On Conditions Under Which It Is Possible To Synchronize Digital Transmission Systems

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J. R. Pierce has recently proposed a system for synchronizing an arbitrary number of geographically separated oscillators, and, under the assumption of zero transmission delays between stations, has shown that a certain linear model of the system is stable in the sense that all of the station frequencies approach a common final value as $t \to \infty$.

This paper reports on some results concerning the dynamic behavior of Pierce's linear model. The results take into account transmission delays. More explicitly, we prove that if a certain set of simple inequalities involving the delays is satisfied, then the system is stable and the oscillator frequencies approach their common final value at an exponential rate. These inequalities have the property that they are always satisfied for sufficiently small delays. This paper presents a simple example showing that the system can be unstable when the inequalities are not met. In addition, we present some information concerning the rate of decay of the natural modes of stable systems, discuss an alternative stability criterion not involving the transmission delays and derive an explicit expression for the final frequency. Finally, we discuss the mathematical relationship between Pierce's model and earlier models of synchronization systems.

I. INTRODUCTION, DISCUSSION, AND SUMMARY OF RESULTS

1.1 *The Model*

It is well known that the problem of synchronizing the frequencies of geographically separated oscillators is of importance in connection with the detection and switching of pulse-code-modulated signals.

J. R. Pierce has recently proposed a system for synchronizing digital transmission networks* The system uses oscillators of adjustable rate of change of frequency, buffers which accept pulses at an incoming rate and which produce corresponding output pulses at the local clock rate, and adequate delay at each station to enable PCM frames to be properly aligned in time.

In Pierce's model the content b_{ij} of the buffer at station i which accepts pulses from station j is assumed to satisfy the equation,†

$$\dot{b}_{ij} = f_j - f_i \tag{1}$$

in which f_j and f_i are the frequencies at stations j and i, respectively, and the overall system of coupled oscillators is assumed to be governed by the equations

$$\dot{f}_i = c_i(f_{0i} - f_i) + \sum_{j \neq i} a_{ij} b_{ij} \qquad i = 1, 2, \cdots, n \tag{2}$$

for $t \geqq 0$, where n is the number of stations, each c_i is a positive constant, f_{0i} is the center frequency of the oscillator at station i, and each a_{ij} is nonnegative and satisfies $a_{ij} = a_{ji}$.

In his discussion of equations (2), Pierce assumes that the transmission delay between stations i and j can be neglected for all $i \neq j$, in which case equation (1) can be interpreted as

$$\dot{b}_{ij}(t) = f_j(t) - f_i(t), \qquad t \geqq 0 \tag{3}$$

instead of as

$$\dot{b}_{ij}(t) = f_j(t - \tau_{ij}) - f_i(t), \qquad t \geqq 0 \tag{4}$$

with each τ_{ij} a nonnegative constant. Under the natural assumption that there is some path from each station to every other station, Pierce has shown, by directing attention to a passive RLC network analog of the equations,‡ that the system is stable in the

* The system is described in an unpublished memorandum which, after this paper was written, was considerably expanded and then published.[1] Since the memorandum did not explicitly consider the "$C = 0$ system" of Ref. 1, it is not mentioned above. A nonlinear version of the "$C = 0$ system" is considered in some detail by this writer in a paper to be published.

† A dot over a mathematical symbol denotes the derivative with respect to time.
The number $b_{ij}(t)$ denotes, to within an additive constant, an approximation to the product of a fixed constant and the number of pulses stored in the ijth buffer at time t.

