Solutions of Fokker-Planck Equation with Applications in Nonlinear Random Vibration

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In the course of analyzing the dynamic behavior of mechanical systems subjected to random excitations, we investigate the associated Fokker–Planck equation. We also discuss the relationship between the characteristics of the random excitation and the nonlinear intensity coefficients governed by the physical properties of the system. This relationship leads to some simplified methods for solving the response probability density of certain nonlinear systems. We present general solutions to a class of multidimensional problems with desirable constraints. The random motion of a single-mode mechanical oscillator with various nonlinear stiffnesses and a charged particle moving in an electromagnetic field are examples. Cosine-power and sech-power distributions are found to be associated with the steady state response of a tangent stiffness system and a hyperbolic tangent stiffness system, respectively. When the influencing magnetic vector potential ***M*** *is irrotational, the stationary probability for the moving particle in the 6-dimensional response phase-space is statistically independent.*

I. INTRODUCTION

Although the theory of stochastic processes has found wide applications in information and communication sciences for many years, only recent advances in rocket propulsion and aerospace industries have made random vibration problems subjects of growing importance in mechanical and civil engineering. These problems involve structural responses due to random loadings and are in general nonlinear resulting from large motions.[1] Such nonlinear random transformation problems often encountered in practice are generally memory-dependent; that is, the equations of motion are described by nonlinear differential equations.[2–4] Under the Markov and Smoluchowski assumptions, it has been shown that the probability density function of a large class of random processes satisfies equations of the Fokker-Planck (F-P) type.[5–9] Recently Pawula showed that generalized Fokker-Planck equations can be derived for many cases with both these assumptions removed.[7] Many interesting problems with their governing equations of the Fokker-Planck type have been investigated by various researchers. Rosenbluth, and others, studied the Fokker-Planck equation for the distribution function for gases with an inverse-square particle interaction force;[10] van Kampen used an Fokker-Planck equations to describe the thermal fluctuations in linear and nonlinear systems;[11] Ariaratnam found the steady state response distribution for a class of nonlinear two-mode mechanical oscillators by applying certain constraints to decouple the governing Fokker-Planck equation;[12] and Hempstead and Lax used Fourier transform techniques to eliminate the phase variable from the Fokker-Planck equation in the polar coordinates for a rotating-wave Van der pol oscillator and found the phase and amplitude spectra.[13]

The Fokker-Planck equation, satisfied by the random response probability density function of a dynamic system, is a parabolic partial differential equation which generally is rather difficult to solve. Although approximate results may be obtained by using the perturbation and equivalent linearization tehcniques (for which brief accounts are given in Appendixes A and B), the formal solution yielded by the appropriate Fokker-Planck equation is still the most sought one.[14–17] In this paper, we investigate the relationship between the random input characteristics and the intensity coefficients of the response process as governed by the physical properties of the system. Based on this relationship, we establish theorems concerning classes of potential-type and uncoupled-type solutions to the multidimensional stationary Fokker-Planck equations. Then, we present simple methods, based on these theorems, for solving such classes of random transformations and we describe the required natural restraints which justify applying these methods on physical grounds.

As examples we analyze two separate cases, a simple mechanical oscillator with various nonlinear spring resistances and a charged particle moving in an electromagnetic field, which we also consider to be subject to random excitation.

II. MARKOV PROCESSES AND RANDOMLY EXCITED DYNAMIC SYSTEMS

A stochastic process which has no aftereffect is called a Markov process. If $\mathbf{y}(t) = [y_1(t), y_2(y), \cdots, y_N(t)]$ is such a process (where bold type indicates a vector) we can write

$$p(\mathbf{y}_1 \mid \mathbf{y}_2, \cdots, \mathbf{y}_N) = p(\mathbf{y}_1 \mid \mathbf{y}_2), \qquad N > 2 \tag{1}$$

and

$$p(\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_N) = p(\mathbf{y}_1 \mid \mathbf{y}_2)p(\mathbf{y}_2 \mid \mathbf{y}_3) \cdots p(\mathbf{y}_{N-1} \mid \mathbf{y}_N)p(\mathbf{y}_N) \tag{2}$$

where $p(\mid)$ and $p(\ )$ represent the conditional probability density and the joint probability density, respectively; and $\mathbf{y}_1 = \mathbf{y}(t_1), \cdots, \mathbf{y}_N = \mathbf{y}(t_N)$.

In examining the motion of a dynamic system under purely random disturbance, it is found that the phase point $\mathbf{y}(t_2)$ of the system at time t_2 depends only on the phase-point position $\mathbf{y}(t_1)$ at the previous time t_1. Therefore, the trajectory of the phase-point of a system under purely random disturbance is described by a Markov process $\mathbf{y}(t) = [y_1(t), y_2(t), \cdots, y_N(t)]$ in the phase space. The components y_i, $i=1, 3, 5, \cdots, N-1$ represent the generalized coordinates of the system and components y_{i+1} represent the first time derivatives of y_i. These N variables completely defined the dynamic state of a viboratory system in the N-dimensional phase-space.

If such a Markov process $\mathbf{y}(t) = [y_1(t), y_2(t), \cdots, y_N(t)]$ satisfies the following conditions

(*i*) the Smoluchowski equation

$$p(\mathbf{y}_1 \mid \mathbf{y}_2, \Delta t) = \int d\mathbf{y}_3\, p(\mathbf{y}_1 \mid \mathbf{y}_3, t + \Delta t)p(\mathbf{y}_3 \mid \mathbf{y}_2, t) \tag{3}$$

holds for every $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ defined in the N-dimensional phase space,

(*ii*) the higher order intensity coefficients vanish, that is,

$$K_s(\mathbf{y}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle (\mathbf{y}_{\Delta t} - \mathbf{y})^s \rangle = 0 \quad \text{for} \quad s \geqq 3 \tag{4}$$

and the first and second intensity coefficients

$$K_1(\mathbf{y}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle \mathbf{y}_{i,\Delta t} - \mathbf{y}_i \rangle = A_i(\mathbf{y}, t) \tag{5}$$

$$K_2(\mathbf{y}) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle (\mathbf{y}_{i,\Delta t} - \mathbf{y}_i)(\mathbf{y}_{j,\Delta t} - \mathbf{y}_j) \rangle = B_{ij}(\mathbf{y}, t) \tag{6}$$

exist, where the symbol $\langle\ \rangle$ indicates ensemble average,*

(*iii*) A_i, B_{ij} are continuous and bounded,

(*iv*) $\partial A_i / \partial y_j$ exist for every y_j, and are continuous and bounded,

* A Markov process which satisfies condition (*ii*) is said to be continuous.

