^2,\end{aligned}$$

which is positive. The same argument with $s_j = 0$ gives a positive amplitude at antenna 1; the two sums are inphase.

In New York City large path differences are observed. There it may be difficult to make f small enough to satisfy always the condition just derived. However if the total number K of paths is small, there is still some tendency for the phases from squarers 1 and 2 to be close. For although the P_kS_j contributions from antennas 1 and 2 differ in the $K(K - 1)$ cases with $j \neq k$, the argument of Section 2.1 shows that the two antennas give equal contributions in the K cases with $j = k$. One can analyze simple models in which L_k and other parameters are randomly chosen and still conclude that the IF outputs from the two squarers are correlated, but to an extent that decreases as K increases. However I omit those details and assume from now on that signal outputs from the squarers add inphase. I also assume that F is large enough, say about 1000 MHz, so that the phases of noise received in antennas placed a few feet apart can be considered independent.

III. RESPONSE TO RANDOM NOISE

This section considers the effect of random noise on diversity reception and gives expressions (16), (17), and (18) for output noise spectra. Multipath fading effects make the output signal to noise ratio, snr, depend on the position of the receiver. A single mathematically convenient figure of merit is the ratio of expected signal power to expected noise power. This ratio is called SNR here. Before the mathematical details begin, some of the results will be summarized.

SNR increases linearly with the number N of antennas [equation (20)]. For a given amount of total transmitter power, the largest output signal power is obtained by transmitting equal amounts of power in the pilot and modulated signal. The diversity system will be compared with a conventional system using the same transmitter power. If N is small, the conventional system has a slight noise advantage because it uses the full transmitter power for the modulated signal (the pilot is generated in the receiver). The diversity system with $N = 3$ has about the same SNR as a conventional system. However, the probability distributions of snr for these receivers are very different; the one for the diversity receiver is more sharply peaked. As a result a diversity system, even with $N = 2$, produces a small snr less often than the conventional system (compare with Table I).

When making SNR comparisons one must also recognize qualitative differences between the output noises from different receivers. The conventional receiver has a steady noise output resulting from input noise beating against the steady local oscillator signal. In the diversity system the output noise results largely from input noise beating against fluctuating pilot and modulated signals. During fades the output noise from the diversity receiver also fades while the noise from the conventional receiver does not. Thus, the diversity receiver has acceptable snr more often than a conventional receiver with the same output SNR.

3.1 *Noise Spectra*

The mathematical treatment will begin with the case $N = 1$; the extension to more antennas will be easy. The input to the squarer is the sum of three voltages:

$$\text{Pilot} \qquad P(t) = A \cos (2\pi Ft + \varphi), \tag{3}$$

$$\text{Signal} \qquad S(t) = AB \cos [2\pi(F + f)t + \psi], \tag{4}$$

$$\text{Noise} \qquad n(t) = \sum n_i \cos (2\pi f_i t + \xi_i). \tag{5}$$

Here the noise is represented, as by S. O. Rice,[2] as a sum of sinusoids with random phases ξ_i and amplitudes n_i. Rice studied the effect of squaring a random noise; this section adapts his work to the present problem.

The received pilot power is $\frac{1}{2}A^2$ (into a one ohm load); likewise the signal has power $\frac{1}{2}A^2B^2$. The noise has a one-sided power spectrum function $w(\nu)$ such that

$$w(\nu)\ \Delta\nu = \tfrac{1}{2} \sum_{\nu < f_i < \nu + \Delta\nu} n_i^2$$

represents the noise power in the frequency band from ν to $\nu + \Delta\nu$. The shape of the function $w(\nu)$ is determined by the tuned circuits (not shown in Fig. 1) which filter the antenna signal before squaring. Figure 3 shows a typical case

$$w(\nu) = \begin{cases} N_o\,, & F - b \leqq \nu \leqq F + f + a, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

which uses a band-pass filter slightly wider than necessary to pass the pilot and signal at frequencies F and $F + f$.