‡ In the Laplace-transform domain, the equations can be written as the system of equations $sF + CF + s^{-1}AF = G$, in which F is the transpose of the Laplace transform of $(f_1, f_2, \ldots, f_n)$ and (with a set of common-ground node-voltage equations analogy in mind) C and A may be interpreted as a diagonal conductance parameter matrix and an elastance parameter matrix, respectively. The "source vector" G takes into account initial conditions, the values of the center frequencies, and the values of the constants c_i.

sense that each frequency f_i approaches the same final frequency as $t \to \infty$.*

1.2 *Example of An Unstable System*

We wish to show now that if transmission delays are taken into account, then the system described above need not be stable. Consider the equations of a two-station system with $c_1 = c_2 = c$, $a_{12} = a_{21} = a$, and $\tau_{12} = \tau_{21} = \bar{\tau}(c, a, \bar{\tau} > 0)$:

$$\dot{f}_1 + cf_1 + a \int_0^t [f_1(\tau) - f_2(\tau - \bar{\tau})] \, d\tau = ab_{12}(0) + cf_{01}$$

$$\dot{f}_2 + cf_2 + a \int_0^t [f_2(\tau) - f_1(\tau - \bar{\tau})] \, d\tau = ab_{21}(0) + cf_{02}$$

for $t \geqq 0$. It is not difficult to verify that these equations posses a solution

$$f_1(t) = \text{Re}\,[v_1 e^{i\omega t}] \qquad f_2(t) = \text{Re}\,[v_2 e^{i\omega t}]$$

for $t \geqq -\bar{\tau}$ with v_1, v_2, and $\omega \neq 0$ constants if and only if

$$-a\,\text{Re}\,[(v_1 - v_2 e^{-i\omega\bar{\tau}})(i\omega)^{-1}] = ab_{12}(0) + cf_{01} \tag{5}$$

$$-a\,\text{Re}\,[(v_2 - v_1 e^{-i\omega\bar{\tau}})(i\omega)^{-1}] = ab_{21}(0) + cf_{02} \tag{6}$$

and $Mv = \theta$, in which v is the transpose of (v_1, v_2), θ is the zero 2-vector, and

$$M = \begin{bmatrix} -\omega^2 + i\omega c + a & -ae^{-i\omega\bar{\tau}} \\ -ae^{-i\omega\bar{\tau}} & -\omega^2 + i\omega c + a \end{bmatrix}.$$

Thus if $\det M = 0$ for some real value of $\omega \neq 0$, then the system is unstable in the sense that for some values of the right sides of equations (5) and (6) there exists a pair of real-valued functions $f_1(\cdot)$ and $f_2(\cdot)$ such that the equations are satisfied for all $t \geqq 0$, and at least one of these functions does not approach a limit as $t \to \infty$. Moreover, if for example $\omega^2 = a$, $c^2 = a$, and $\omega\bar{\tau} = \frac{1}{2}\pi$, then $\det M = 0$, which shows that the two-station system can be unstable if additional restrictions are not imposed on c, a, and $\bar{\tau}$.†

* He has also exploited the network analogy further in order to obtain an expression for the final frequency and to make assertions concerning the behavior of the system when certain elements are nonlinear.

† Note that $f_1(t)$ or $f_2(t)$ can be negative in this example. While it is certainly true that instantaneous frequencies are not negative, our analysis is intended to show only that the solution of the equations can possess a sinusoidal mode. Thus, the conclusion and the essential details of the analysis are unchanged if we add a constant u to $f_1(t)$ and $f_2(t)$ provided that we subtract the constant cu from the right side of equation (5) and the right side of equation (6).

1.3 *Summary of Results*

The purpose of this paper is to report on some results concerning the dynamic behavior of Pierce's model. The results take into account transmission delays. In particular, we prove that the (assumed connected) system is stable whenever

$$c_i > \sum_{j \neq i} a_{ij} \tfrac{1}{2}(\tau_{ij} + \tau_{ji}) \qquad i = 1, 2, \cdots, n. \tag{7}$$

In fact we prove that if condition (7) is satisfied, then the frequencies f_i approach their final value ρ at an exponential rate.* Concerning the final frequency ρ, we derive the explicit expression

$$\rho = \frac{\sum_i c_i f_{0i} + \sum_i \sum_{j \neq i} a_{ij} b_{ij}(0) + \sum_i \sum_{j \neq i} a_{ij} \int_{-\tau_{ij}}^{0} f_j(u)\, du}{\sum_i c_i + \sum_i \sum_{j \neq i} a_{ij} \tau_{ij}}. \tag{8}$$

This paper also presents some material concerning bounds on the rate of decay of transients in stable systems. More explicitly, we derive a lower bound on the rate of decay of all complex natural modes.