(*v*) $\partial B_{ij}/\partial y_i$ and $\partial^2 B_{ij}/\partial y_i \partial y_j$ exist for every y_i and y_j, and are continuous and bounded, and

(*vi*) $B_{ij}(\mathbf{y}, t)$ form a positive definite matrix,

then the conditional probability $p(\mathbf{y}_0 \mid \mathbf{y}, t)$ of the process $y(t)$ satisfies the Fokker–Planck equation

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial y_i}[A_i p] + \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2}{\partial y_i \, \partial y_j}[B_{ij} p] \tag{7}$$

and the initial condition

$$p(\mathbf{y}_0 \mid \mathbf{y}, t) = \prod_{i=1}^{N} \delta(y_i - y_{i0}) \quad \text{as} \quad t \to 0, \tag{8}$$

where $\mathbf{y}_0$ is the initial state of $\mathbf{y}$ and δ represents the Dirac delta function.

Consider a class of n-degree-of-freedom nonlinear discrete dynamic systems whose motions are defined by the following system of differential equations

$$\begin{aligned} \ddot{x}_i + a_i \dot{x}_i[1 + \epsilon_i D_i(x_1, x_2, \cdots, x_n, \dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n)] \\ + b_i x_i[1 + \mu_i S_i(x_1, x_2, \cdots, \dot{x}_n, \dot{x}_1, \cdots, \dot{x}_n)] = \eta_i(t) \\ i = 1, 2, \cdots, n \end{aligned} \tag{9}$$

where a_i and b_i are linear damping and stiffness coefficients, respectively; ϵ_i and μ_i are nonlinear parameters; D_i and S_i are nonlinear functions; and $\eta_i(t)$ are excitations and are, in general, random processes.

It is always convenient to transform the motion in n-dimensional generalized coordinates space into a $2n$-dimensional phase-space, that is, let $y_i = x_i$ and $y_{i+1} = \dot{x}_i$. Equation (9) can be written in a system of $2n$ first order differential equations as

$$\dot{y}_i = f_i(y_1, y_2, \cdots, y_{2n}) + \eta_i(t) \qquad i = 1, 2, \cdots, 2n. \tag{10}$$

Assuming $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t_1)\eta_j(t_2) \rangle = S_{ij}\delta_{ij}(t_1 - t_2)$ for constant D_{ij} and applying equations (4) through (6) to equation (9), we obtain

$$\begin{aligned} A_i(\mathbf{y}, t) &= y_{i+1}, \\ A_{i+1}(\mathbf{y}, t) &= -a_i y_{i+1}[1 + \epsilon_i D_i(y_1, \cdots, y_{2n})] \\ &\qquad - b_i y_i[1 + \mu_i S_i(y_1, \cdots, y_{2n})], \\ B_{ii}(\mathbf{y}, t) &= 0, \\ B_{ij}(\mathbf{y}, t) &= 0, \end{aligned}$$

and

$$B_{i+1,i+1}(\mathbf{y}, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left\{ \int_t^{t+\Delta t} \eta_i(\tau)\, d\tau - a_i y_{i+1}[1 + \epsilon_i D_i]\, \Delta t - b_i y_i [1 + \mu_i S_i]\, \Delta t \right\}^2 \right\rangle$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \int_t^{t+\Delta t} \int_t^{t+\Delta t} \eta_i(\tau_i) \eta_i(\tau_2)\, d\tau_1\, d\tau_2 + O(\Delta t^2) \right\rangle$$

$$= S_{ii} \ .$$

From the above results we notice that the coefficients A_{i+1} are determined by the specific nonlinear damping and nonlinear spring functions while coefficients B_{ij} depend only upon the statistical properties of the random excitation functions. Therefore, we can conclude that for the random response components $y_i(t)$ in equation (10) if the limits of their first and second increments

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle \Delta y_i(t) \rangle \quad \text{and} \quad \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle \Delta y_i(t)\ \Delta y_j(t) \rangle$$

exist, then the probability density function of the many random functions $y_1, \cdots, y_{2n}$ satisfies the Fokker-Planck equation (7).

Equation (7) can be written as

$$\dot{p} = -\sum_{i=1}^{2n} \frac{\partial G_i(\mathbf{y})}{\partial y_i} \tag{11}$$

where

$$G_i(\mathbf{y}) = A_i(\mathbf{y})p - \frac{1}{2} \sum_{j=1}^{2n} \frac{\partial}{\partial y_j} [B_{ij}(\mathbf{y})p] \tag{12}$$

describe the components of a probability current vector

$$\mathbf{G} = (G_1\ , G_2\ , \cdots\ , G_{2n}).$$

Because the general solution to the above multidimensional parabolic partial differential equation with arbitrary boundary condition is difficult and impracticable to find, no such attempt is made in this study. However, with certain constraints on the properties of the system as well as on the characteristics of the random input, equation (11) may be reduced to simpler and more readily solvable forms. Such forms are considered in Section III.

III. STATIONARY SOLUTION OF JOINT PROBABILITY DISTRIBUTION

In the limiting case of equation (11) when $t \to \infty$ the transition probability density $p(\mathbf{y}_0 \mid \mathbf{y})$ tends to a stationary joint probability density $p_{st}(\mathbf{y}) = p_{st}(y_1, y_2, \cdots, y_{2n})$ which is independent of the initial conditions and approaches a stationary (steady state) value as the time of passage is sufficiently large. Under this stationary situation, equation (11) becomes

$$\sum_{i=1}^{2n} \frac{\partial G_i(\mathbf{y})}{\partial y_i} = 0 \tag{13}$$

or

$$\overline{\nabla}_{y_i} \cdot \mathbf{G} = 0, \qquad i = 1, 2, \cdots, 2n. \tag{14}$$

Therefore $\mathbf{G}$ may be regarded as being incompressible and there are no sources or sinks in the region R. Notice that since rotational probability flows can occur even for cases of zero boundary conditions, that is, the probability current $\mathbf{G}(G_1, G_2, \cdots, G_{2n})$ satisfies

$$G_i(\mathbf{y}) = 0, \qquad i = 1, 2, \cdots, 2n \tag{15}$$

on the boundary of the region R under consideration, $\mathbf{G}$ may not vanish within R. In a special case, however, discussed in Section 3.1, the current vector $\mathbf{G}$ vanishes in the whole region R.