Squaring $P + S + n$ produces six terms; P^2, S^2, n^2, $2PS$, $2Pn$, $2Sn$. P^2 and S^2 contribute nothing to the output after the output filter removes components remote from frequency f. The other contributions are

$$A^2B \cos (2\pi ft + \psi - \varphi) \quad \text{from} \quad 2PS, \tag{7}$$

$$A \sum n_i \cos [2\pi(f_i - F)t + \xi_i - \varphi] \quad \text{from} \quad 2Pn, \tag{8}$$

$$AB \sum n_i \cos [2\pi(f_i - F - f)t + \xi_i - \psi] \quad \text{from} \quad 2Sn, \tag{9}$$

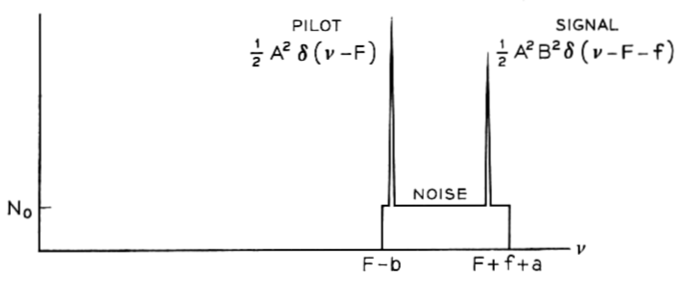

Fig. 3 — Power spectra at the input to a squarer.

$$\sum_{j<i} n_i n_j \cos [2\pi(f_i - f_j)t + \xi_i - \xi_j] \qquad \text{from} \quad n^2. \tag{10}$$

The $2PS$ contribution is the desired output; its power is $\frac{1}{2}A^4B^2$. The spectra of the other contributions appear in Fig. 4. The spectral density functions are

$$A^2 w(\nu + F) \qquad \text{from} \quad 2Pn, \tag{11}$$

$$A^2B^2 w(F + f + \nu) \qquad \text{from} \quad 2Sn, \tag{12}$$

$$2 \int_0^\infty w(x)w(\nu + x)\, dx \quad \text{from} \quad n^2. \tag{13}$$

Functions (11), (12), and (13) assign some power to negative values of ν; these are to be aliased to positive frequencies. This aliasing accounts for the peculiar discontinuities in the spectra at low frequencies. The dotted lines show functions (11), (12), and (13) before aliasing. The values of a and b will be assumed smaller than f so that, as in Fig. 4, the noise power densities at frequency f are A^2N_o for Pn noise and $A^2B^2N_o$ for Sn noise.

In the case of gaussian noise, the phases ξ_i in functions (8), (9), and (10) are independent. It then follows that the three kinds of output noise components at a given frequency ν are uncorrelated. Then these noises add powerwise and the total noise spectral density is the sum of functions (11), (12), and (13).

3.2 *Noise in Diversity System*

In a diversity system the same kind of analysis applies for each of N antennas. The amplitudes and phases would now be written as A_k, n_{ik}, ψ_k, φ_k, and ξ_{ik} where the subscript k ($k = 1, \cdots, N$) specifies the antenna. All these random variables are independent of one another except for ψ_k and φ_k which satisfy $\psi_1 - \varphi_1 = \psi_2 - \varphi_2 = \cdots = \psi_N - \varphi_N = \theta$ because, as discussed in Section II, the $2PS$ terms have a common phase angle θ. Thus the N signal components add voltage-wise and the expected signal power at the output is

$$\tfrac{1}{2}B^2E(\sum A_k^2)^2 = \tfrac{1}{2}B^2E\{NE(A^4) + N(N-1)[E(A^2)]^2\}.$$

Let k_o denote the (dimensionless) ratio

$$k_o = E(A^4)/[E(A^2)]^2. \tag{14}$$

For rayleigh fading, $k_o = 2$. For no fading $k_o = 1$. The expected output signal power is