We prove also that the system is stable whenever

$$c_i \geqq \Big(2 \sum_{j \neq i} a_{ij}\Big)^{\frac{1}{2}} \qquad i = 1, 2, \cdots, n \tag{9}$$

and that if condition (9) is satisfied, then [as in the case of condition (7)] the frequencies f_i approach ρ at an exponential rate.

Unlike condition (7), condition (9) is obviously not always satisfied for sufficiently small delays. On the other hand, if condition (9) is satisfied, then the system is stable independent of the values of the τ_{ij}. In this sense the results corresponding to conditions (7) and (9) are complementary.

Our results described above are stated in a more precise manner in Section II, and proofs are given in Section III.

1.4 *The Relation to Earlier Work*

The system governed by equations (2) and (4) is closely related to synchronization systems which have been studied earlier. This relation is made clear as follows. From equations (2) and (4)

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* Of course in the vast majority of cases it is reasonable to assume that $\tau_{ij} = \tau_{ji}$ for all $i \neq j$. We have proceeded without this assumption in order to show that our stability result is not critically dependent on it.

$$\dot{f}_i = c_i(f_{0i} - f_i) + \sum_{j \neq i} a_{ij} \int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)] \, d\tau + \sum_{j \neq i} a_{ij} b_{ij}(0) \qquad i = 1, 2, \cdots, n \tag{10}$$

for all $t \geqq 0$. We write equations (10) as

$$\dot{f}_i + c_i f_i = \sum_{j \neq i} a_{ij} \int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)] \, d\tau + c_i f_{0i} + \sum_{j \neq i} a_{ij} b_{ij}(0) \qquad i = 1, 2, \cdots, n \tag{11}$$

for all $t \geqq 0$. Then, treating the right sides of equations (11) as "driving functions," we can solve the n first-order differential equations (11) to obtain

$$f_i = \int_0^t \exp[-c_i(t - u)] \sum_{j \neq i} a_{ij} \int_0^u [f_j(\tau - \tau_{ij}) - f_i(\tau)] \, d\tau \, du + f_i(0) \exp(-c_i t) + \int_0^t \exp[-c_i(t - \tau)][c_i f_{0i} + \sum_{j \neq i} a_{ij} b_{ij}(0)] \, d\tau$$

for all i and all $t \geqq 0$. But

$$\int_0^u f_j(\tau - \tau_{ij}) \, d\tau = \int_0^{(u - \tau_{ij})} f_j(\tau) \, d\tau + \int_{-\tau_{ij}}^0 f_j(v) \, dv$$

for all j and all $u \geqq 0$. Thus, with

$$p_i(t) = \int_0^t f_i(\tau) \, d\tau$$

for all i, we have (after some simplification)

$$\dot{p}_i = \int_0^t k_i(t - \tau) \sum_{j \neq i} a_{ij}[p_j(\tau - \tau_{ij}) - p_i(\tau)] \, d\tau + v_i + u_i(t), \qquad t \geqq 0 \quad i = 1, 2, \cdots, n \tag{12}$$

in which each v_i is a constant and each $u_i(t)$ approaches zero as $t \to \infty$. More explicitly, $k_i(t) = \exp(-c_i t)$,

$$v_i = f_{0i} + (c_i)^{-1} \sum_{j \neq i} a_{ij} \left[\int_{-\tau_{ij}}^0 f_j(u) \, du + b_{ij}(0) \right],$$

and $u_i(t) = [f_i(0) - v_i] \exp(-c_i t)$.