Two situations which readily yield steady state solutions to equation (10) are investigated in Section 3.1.

3.1 *Potential Distribution*

Under the assumption that the probability current vector $\mathbf{G}$ vanishes everywhere in R, a solution of the potential form $\exp(-U)$ can be constructed where U is a Liapunov type potential function of the system.

If the equations of motion of a dynamic system satisfy the following conditions:

(*i*) $G_i =$ for all i,
(*ii*) the matrix $[B_{ij}]$ is not singular,

$$(iii)\ \frac{\partial}{\partial y_\alpha} \sum_i D_{\beta i}\left(\sum_j \frac{\partial B_{ij}}{\partial y_j} - 2A_i\right) = \frac{\partial}{\partial y_\beta} \sum_i D_{\alpha i}\left(\sum_j \frac{\partial B_{ij}}{\partial y_j} - 2A_i\right), \qquad \alpha, \beta = 1, 2, \cdots, 2n$$

and

(*iv*) there is no probability flow through the boundary of R,

then the solution to the steady state Fokker-Planck equation (13) is

$$p_{st}(\mathbf{y}) = C \exp\left\{-\int_{a_1,a_2,\cdots,a_{2n}}^{y_i,y_2,\cdots,y_{2n}} \sum_{\alpha}\left[\sum_{i,j} D_{\alpha i}\frac{\partial B_{ij}}{\partial y_j} - 2\sum_i D_{\alpha i}A_i\right] dy_\alpha\right\} \tag{16}$$

where $[D_{\alpha i}] = [B_{ij}]^{-1}$, that is,

$$\sum_i D_{\alpha i}B_{ij} = \delta_{\alpha j} ,$$

and C is the normalization constant determined by

$$\int_{2n\text{-fold}} \cdots \int p_{st}\, dy_1 , \cdots , dy_{2n} = 1. \tag{17}$$

Equation (16) can be easily verified by a direct substitution of $p(\mathbf{y}) = \exp(-U)$ into equation (13) to solve for U. A special case of interest is when the matrix $[B_{ij}]$ is isotropic, that is, when

$$[B_{ij}] = B(\mathbf{y})[\delta_{ij}]. \tag{18}$$

The solution for such a case, which can be obtained by substituting equation (18) into (16), is

$$\begin{aligned} p_{st}(\mathbf{y}) &= C \exp(-U) \\ U &= \log\frac{1}{B(\mathbf{y})} - 2\int_{a_1,\cdots,a_{2n}}^{y_i,\cdots,y_{2n}} \sum_i^{2n} \frac{A_i}{B(\mathbf{y})}\, dy_i \end{aligned} \tag{19}$$

where C is the usual normalization factor.

3.2 *Uncoupled Distribution*

There are cases when all generalized response variables of a system in the $2n$ phase-space coordinates are independent of one another. There are two approaches by which the solutions can be achieved.

3.2.1 *Forward Approach*

The governing Fokker–Planck equation may be reduced to a form in which the final probability distribution function clearly shows a statistical independent character. For this type of motion it is sometimes possible to find appropriate partial operators which, when linearly operated on functions of the type $g_i(y_i)p + h_i(y_i)(\partial p/\partial y_i)$, generate an equivalent nonlinear partial differential equation. If equation (13) can be put in the form

$$\sum_{i=1}^{2n} L_i \left[g_i(y_i)p + h_i(y_i) \frac{\partial p}{\partial y_i} \right] = 0 \tag{20}$$

where the coefficients L_i are arbitrary first order partial differential operators, and if there exists a $p(\mathbf{y})$ satisfying each

$$g_i(y_i)p + h_i(y_i) \frac{\partial p}{\partial y_i} = 0, \qquad (i = 1, 2, \cdots, 2n),$$

then by Gray's uniqueness theorem such $p(\mathbf{y})$ is the unique solution of equation (20).[18] Such a solution is

$$p_{st}(\mathbf{y}) = C \prod_{i=1}^{2n} \exp \left[- \int_0^{y_i} \frac{g_i(\lambda_i)}{h_i(\lambda_i)} \, d\lambda_i \right] \tag{21}$$

and C is the normalization factor.

Notice that previous investigators such as Ariaratnam and Klein had their problems satisfying equation (20) and therefore obtained their solutions in a form similar to equation (21).

3.2.2 *Backward Approach*

Sometimes it is more convenient to work backward. By this procedure, the statistically independent property is assumed in order to derive the desirable solution for the Fokker-Planck equation under investigation. This method gives a close insight into the physical properties of the system and enables the natural boundary conditions to be deduced. These deduced boundary conditions provide the necessary constraints for randomly excited systems which have independent distributions in their response variables.

If equation (13) satisfies the conditions (22)

(*i*) $[B_{ij}]$ is a constant diagonal matrix for $i, j = 1, 2, \cdots, 2n$, and,

(*ii*) the first order intensity coefficients A_i are functions of y_i and y_{i+1} only, and there are no cross terms in A_i, that is,

$$\left.\begin{aligned} A_i &= A_{i,i}(y_i) + A_{i,i+1}(y_{i+1}) \\ A_{i+1} &= A_{i+1,i}(y_i) + A_{i+1,i+1}(y_{i+1}) \end{aligned}\right\} \quad \text{for} \quad i = 1, 3, \cdots, 2n-1; \tag{23}$$

then by setting $p_{st}(y) = \prod_{i=1}^{2n} p_i(y_i)$ in equation (13), we obtain

$$\frac{1}{2} \sum_{i=1}^{2n} B_{ii} \frac{p(\mathbf{y})}{p_i(y_i)} \frac{\partial^2}{\partial y_i^2} p_i(y_i) = \sum_{i=1}^{2n} \left[\frac{p(\mathbf{y})}{p_i(y_i)} (A_{i,i} + A_{i,i+1}) \frac{\partial p_i(y_i)}{\partial y_i} + p(\mathbf{y}) \frac{\partial A_{i,i}(y_i)}{\partial y_i} \right].$$

Dividing the above equation by $p(\mathbf{y})$, we obtain

$$\frac{1}{2}\sum_{i=1}^{2n} B_{ii}\frac{p_i''}{p_i} = \sum_{i=1}^{2n}\left[(A_{i,i} + A_{i,i+1})\frac{p_i'}{p_i} + A_{i,i}'\right]$$

where the $'$ denotes $\partial/\partial y_i$.