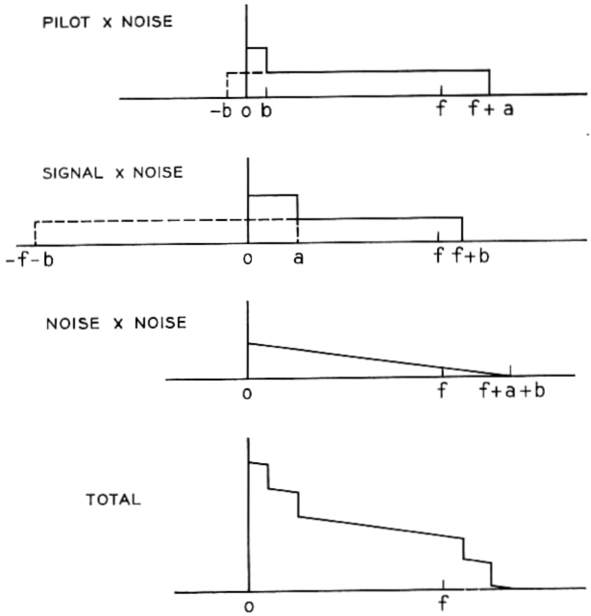

Fig. 4 — Output noise spectra.

$$E(\text{Signal Power Out}) = \tfrac{1}{2}N(N + k_o - 1)[E(A^2)]^2B^2$$
$$= \tfrac{1}{2}N(N + 1)[E(A^2)]^2B^2(\text{rayleigh}). \tag{15}$$

According to equations (3) and (4), $\frac{1}{2}E(A^2)$ and $\frac{1}{2}B^2E(A^2)$ are the expected received powers of the pilot and signal. With fixed transmitter power [fixed $\frac{1}{2}(1 + B^2)E(A^2) = P_o$], the output signal, equation (15), is maximized when the transmitted power is divided equally between pilot and signal [$B = 1$, $E(A^2) = P_o$]. Then equation (15) becomes $E(\text{Signal Power Out}) = \frac{1}{2}N(N + k_o - 1)P_o^2$.

The noise terms (11), (12), and (13) for the N antennas add powerwise and the expected output noise power spectrum is a sum of three terms

$$NE(A^2)w(\nu + F) \qquad \text{from} \qquad 2Pn, \tag{16}$$

$$NB^2E(A^2)w(F + f + \nu) \qquad \text{from} \qquad 2Sn, \tag{17}$$

and

$$2N \int_0^\infty w(x)w(\nu + x)\, dx \quad \text{from} \quad n^2. \tag{18}$$

3.3 *SNR Formulas*

For a typical case, suppose $w(\nu)$ is the function (6) with $a < f$ and $b < f$. Suppose also that the output filter in Fig. 1 has a narrow rectangular transfer function with bandwidth Δf about frequency f. Then the expected output noise power is

$$E(\text{Noise Power Out}) = 2NN_o[P_o + N_o(a + b)]\Delta f \qquad (19)$$

where again $P_o = \frac{1}{2}(1 + B^2)E(A^2)$ is the total expected power which an antenna receives from pilot and signal. In this case the output noise power does not depend on B and the choice $B = 1$ maximizes not only the output signal power but the output signal to noise ratio as well. With $B = 1$, equations (15) and (19) combine to give

$$\text{SNR} = \frac{N + k_o - 1}{4} \frac{P_o/(N_o \Delta f)}{1 + (a + b)N_o/P_o}, \qquad (20)$$

where k_o is given by equation (14) ($k_o = 2$ for Rayleigh fading).

If the input noise spectrum is not flat as in Fig. 3, the output noise contributions (16) and (17) do not combine into the term $2NN_oP_o\Delta f$ which appears in equation (19). In that case the value of B which gives the best output SNR may not be one but will depend on the input noise power densities at F and $F + f$.

Formulas (11), (12), and (13) also apply, with slight reinterpretations, to the conventional receiver without diversity. $\frac{1}{2}A^2$ is the power of a local oscillator and $\frac{1}{2}A^2B^2$ is the received signal power. Then A has a well determined value, but B is a random variable having perhaps a Rayleigh distribution. Now P_o is $E(\frac{1}{2}A^2B^2)$. The desired output signal has amplitude A^2B and so has expected power $E(\text{Signal Power Out}) = \frac{1}{2}A^4E(B^2) = A^2P_o$. The local oscillator is deliberately made much stronger than the incoming signal or noise; then the output noise components (12) and (13) are negligible compared with formula (11). For the output filter of bandwidth Δf, $E(\text{Noise Power Out}) = A^2w(F + f)\Delta f$. When $w(\nu)$ is the function (6) again,

$$\text{SNR} = P_o/(N_o\Delta f). \qquad (21)$$

The output signal to noise ratio in equations (20) and (21) differ by a factor

$$(\text{SNR})_{\text{conventional}}/(\text{SNR})_{\text{diversity}} = 4[1 + (a + b)N_o/P_o]/(N + k_o - 1). \qquad (22)$$

The term $(a + b)N_o/P_o$ represents that part of the input noise to signal ratio which results from noise arriving outside the band $F \leqq \nu \leqq F + f$. Then this term will be small in any useful case. The remaining factor $4/(N + k_o - 1)$ gives the conventional system the advantage unless $N \geqq 5 - k_o$. When Rayleigh fading holds, a three-antenna diversity system has the same output SNR as the conventional system.