The above equation can be reorganized as

$$\sum_{i=1}^{2n}\left[\frac{B_{ii}}{2}\frac{p_i''}{p_i} - \frac{p_i'}{p_i}A_{i,i} - A_{i,i}'\right] = \sum_{i=1,3,\cdots}^{2n-1}\left[\frac{p_i'}{p_i}A_{i,i+1} + \frac{p_{i+1}'}{p_{i+1}}A_{i+1,i}\right].$$

A sufficient solution $p(\mathbf{y})$ for the above equation requires that it satisfies the following relations

$$\text{(i)}\quad \frac{B_{ii}}{2}\frac{p_i''}{p_i} - \frac{p_i'}{p_i}A_{i,i} - A_{i,i}' = 0 \quad \text{for} \quad i = 1, 2, \cdots, 2n \tag{24}$$

and

$$\text{(ii)}\quad \frac{p_i'}{p_i}\frac{1}{A_{i+1,i}} = -\frac{p_{i+1}'}{p_{i+1}}\frac{1}{A_{i,i+1}} = E = \text{constant}$$

$$\text{for} \quad i = 1, 3, \cdots, 2n - 1. \tag{25}$$

Equation (24) can be reduced to

$$\frac{B_{ii}}{2}\frac{d}{dy_i}(p_i') = \frac{d}{dy_i}(p_iA_{i,i}).$$

Integration with respect to y_i yields

$$\frac{B_{ii}}{2}p_i' = p_iA_{i,i} + c_1 .$$

When compared with equation (25), c_1 vanishes and the following condition must hold

$$\frac{2A_{i,i}}{A_{i+1,i}B_{i,i}} = -\frac{2A_{i+1,i+1}}{A_{i,i+1}B_{i+1,i+1}} = E, \quad (i = 1, 3, \cdots, 2n - 1). \tag{26}$$

Equation (26) is the natural restraint under which the backward method can be applied. From the above analysis and the uniqueness theorem stated, we can claim. If the steady state Fokker–Planck equation satisfies conditions (22), (23), and (26), and if the phase-space coordinates y_1 , y_2 , $\cdots$, y_{2n} are statistically independent, then the unique solution $p_{st}(\mathbf{y})$ of equation (13) is

$$p_{st}(\mathbf{y}) = C \prod_{i=1,3,\cdots}^{2n-1} \exp\left[\int_0^{y_i} EA_{i+1,i}(\lambda_i)\, d\lambda_i - \int_0^{y_{i+1}} EA_{i,i+1}(\lambda_{i+1})\, d\lambda_{i+1}\right] \tag{27}$$

where C is the normalized constant.

IV. EXAMPLES

4.1 *Randomly Excited Nonlinear Simple Mechanical Oscillator*

When subjected to dynamic loadings $\eta(t)$ the equation of motion for a single-mode oscillator with nonlinear spring $F(x)$ is

$$\ddot{x} + \beta\dot{x} + F(x) = \eta(t) \tag{28}$$

where the excitation $\eta(t)$ is a gaussian stationary process with

$$\left.\begin{aligned} \langle \eta(t) \rangle &= 0 \\ \langle \eta(t_1)\eta(t_2) \rangle &= 2\pi S_o\, \delta(t_1 - t_2) \end{aligned}\right\} \tag{29}$$

in which S_o is a constant power spectral density.

Letting $y_1 = x$ and $y_2 = \dot{x} = \dot{y}_1$, the intensity coefficients A_i and B_{ij} can be found by using equations (4) and (5) as follows

$$A_1 = A_{11} + A_{12} = y_2, \quad \text{hence} \quad A_{11} = 0, \qquad A_{12} = y_2;$$

$$A_2 = A_{21} + A_{22} = -F(y_1) - \beta y_2, \quad \text{hence} \quad A_{21} = -F(y_1), \qquad A_{22} = -\beta y_2;$$

$$B_{22} = 2\pi S_o .$$

Therefore the governing Fokker-Planck equation is

$$\pi S_o \frac{\partial^2 p}{\partial y_2^2} - \frac{\partial}{\partial y_1}(y_2 p) + \frac{\partial}{\partial y_2}\{[F(y_1) + \beta y_2]p\} = 0. \tag{30}$$

Forward approach:

Equation 30 can be written in the form of equation (20) as

$$\frac{\partial}{\partial y_2}\left[F(y_1)p + \frac{\pi S_o}{\beta}\frac{\partial p}{\partial y_1}\right] + \left[\beta\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_1}\right]\left[y_2 p + \frac{\pi S_o}{\beta}\frac{\partial p}{\partial y_2}\right] = 0,$$

from which we see that

$$L_1 = \frac{\partial}{\partial y_2}, \qquad L_2 = \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_1},$$

$$g_1(y_1) = F(y_1), \qquad h_1(y_1) = \frac{\pi S_o}{\beta},$$

and

$$g_2(y_2) = y_2, \qquad h_2(y_2) = \frac{\pi S_o}{\beta}.$$

Substitution of g_1, g_2, h_1 and h_2 into equation (21) gives the following steady state solution for the system described by equation (28)

$$\begin{aligned} p_{st}(\mathbf{y}) &= C \exp\left\{-\int_0^{y_1} \frac{\beta F(\zeta_1)}{\pi S_o}\, d\zeta_1 - \int_0^{y_2} \frac{\beta \zeta_2}{\pi S_o}\, d\zeta_2\right\} \\ &= C \exp\left\{-\frac{\beta}{\pi S_o}\left[\int_0^{y_1} F(\zeta_1)\, d\zeta_1 + \frac{y_2^2}{2}\right]\right\}. \end{aligned} \tag{31}$$

Backward approach:

In this two-dimensional case, it can be shown that condition (26) is satisfied:

$$E = \frac{2A_{11}}{B_{11}A_{21}} = \frac{-2A_{22}}{B_{22}A_{12}} = \frac{-2(-\beta y_2)}{2\pi S_o y_2} = \frac{\beta}{\pi S_o}.$$

Therefore, according equations (22) through (27) the stationary solution can be written:

$$\begin{aligned} p_{st}(y_1, y_2) &= C \exp\left[\int_0^{y_1} EA_{21}(\zeta_1)\, d\zeta_1 - \int_0^{y_2} EA_{12}(\zeta_2)\, d\zeta_2\right] \\ &= C \exp\left\{-\frac{\beta}{\pi S_o}\left[\int_0^{y_1} F(\zeta_1)\, d\zeta_1 + \frac{y_2^2}{2}\right]\right\} \end{aligned}$$

which is the same as equation (31).