3.4 *snr Distribution*

As mentioned in Section 3.3, the signal and noise levels of conventional and diversity receivers fluctuate differently as the receiver moves. In the case of Rayleigh fading one can obtain the probability distribution functions for snr = (Signal Power Out/Noise Power Out) for the two receivers. Again take the simple input noise spectrum of equation (6) with small values of a and b.

Expressions (7), (11), and (12) show that snr $= (4N_o\Delta f)^{-1} \sum A_k^2$ for the diversity receiver ($B = 1$). Each Rayleigh amplitude A_k may be expressed in terms of independent gaussian variables x_k, y_k of mean zero and unit variance by means of $A_k^2 = \frac{1}{2}P_o(x_k^2 + y_k^2)$.

Then snr $= (\chi_{2N}^2/8)(P_o/N_o\Delta f)$ where $\chi_{2N}^2 = X_1^2 + \cdots + X_N^2 + Y_1^2 + \cdots + Y_N^2$ has the chi-squared probability distribution with $2N$ degrees of freedom. The same result might be obtained by interpreting the receiver as a maximal ratio combiner.[3]

Expressions (7), (11), and (12) also apply to the conventional receiver if, as explained above, A is a fixed number while B is a small rayleigh variable. Only $2Pn$ noise need be considered; then snr $= (2N_o\Delta f)^{-1}A^2B^2 = \frac{1}{2}\chi_2^2(P_o/N_o\Delta f)$, where again χ_2^2 has the chi-squared distribution, now with two degrees of freedom.

Suppose the system fails when snr is below some known critical value. Suppose such failure can be tolerated only a small fraction Q of the time. The given value of Q is reached at some χ^2 value which can be read from probability tables. To achieve the desired small failure probability the ratio $P_o/(N_o\Delta f)$ (a kind of input SNR) must be

$$P_o/(N_o\,\Delta f) = \begin{cases} (8/\chi_{2N}^2)\ \text{snr} & \text{(diversity)} \\ \frac{1}{4}(8/\chi_2^2)\ \text{snr} & \text{(conventional)} \end{cases}.$$

Table II gives $10 \log (8/\chi_{2N}^2)$ as a function of Q. Thus for a 0.01 probability of failure, $P_o/(N_o\Delta f)$ must exceed the critical snr by 26.0 dB, 14.3 dB, 6.87 dB, and 1.39 dB for diversity systems of one, two, four, and eight antennas. The conventional receiver requires $26.0 - 6.0 = 20.0$ dB and so is intermediate between diversity systems with $N = 1$ and 2.

TABLE II—VALUES OF 10 LOG $(8/\chi_{2N}^2)$ FOR WHICH PROBABILITY OF FAILURE = Q

Number of Antennas	Q					
	0.001	0.005	0.01	0.025	0.05	0.1
1	36.0	29.0	26.0	22.0	18.9	15.8
2	19.5	15.9	14.3	12.1	10.5	8.78
3	13.3	10.7	9.62	8.01	6.89	5.60
4	9.74	7.75	6.57	5.65	4.67	3.60
5	7.36	5.70	4.95	3.92	3.08	2.16
6	5.61	4.15	3.50	2.59	1.85	1.03
7	4.23	2.96	2.35	1.53	0.86	0.12
8	3.15	1.92	1.39	0.64	0.02	−0.63

IV. INTERFERENCE FROM A SECOND STATION

Suppose a diversity system tries to receive a desired signal while another station uses the same channel. The pilots and modulated signals of the two stations produce a variety of beat components, three of which cause interference at IF [functions (23), (24), and (25) below]. Two sound like doppler-distorted versions of the modulated signals from the desired station and its competitor. The third is an undistorted copy of the modulated signal from the competing station.