4.1.1 *Cubic Stiffness Characteristics*

Let us consider various nonlinear spring resistance functions $F(x)$. The group of cubic stiffness characteristics is the classical case concerned with the hardening spring type of nonlinearity, which is represented by

$$F(x) = k_o x + \epsilon x^3, \tag{32}$$

where k_o is the initial linear stiffness and ϵ is the nonlinear coefficient.

Substituting equation (32) into (31), we obtain

$$p_{st}(x, \dot{x}) = C \exp\left[-\frac{\beta}{\pi S_o}\left(\frac{k_o x^2}{2} + \frac{\epsilon x^4}{4} + \frac{\dot{x}^2}{2}\right)\right]. \qquad (33)$$

Notice from equation (33) that the marginal probability density disitrbutions for x and $\dot{x}$ are statistically independent and that $p(\dot{x})$ follows the gaussian law. As $\epsilon \to 0$, the system becomes linear and $p(x)$ approaches gaussian.

4.1.2 *Tangent Stiffness Characteristics*[19]

The spring function $F(x)$ in this case is shown in Fig. 1 and is represented by

$$F(x) = \left(\frac{2k_o d}{\pi m}\right) \tan\left(\frac{\pi x}{2d}\right), \qquad -d < x < d \qquad (34)$$

in which m and d are constants. Notice that d is the threshold of the oscillator and m can be regarded as the mass of the oscillator. Again it follows from equation (31) that

$$p_{st}(x, \dot{x}) = p(\dot{x})p(x)$$
$$= C\left[\exp\left(-\frac{\dot{x}^2}{2\sigma_o^2\omega_o^2}\right)\exp\left(\frac{4d^2}{\pi^2\sigma_o^2}\ln\cos\frac{\pi x}{2d}\right)\right], \qquad (35)$$

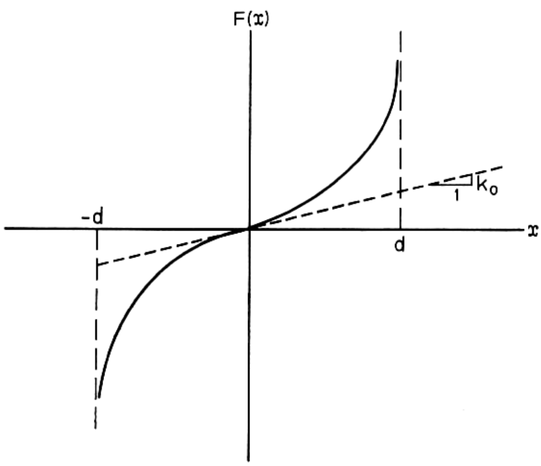

Fig. 1 — Tangent stiffness characteristics.

where $\omega_o^2 = k_o/m$ and $\sigma_o^2 = \pi S_o/2\beta\omega_o^2$ is the corresponding linear mean square response [that is, if $F(x) = k_o x$].

The cosine-power distribution $p(x)$ is shown in Fig. 2 for various values of $n = 4d^2/\pi^2\sigma_o^2$. It is interesting that for fixed σ_o^2, $p(x)$ approaches the gaussian distribution as $d \to \infty$ and approaches the uniform distribution as $d \to 0$.

4.1.3 *Hyperbolic Tangent Stiffness Characteristics*

Figure 3 shows a hyperbolic tangent stiffness model; the spring resistance function $F(x)$ is given by

$$F(x) = \frac{k_o}{mb} \tanh (bx), \qquad b, k_o, m > 0. \tag{36}$$

Notice that the resistance $F(x)$ developed during the vibration is bounded between k_o/bm and $-k_o/bm$. Therefore k_o/bm may be regarded as the yield level and $1/bm$ the corresponding yielding response; equation (36) can be used to model the familiar elastic-perfect-plastic behavior observed in many physical realities. The joint probability density function for this can be obtained from equation (31),

$$p_{st}(x, \dot{x}) = p(\dot{x})p(x) = C \exp\left[-\frac{\dot{x}^2}{2\omega_o^2\sigma_o^2} - \frac{1}{b^2\sigma_o^2}\ln\cosh(bx)\right]. \tag{37}$$

There are some interesting features about the limiting behavior of the marginal distribution $p(x)$ which is of sech-power type

$$p(x) = C_1(b)\,\mathrm{sech}^{(1/\sigma_o^2 b^2)}\, bx. \tag{38}$$

The sech-power term in equation (38) belongs to a class of distribution function $f_n(x)$ of a monotone decreasing sequence, integrable on $[-\infty, \infty]$ such that $f_n(x) \leqq 1$ for all x. It can be shown that

$$\lim_{n\to\infty}\left[f_n(x)\Big/\int_{-\infty}^{\infty} f_n(x)\,dx\right] = 0;^{20} \text{ and for } \lim_{b\to 0} p(x) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_o}\exp\left(-\frac{x^2}{2\sigma_o^2}\right),$$

the limit of the distribution function in equation (38) at zero, that is, $\lim_{b\to 0} p(x)$ converges positively to a normal distribution with zero mean and variance σ_o^2. The sech-power distribution $p(x)$ is shown in Fig. 4 for various values of $m' = 1/\sigma_o^2 b^2$.