Under multipath conditions, the two doppler-distorted beats have phases which are uncorrelated from antenna to antenna. The output SNR's for these noises grow linearly with the number N of antennas [equation (26)]. The third components from the separate squarers add in phase. Then the SNR for this interference is not reduced by increasing N [equation (27)]. However, increasing N reduces the variability of the power levels of the output signal and noise. Thus, if the desired station is a few decibels stronger than the competing station, increasing N reduces the chance that multipath fading will allow the competing station to override the desired station (Table III).

One may reuse much of the formalism of Section III. A single antenna again receives a pilot [equation (3)], modulated signal [equation (4)], and a noise which is a special case of equation (5). The noise now has only two components. One is a pilot $P'(t)$ of frequency F, phase φ', and amplitude A'. The other is a modulated signal $S'(t)$ of frequency $F + f$, phase ψ', and amplitude $A'B'$.

Squaring produces IF components which are obtainable from func-

TABLE III—VALUES OF 10 LOG F SUCH THAT PROBABILITY OF FAILURE = Q ($2P'S'$ NOISE)

Number of Antennas	Q							
	0.001	0.005	0.01	0.025	0.05	0.1	0.25	0.50
1	30.0	23.0	20.0	15.9	12.8	9.54	4.77	0
2	17.3	13.6	12.0	9.82	8.05	6.14	3.14	0
3	13.0	10.4	9.27	7.64	6.31	4.84	2.50	0
4	10.8	8.74	7.80	6.46	5.36	4.13	2.14	0
5	9.56	7.70	6.88	5.71	4.79	3.67	1.90	0
6	8.35	6.92	6.19	5.16	4.29	3.32	1.73	0
7	7.73	6.33	5.70	4.75	3.94	3.04	1.58	0
8	7.17	5.86	5.29	4.42	3.69	2.88	1.49	0

tions (7), (8), (9), and (10). The desired signal component is function (7) again. The $2Pn$ component, function (8), has two parts, one of which [$P(t)$ beating against $P'(t)$] contributes nothing. The remaining IF contribution from function (8) is

$$AA'B' \cos (2\pi ft + \psi' - \varphi) \quad \text{from} \quad 2PS'. \tag{23}$$

Likewise functions (9) and (10) contribute only

$$AA'B \cos (2\pi ft + \psi - \varphi') \quad \text{from} \quad 2SP', \tag{24}$$

and

$$A'^2B' \cos (2\pi ft + \psi' - \varphi') \quad \text{from} \quad 2P'S'; \tag{25}$$

the $2SS'$ and S'^2 terms do not contribute at IF.

The three interference terms (23), (24), and (25) have different characteristics. The $2PS'$ and $2P'S'$ components carry the modulation (AM or FM) of $S'(t)$ and act like interfering stations at IF. Likewise the $2SP'$ term sounds like a station with the desired modulation of $S(t)$. As the receiver moves, the two angles ψ', and φ undergo different doppler shifts. Then the $2PS'$ component contains a residual doppler distortion. Likewise the $2SP'$ component is doppler distorted and so will be considered a noise. By contrast, as in the $2PS$ term. the doppler shifts in the $2P'S'$ term cancel out leaving an undistorted interfering signal.

Because the $2P'S$ component has both the desired modulation and doppler distortion it is not clear whether it should be treated as a signal term or as a noise term. If it were counted as part of the signal, the $2P'S$ term would be a source of fluctuation of the output signal level (it differs in phase from the $2PS$ term by a random amount).

To call the $2P'S$ term a kind of noise is probably overconservative if the system uses FM of index high enough to make the doppler distortion unimportant. It turns out that the power levels of the $2PS'$ and $2P'S$ terms have the same probability distribution. Thus, whenever other interference terms are small it does not matter much whether $2P'S$ components are treated as signal or as noise.

4.1 *SNR Formulas*

As in Section 3.3 one can compute an SNR, defined as E(Signal Power Out)/E(Noise Power Out), for each of the three interferences. Again multipath fading conditions will be assumed so that pilot amplitudes and phases from the N antennas are independent variables. The conditions $\psi_1 - \varphi_1 = \psi_2 - \varphi_2 = \cdots = \psi_N - \varphi_N = \theta$ and $\psi_1' - \varphi_1' = \psi_2' - \varphi_2' = \cdots = \psi_N' - \varphi_N' = \theta'$ relate the signal phases to the pilot phases.