4.2 *Random Vibration of a Charged Particle Moving in an Electromagnetic Field*

As a second example, we consider a particle of mass m carrying charge q subjected to a random loading $\mathbf{n}(t) = \eta_1(t)\hat{\imath} + \eta_2(t)\hat{\jmath} + \eta_3(t)\hat{k}$

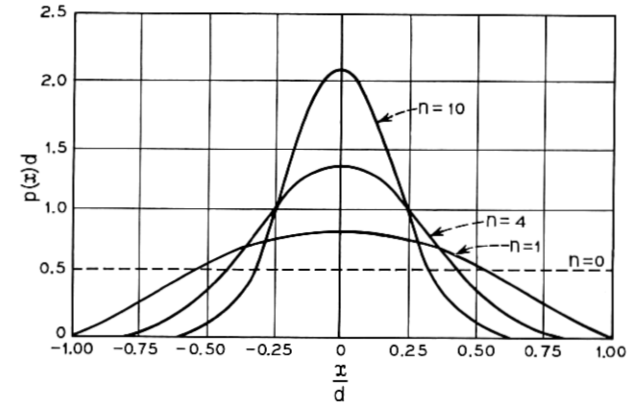

Fig. 2 — Cosine-power probability density distribution function.

where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are base vectors in the Cartesian coordinates. The particle, moving in an electromagnetic field $\mathbf{M} = M_1(x, y, z)\hat{\imath} + M_2(x, y, z)\hat{\jmath} + M_3(x, y, z)\hat{k}$, is also subjected to a friction force $F_f = -\nabla_v \mathfrak{F}$ where $\mathfrak{F} = \frac{1}{2}(\lambda_1 v_x^2 + \lambda_2 v_y^2 + \lambda_3 v_z^2)$ is an energy dissipation function. The complete force on the particle is therefore,

$$\mathbf{F} = q\left\{-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{M}}{\partial t} + \frac{1}{c}(\mathbf{v} \times \nabla \times \mathbf{M})\right\} \tag{39}$$

where ϕ is the scalar potential and c is a constant.

If we introduce a velocity-dependent potential w, such that

$$w = q\phi - \frac{q}{c}\mathbf{M}\cdot\mathbf{v} \tag{40}$$

then equation (39) can be written as

$$\mathbf{F} = -\nabla_d w + \frac{d}{dt}\nabla_v w. \tag{41}$$

Therefore the Lagrangian function L can be expressed in terms of w as

$$L = T - q\phi + \frac{q}{c}\mathbf{M}\cdot\mathbf{v} \tag{42}$$

where T represents the kinetic energy of the particle, and the equation of motion of the charged particle can be derived from the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathfrak{F}}{\partial \dot{q}_i} = \eta_i(t) \tag{43}$$

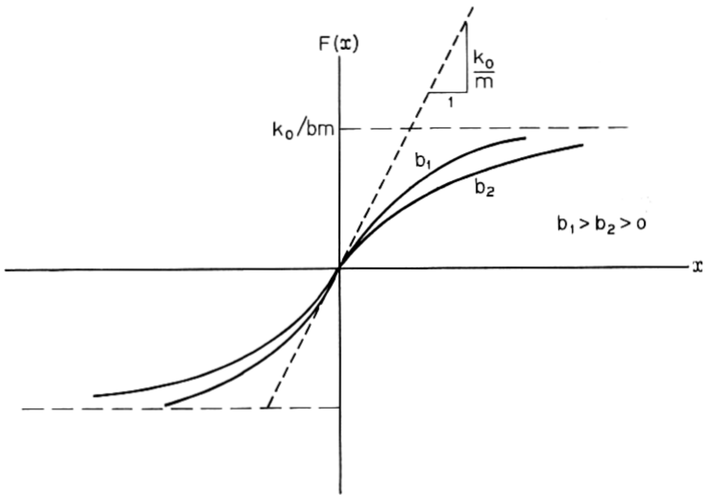

Fig. 3 — Hyperbolic tangent stiffness characteristics.

where q_i represent the generalized coordinates of the motion. Using equations (42) and (43), one obtains

$$m\ddot{u}_1 + q\,\frac{\partial \phi}{\partial u_1} + \frac{q}{c}\left(\frac{\partial M_1}{\partial t} + \frac{\partial M_1}{\partial u_3}\,u_4 + \frac{\partial M_1}{\partial u_5}\,u_6 - \frac{\partial M_2}{\partial u_1}\,u_4 - \frac{\partial M_3}{\partial u_1}\,u_6\right) + \lambda_1 u_2 = \eta_1$$

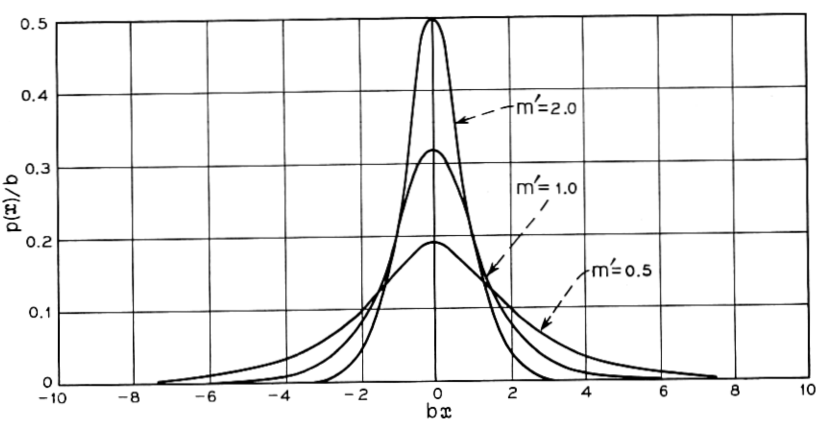

Fig. 4 — Sech-power probability density distribution function.

$$m\ddot{u}_3 + q\frac{\partial\phi}{\partial u_3} + \frac{q}{c}\left(\frac{\partial M_2}{\partial t} + \frac{\partial M_2}{\partial u_1}u_2 + \frac{\partial M_2}{\partial u_5}u_6 - \frac{\partial M_1}{\partial u_3}u_2 - \frac{\partial M_3}{\partial u_3}u_6\right) + \lambda_2 u_4 = \eta_2 \tag{44}$$

$$m\ddot{u}_5 + q\frac{\partial\phi}{\partial u_5} + \frac{q}{c}\left(\frac{\partial M_5}{\partial t} + \frac{\partial M_3}{\partial u_3}u_4 + \frac{\partial M_3}{\partial u_1}u_2 - \frac{\partial M_1}{\partial u_5}u_2 - \frac{\partial M_2}{\partial u_5}u_4\right) + \lambda_3 u_6 = \eta_3$$

where in equation (44), transformations $u_1 = x$, $u_2 = \dot{x} = \dot{u}_1$, $u_3 = y$, $u_4 = \dot{y} = \dot{u}_3$, $u_5 = z$, and $u_6 = \dot{z} = \dot{u}_5$ have been made.