The expected signal output power is given by equation (15) as before. The expected power from the N terms of type $2PS'$ is $E(\sum \frac{1}{2}A_k^2 A_k'^2 B'^2) = \frac{1}{2}NB'^2 E(A^2)E(A'^2)$. Likewise the $2SP'$ power has expected value $\frac{1}{2}NB^2 E(A^2)E(A'^2)$. The SNR's are SNR $= (N + k_o - 1)(B/B')^2 E(A^2)/E(A'^2)$ for $2PS'$ interference and SNR $= (N + k_o - 1)E(A^2)/E(A'^2)$ for $2SP'$ interference [recall the definition of k_o given by equation (14)]. When $B = B' = 1$ and the expected received powers from the two stations are P_o and P_o', both interferences have

$$\text{SNR} = (N + k_o - 1)P_o/P_o' . \tag{26}$$

The expected power of $2P'S'$ interference is given by equation (15) with A' and B' replacing A and B. Then, if $B' = B$, the SNR for $2P'S'$ is

$$\text{SNR} = (P_o/P_o')^2. \tag{27}$$

The two expressions (26) and (27) have interesting differences. They depend on N in different ways because the $2SP'$ and $2S'P$ components from separate antennas add with random phases while the $2S'P'$ components add in phase. The input signal to noise ratio P_o/P_o' appears with different exponents in equations (26) and (27) because equation (26) relates to beats between the desired station and the interfering one, while equation (27) relates to beats of the interfering station with itself.

Because of these differences, either kind of output noise can be the more serious one, depending on the situation. For a given number of antennas, the $2S'P$ and $2SP'$ noises are stronger than the $2S'P'$ noise when P_o/P_o' is large. As P_o/P_o' becomes smaller, all noises increase and, at $P_o/P_o' = N + k_o - 1$, they have equal powers. When P_o/P_o' is still smaller, the $2S'P'$ noise (undistorted copy of the interfering signal) predominates. With Rayleigh fading and $N = 4$ antennas, the $2S'P'$ noise predominates at input signal to noise ratios of 7 dB or less.

In conventional systems, an interfering station produces only one output noise component. It has

$$\text{SNR} = P_o/P_o' . \tag{28}$$

None of the noise components of the diversity system are as bad as this unless the interfering station is stronger than the desired one.

4.2 *snr Distributions, $2P'S'$ Noise*

Equation (27) shows that adding more antennas does not improve the SNR for $2P'S'$ noise. However, diversity helps by reducing the chance that a large fluctuation of the interfering signal level will cause the system to fail. To study this effect let $A_1, \cdots, A_N$ be signal amplitudes, as in expressions (7) and (8), received by the N antennas. Likewise let these antennas receive $A_1', \cdots, A_N'$ from the interfering station. Under severe multipath conditions these $2N$ amplitudes may be regarded as independent random variables. Again take $B = B' = 1$ so that $E(A_k^2) = P_o$, $E(A_k'^2) = P_o'$. The desired and interfering stations produce output signals with amplitudes $\sum A_k^2$ and $\sum A_k'^2$. Then

$$\text{snr} = (\sum A_k^2 / \sum A_k'^2)^2 \tag{29}$$

is the random variable which must be studied.

The probability distribution function for snr can be obtained easily in the case of rayleigh fading. Each Rayleigh amplitude A may be represented by the formula $A^2 = X^2 + Y^2$ where X and Y are independent gaussian variables with variance $E(X^2) = E(Y^2) = \frac{1}{2}P_o$.