Define

$$\left.\begin{aligned} D_1 &= (d_{13} - d_{21})u_4 + (d_{15} - d_{31})u_6 \\ D_2 &= (d_{21} - d_{13})u_2 + (d_{25} - d_{33})u_6 \\ D_3 &= (d_{31} - d_{15})u_2 + (d_{33} - d_{25})u_4 \end{aligned}\right\} \tag{45}$$

where $d_{\alpha\beta} = d_{\alpha\beta}(u_1, u_3, u_5) = \partial M_\alpha/\partial u_\beta$, $\alpha = 1, 2$, or 3 and $\beta = 1, 3$, or 5, are independent of the velocity variables.

The following properties are assumed for the forcing function $\mathbf{n}(t)$

$$\langle \eta_\alpha \rangle = 0 \qquad \text{for} \quad \alpha = 1, 2, \text{ or } 3 \tag{46}$$

$$\left.\begin{aligned} \langle \eta_\alpha(t_1)\eta_\alpha(t_2)\rangle &= 0 & \text{for} \quad \alpha \neq \beta \\ &= 2\pi S_\alpha(t_1 - t_2) & \text{for} \quad \alpha = \beta \end{aligned}\right\}. \tag{47}$$

From equations (44) through (47), the six-dimensional Fokker–Planck equation governing the transition probability density $p(u_1, u_2, \cdots, u_6)$, derived by the standard technique, is

$$\begin{aligned} \dot{p}(u_1, \cdots, u_6) &= -\frac{\partial}{\partial u_1}(u_2 p) - \frac{\partial}{\partial u_2}\left\{\left[-\frac{\lambda_1}{m}u_2 - \frac{q}{m}\frac{\partial\phi}{\partial u_1} - \frac{q}{mc}\left(\frac{\partial M_1}{\partial t} + D_1\right)\right]p\right\} \\ &\quad - \frac{\partial}{\partial u_3}(u_4 p) - \frac{\partial}{\partial u_4}\left\{\left[-\frac{\lambda_2}{m}u_4 - \frac{q}{m}\frac{\partial\phi}{\partial u_3} - \frac{q}{mc}\left(\frac{\partial M_2}{\partial t} + D_2\right)\right]p\right\} \\ &\quad - \frac{\partial}{\partial u_5}(u_6 p) - \frac{\partial}{\partial u_6}\left\{\left[-\frac{\lambda_3}{m}u_6 - \frac{q}{m}\frac{\partial\phi}{\partial u_5} - \frac{q}{mc}\left(\frac{\partial M_3}{\partial t} + D_3\right)\right]p\right\} \\ &\quad + \frac{\pi}{m^2}\left[S_1\frac{\partial^2 p}{\partial u_2^2} + S_2\frac{\partial^2 p}{\partial u_4^2} + S_3\frac{\partial^2 p}{\partial u_6^2}\right]. \end{aligned} \tag{48}$$

The initial condition for equation (48) is

$$p(u_1, u_2, \cdots, u_6)\,|_{t_o} = \prod_{i=1}^{6} \delta[u_i(t) - u_i(t_o)]. \tag{49}$$

At $t \to \infty$, the motion becomes stationary; therefore $\dot{p} = 0$ and $p = p_{st}$ if $\partial \mathbf{M}/\partial t = 0$. Imposing these conditions on equation (48) and making use of equation (45), the corresponding steady state equation for (48) is:

$$\left(\frac{\lambda_1}{m^2}\frac{\partial}{\partial u_2} - \frac{1}{m}\frac{\partial}{\partial u_1}\right)\left(mu_2 p + \frac{\pi S_1}{\lambda_1}\frac{\partial p}{\partial u_2}\right) + \frac{1}{m}\frac{\partial}{\partial u_2}\left(q\frac{\partial \phi}{\partial u_1}p + \frac{\pi S_1}{\lambda_1}\frac{\partial p}{\partial u_1}\right)$$
$$+ \left(\frac{\lambda_2}{m^2}\frac{\partial}{\partial u_4} - \frac{1}{m}\frac{\partial}{\partial u_3}\right)\left(mu_4 p + \frac{\pi S_2}{\lambda_2}\frac{\partial p}{\partial u_4}\right) + \frac{1}{m}\frac{\partial}{\partial u_4}\left(q\frac{\partial \phi}{\partial u_3}p + \frac{\pi S_2}{\lambda_2}\frac{\partial p}{\partial u_3}\right)$$
$$+ \left(\frac{\lambda_3}{m^2}\frac{\partial}{\partial u_6} - \frac{1}{m}\frac{\partial}{\partial u_5}\right)\left(mu_6 p + \frac{\pi S_3}{\lambda_3}\frac{\partial p}{\partial u_6}\right) + \frac{1}{m}\frac{\partial}{\partial u_6}\left(q\frac{\partial \phi}{\partial u_5}p + \frac{\pi S_3}{\lambda_3}\frac{\partial p}{\partial u_5}\right)$$
$$+ q\left(D_1\frac{\partial p}{\partial u_2} + D_2\frac{\partial p}{\partial u_4} + D_3\frac{\partial p}{\partial u_6}\right) = 0. \tag{50}$$

Notice that in the above equation $p = p_{st}$. Terms in the last parentheses of equation (50) vanish if

$$(i) \quad D_1 = D_2 = D_3 = 0 \tag{51a}$$

or

$$(ii) \quad u_2 : u_4 : u_6 = (d_{25} - d_{33}) : (d_{31} - d_{15}) : (d_{13} - d_{21}) \tag{51b}$$

or

$$(iii) \quad \nabla \times \mathbf{M} = 0 \quad \text{or} \quad \mathbf{M} = G \tag{51c}$$

where G is a scalar potential function of x, y and z.