In these terms, the quantity

$$\begin{aligned} F &= \frac{(X_1'^2 + Y_1'^2 + X_2'^2 + \cdots + Y_N'^2)/(\frac{1}{2}P_o')}{(X_1^2 + Y_1^2 + X_2^2 + \cdots + Y_N^2)/(\frac{1}{2}P_o)} \\ F &= (P_o/P_o')\,\text{snr}^{-\frac{1}{2}}, \end{aligned} \tag{30}$$

is the ratio of two sums of $2N$ independent squares of gaussian variables of unit variance. Statisticians use such ratios frequently and have tabulated their probability distributions. Abramowitz and Stegun give such a table.[4] In their notation the cumulative probability function for F is $P(F \mid 2N, 2N)$, a special case of their $P(F \mid \nu_1, \nu_2)$. Their Table 26.9 gives $Q(F \mid \nu_1, \nu_2) = 1 - P(F \mid \nu_1, \nu_2)$, so that snr has the distribution function

$$\text{Prob}\,\{\text{snr} \leqq (P_o/P_o')^2 F^{-2}\} = Q(F \mid 2N, 2N). \tag{31}$$

Table III reproduces part of Abramowitz and Stegun's table after converting F values to decibels. The numbers tabulated are values 10 $\log_{10} F$ which are needed to make the probability of equation (31) a small value $Q = 0.001, 0.005, 0.01, 0.025, 0.05, 0.1, 0.25$, or 0.5. To use Table III one must first know how small snr can become before the system will fail; one also decides on an acceptable probability Q of failure. The table gives a corresponding value of F and the conditions for not failing are met as long as the input signal to noise ratio P_o/P_o' satisfies

$$F \text{ snr}^{1/2} \leqq P_o/P_o' . \tag{32}$$

For example, suppose the system fails if snr becomes as small as 3 dB. Suppose failure can be tolerated only 1 percent of the time. The tabulated values of F for $Q = 0.01$ and $N = 1, 4, 8$ are 20.0, 7.80, and 5.29 dB. Then inequality (32) requires the input signal to noise ratio to be

$$\begin{aligned} 20.0 + 1.50 &= 21.5 \text{ dB} && \text{for one antenna,} \\ 7.80 + 1.50 &= 9.30 \text{ dB} && \text{for four antennas,} \\ 5.29 + 1.50 &= 6.79 \text{ dB} && \text{for eight antennas.} \end{aligned}$$

In the case of one and four antennas at these signal levels, equations (26) and (27) show that the other noise components $2SP'$ and $2PS'$ are stronger than the $2P'S'$ component. Thus the snr for $2SP'$ and $2PS'$ noises must be considered later.

To show the advantage of diversity over a conventional system, one may examine the probability distribution function for the conventional snr. This function is not just equation (31) with $N = 1$. A conventional system has snr $= (AB)^2/(A'B')^2 = (X^2 + Y^2)/(X'^2 + Y'^2)$ instead of equation (29). To get a ratio of sums of squares of gaussian variables with unit variance, one must now define $F = (P_o/P_o')/\text{snr}$ instead of equation (30). The value of F for a given failure probability Q is again obtained from Table III with $N = 1$. The input signal to noise ratio P_o/P_o' must then satisfy

$$F \text{ snr} \leqq P_o/P_o' \tag{33}$$

instead of inequality (32). To have snr as low as 3 dB for only a fraction $Q = 0.01$ of the time, the input signal to noise ratio must now be 23 dB or more.

4.3 *snr Distributions, $2SP'$ and $2P'$ Noises*

The SNR calculation showed that $2SP'$ and $2PS'$ components are apt to be the strongest noises when N is small. The distribution functions for their snr may also be derived. Again rayleigh fading is assumed and $B' = B = 1$. The latter assumption makes the $2PS'$ and $2SP'$ components have the same snr distribution [compare expressions (23) and (24)].

It is convenient to rewrite the $2PS'$ component (23) in terms of cosine and sine amplitudes

$$X' = A' \cos(\psi' - \varphi), \qquad Y' = -A' \sin(\psi' - \varphi). \tag{34}$$

Then expression (23) becomes $AX' \cos 2\pi ft + AY' \sin 2\pi ft$. Now X' and Y' are independent gaussian random variables with mean zero and variance $\frac{1}{2}P'_o$. When there are N antennas, equations (34) give amplitudes X'_k and Y'_k for the kth antenna. The kind of argument that produced equations (28) and (29) now leads to

$$\text{snr} = \frac{(\sum A_k^2)^2}{(\sum A_k X'_k)^2 + (\sum A_k Y'_k)^2}. \tag{35}$$

It is possible to transform equation (35) into a form to which an F-distribution again applies. As a first step, introduce two new random variables

$$x' = \sum A_k X'_k / (\tfrac{1}{2} P'_o \sum A_j^2)^{\frac{1}{2}}, \qquad y' = \sum A_k Y'_k / (\tfrac{1}{2} P'_o \sum A_j^2)^{\frac{1}{2}}.$$

For any $A_1, \cdots, A_N$, x' and y' are independent gaussian variables of mean zero and variance 1. Now equation (35) becomes

$$\text{snr} = 2 \sum A_k^2 / [P'_o(x'^2 + y'^2)]. \tag{36}$$

Next one can express the pilot $P(t)$ in terms of cosine and sine amplitudes. In this way one obtains $A_k^2 = \frac{1}{2}P_o(x_k^2 + y_k^2)$, where x_k and y_k are independent gaussian random variables of mean zero and variance 1. Finally equation (36) becomes

$$\text{snr} = (P_o/P'_0)/G, \tag{37}$$

where $G = (x'^2 + y'^2)/\sum (x_k^2 + y_k^2)$.

Again the snr involves a ratio G of sums of squares of gaussian variables and formulas for a suitable F-distribution are applicable. This time the numerator and denominator of the ratio contain unequal numbers of terms; the appropriate definition of F is $F = NG$. In the notation of Abramowitz and Stegun[4], the cumulative probability function for F is $1 - Q(F \mid 2, 2N)$. From their table, I obtain Table IV which gives values of $10 \log G$ which may be used with equation (37). Thus if

TABLE IV—VALUES OF 10 LOG G SUCH THAT PROBABILITY OF FAILURE = Q ($2SP'$ AND $2PS'$ NOISES)

Number of Antennas	Q							
	0.001	0.005	0.01	0.025	0.05	0.1	0.25	0.50
1	30.0	23.0	20.0	15.9	12.8	9.54	4.77	0
2	14.9	11.2	9.54	7.26	5.40	3.34	0	−3.83
3	9.54	6.86	5.61	3.84	2.33	.61	−2.32	−5.85
4	6.64	4.41	3.34	1.79	.45	−1.09	−3.82	−7.24
5	4.74	2.76	1.79	.37	−.86	−2.34	−4.95	−8.28
6	3.34	1.52	0.61	−.70	−1.88	−3.31	−5.85	−9.12
7	2.25	0.53	−.32	−1.59	−2.72	−4.09	−6.64	−9.83
8	1.37	−.27	−1.08	−2.32	−3.43	−4.76	−7.24	−10.5

a given output snr must be maintained for all but a fraction Q of the time, Table IV determines G. Then equation (37) determines th einput signal to noise ratio $P_o/P_o' = G$ snr.

Continuing the earlier example with snr = 3 dB, and Q = 1 percent, Table IV gives G values of 20.0, 3.34, and −1.08 dB for 1, 4, and 8 antennas. The required input signal to noise ratios are

$$\begin{aligned} 20.0 + 3.0 &= 23.0 \text{ dB} && \text{for one antenna,} \\ 3.34 + 3.00 &= 6.34 \text{ dB} && \text{for four antennas,} \\ -1.08 + 3.00 &= 1.92 \text{ dB} && \text{for eight antennas.} \end{aligned}$$

4.4 *Transmission Path Lengths*

Suppose that a vehicle receives a station D miles away while a second station D' miles away interferes. If the two stations radiate equal powers, the ratio P_o/P_o' is determined by the path losses to the two stations. For example, with isotropic antennas and inverse square law propagation $P_o/P_o' = D'^2/D^2$.

The numbers in Tables III and IV can be used to set limits on D'. For example, suppose snr must be above 3 dB with probability 0.99; then $2P'S'$ noise is the most serious one. P_o and P_o' must differ by at least 9.3 dB for a four-antenna diversity receiver or by 23 dB for a conventional receiver. If the inverse square law held, D' would have to be at least 2.9 D for four-antenna diversity reception and 14.1 D for conventional reception. While the inverse square law holds in free space, waves near the earth's surface attenuate more rapidly. Measurements by W. C. Jakes followed roughly an inverse fourth power law for ranges between 2 and 15 miles. Then, allowed values of D' can be as small as 1.7 for four-antenna diversity receivers and 3.8 for conventional receivers.

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