If any one of these conditions is satisfied, equation (50) is of the same form as equation (20) and its solution can be found immediately by using equation (21), that is:

$$p_{st}(u_1, \cdots, u_6)$$
$$= C \exp\left[-\frac{m\lambda_1}{\pi S_1}\left(\frac{u_2^2}{2}\right) - \frac{q\lambda_1}{\pi S_1}\int_0^{u_1}\frac{\partial \phi}{\partial \zeta_1}\,d\zeta_1 - \frac{m\lambda_2}{\pi S_2}\left(\frac{u_4^2}{2}\right)\right.$$
$$\left. - \frac{q\lambda_2}{\pi S_2}\int_0^{u_3}\frac{\partial \phi}{\partial \zeta_3}\,d\zeta_3 - \frac{m\lambda_3}{\pi S_3}\left(\frac{u_6^2}{2}\right) - \frac{q\lambda_3}{\pi S_3}\int_0^{u_5}\frac{\partial \phi}{\pi \zeta_5}\,d\zeta_5\right], \tag{52}$$

where C is the normalization factor determined by

$$\int \cdots \int p_{st}\, du_1 , \cdots , du_6 = 1.$$

If damping factors in three directions are identical and the random loading $\mathbf{n}$ is uniform, that is, if

$$\mathfrak{F} = \tfrac{1}{2}\lambda(v_x^2 + v_y^2 + v_z^2) \tag{53}$$

and

$$S_1 = S_2 = S_3 = S, \tag{54}$$

then equation (52) becomes

$$p_{st} = C \exp\left[\frac{-\lambda}{\pi S}(T + q\phi)\right]. \tag{55}$$

Applying this result to single-mode conservative oscillators with potential $V(x)$ and subjected to gaussian white random loadings, the stationary response probability density is

$$p_{st} = C \exp\left(-\frac{\lambda}{\pi S} H\right) \tag{56}$$

where $H = T + V$ is the Hamiltonian function of the system.

It is interesting that if the magnetic vector potential is irrotational, the steady state response probability density of a charged particle under white noise type random disturbances is statistically independent. The solution can be immediately written down in terms of a quadrature. Extension of this result to a conservative dynamic system shows that stationary probability solution is of the form $p_{st} = C \exp[-f(\gamma, H)]$, where H is the Hamiltonian function of the system and γ is a coefficient depending on the random input characteristics and the energy dissipation mechanism of the system.

APPENDIX A

Perturbation Technique

The perturbation method is based on a series expansion in powers of the nonlinearity coefficient ϵ. This method is valid for small values of ϵ only. For example, consider a single-mode oscillator with nonlinear damping and stiffness, when subjected to random force $\eta(t)$, the equation of motion is

$$\ddot{x} + 2\beta\dot{x}(1 + \epsilon \sum_n b_n\dot{x}^n) + \omega_o^2 x(1 + \epsilon \sum_n a_n x^n) = \eta(t). \tag{57}$$

A series solution can be assumed*

$$x(t) = x_o + \epsilon x_1 + \epsilon^2 x_2 + \cdots + \epsilon^n x_n . \tag{58}$$

From equations (57) and (58) it follows that

$$\ddot{x}_o + 2\beta \dot{x}_o + \omega_o^2 x_o = \eta(t) \tag{59}$$

$$\ddot{x}_1 + 2\beta \dot{x}_1 + \omega_o^2 x_1 = -\sum_n (2\beta b_n \dot{x}_o^{n+1} + \omega_o^2 a_n x_o^{n+1}). \tag{60}$$

Therefore the correlation function is

$$R_{xx}(\tau) = R_{x_o x_o}(\tau) + \epsilon R_{x_o x_1}(\tau) + \epsilon R_{x_1 x_o}(\tau) \tag{61}$$

where

$$R_{xy} = \langle x(t) y(t + \tau) \rangle . \tag{62}$$

Equations (59) and (60) are linear and their solutions can be readily obtained

$$x_o(t) = \int_0^\infty h(\tau) \eta(t - \tau) \, d\tau, \tag{63}$$

and

$$x_1(t) = -\int_0^\infty h(\tau) \sum_n [2\beta b_n \dot{x}_o^{n+1}(t - \tau) + \omega_o^2 a_n x_o^{n+1}(t - \tau)] \, d\tau \tag{64}$$

where $h(\tau) = [e^{-\beta\tau}/\omega_o(1 - \beta^2)^{\frac{1}{2}}] \sin (1 - \beta^2)^{\frac{1}{2}} \omega_o \tau$ is the transfer function for system described by equation (59).

From equations (63) and (64), the nonlinear response moments can be found. Considering only the first-order perturbation,

$$R_{x_o x_1}(t) = -\int_0^\infty h(\tau) \sum_n [2\beta b_n \langle x_o(t) \dot{x}_o^{n+1}(t - \tau)\rangle + \omega_o^2 a_n \langle x_o(t) x_o^{n+1}(t - \tau)\rangle] \, d\tau,$$

which can be evaluated explicitly if $x_o(t)$ is gaussian.

APPENDIX B

Equivalent Linearization Technique

Consider the equation of motion

$$\ddot{x} + \beta \dot{x} + \omega_o^2 x + \epsilon g(x, \dot{x}, t) = \eta(t), \tag{65}$$

* Under fairly general conditions it can be shown that this series solution is convergent.

for a nonlinear function g, dependent on both x and $\dot{x}$. The following equation is said to be equivalent to equation (65) in the sense that the mean square deficiency is minimized:

$$\ddot{x} + \beta_e \dot{x} + \omega_e^2 x + e(x, \dot{x}, t) = \eta(t). \tag{66}$$

The deficiency $e(x, \dot{x}, t) = (\beta - \beta_e)\dot{x} + (\omega_o^2 - \omega_e^2)x + \epsilon g(x, \dot{x}, t)$, in which β_e and ω_e are equivalent damping and frequency is determined by,

$$\frac{\partial \langle e^2 \rangle_{\text{av}}}{\partial \beta_e} = 0 \quad \text{and} \quad \frac{\partial \langle e^2 \rangle_{\text{av}}}{\partial \omega_e^2} = 0, \tag{67}$$

where the $\langle\ \rangle_{\text{av}}$ indicates time average, that is,

$$\langle e^2 \rangle_{\text{av}} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^2(x, \dot{x}, t)\, dt. \tag{68}$$

Using equations (67) and (68), we obtain

$$\beta_e = \beta + \epsilon \frac{\langle \dot{x} g \rangle_{\text{av}}}{\langle \dot{x}^2 \rangle_{\text{av}}} \tag{69}$$

$$\omega_e^2 = \omega_o^2 + \epsilon \frac{\langle \dot{x} g \rangle_{\text{av}}}{\langle x^2 \rangle_{\text{av}}} \tag{70}$$

From equations (69) and (70) and by neglecting the deficiency term e, equation (66) can be solved by using standard linear theory. If the system is nonhereditary, the time averages in equations (67) through (70) are replaced by ensemble averages. β_e and ω_e^2 for this situation can be solved by a prior assumption for the probability density function of $x(t)$.

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