Equations of the same type as (12) have been studied extensively in connection with linear models of synchronized networks, and many interesting and informative results have been obtained (see Refs. 2–6). In particular, Beneš has derived a sufficient condition for stability (see Refs. 2 and 3). His condition does not depend on the delays and is therefore of a very different type than condition (7). When applied to our system of equations (10) via the relation between equations (10) and (12), Beneš' condition reduces to our condition (9).* It therefore does not yield Pierce's result obtained via the network analog.

As is often true of pioneering work, Beneš' proof is long and involved. Our derivation of what amounts to his condition applied to our system is relatively simple and is much less involved compared with the rather long proof of our main result which asserts that if condition (7) is satisfied, then the system is stable. To a considerable extent, the methods of proof used here are very different from those of earlier writers concerned with the synchronization problem, and they provide some grip on the problem of estimating the rate of decay of the natural modes of the system. On the other hand, in this paper we do not consider several other important practical problems such as that of obtaining a useful upper bound on the contents of the buffers or of predicting the effects of variable transmission delays (resulting from temperature changes). There is a clear need for much more work in this area, especially in connection with models which take into account purposely-inserted nonlinearities.

II. STATEMENT OF RESULTS

2.1 *Definitions and Assumptions*

Let M denote an arbitrary complex matrix. We denote by M^{tr} and M^* the transpose of M and the complex-conjugate transpose of M, respectively. If M is not a row vector or a column vector, then $(M)_{ij}$ denotes the ijth element of M. If M is a column vector then $(M)_j$ denotes the jth element of M. If M is square, then $M^{(j,j)}$ denotes the matrix obtained from M by deleting the jth row and column. The zero element of complex Euclidean n-space is indicated by θ, and 1_n denotes the identity matrix of order n.

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* Beneš proves that a suitable equilibrium state is approached. He makes no assertions concerning the rate at which it is approached. However, it is possible to modify the alternative proof[3] of Beneš' result due to Gersho and Karafin to show that the equilibrium state is approached exponentially under reasonable additional assumptions. For example, it suffices to assume that each transfer function $H_i(s)$ of Ref. 3 is a rational function of s. See the appendix.

The statement "for all i" means for all $i = 1, 2, \cdots, n$ in which n denotes an arbitrary fixed positive integer (the number of geographically separated stations) such that $n \geqq 2$. If f_i or $F = (f_1, f_2, \cdots, f_n)^{tr}$ denotes a differentiable function of t or a differentiable n-vector-valued function of t, then $\dot{f}_i$ or $\dot{F}$ indicates the derivative with respect to t of f_i or F, respectively.

We assume throughout the paper that

(*i*) A is the real $n \times n$ matrix defined by

$$(A)_{ii} = \sum_{j \neq i} a_{ij} \qquad \text{for all } i \tag{13}$$

$$(A)_{ij} = -a_{ij} \qquad \text{for all } i \neq j \tag{14}$$

in which $a_{ij} = a_{ji} \geqq 0$, for all $i \neq j$.

(*ii*)
$$\det A^{(n,n)} > 0 \tag{15}$$

(*iii*)
$$\tau_{ij} \geqq 0 \quad \text{for all} \quad i \neq j \tag{16}$$

(*iv*) the operator $\underline{A}$ is a mapping of the set of n-vector-valued functions of t into itself defined by the condition that if $F(t) = [f_1(t), f_2(t), \cdots, f_n(t)]^{tr}$, then

$$[(\underline{A}F)(t)]_i = \sum_{j \neq i} a_{ij}[f_i(t) - f_j(t - \tau_{ij})] \tag{17}$$

for all i.

(*v*) $C = \text{diag}(c_1, c_2, \cdots, c_n)$ with $c_i > 0$ for all i

(*vi*) $\tilde{A}$ is the complex $n \times n$ matrix defined by

$$(\tilde{A})_{ii} = \sum_{j \neq i} a_{ij} \qquad \text{for all } i \tag{18}$$

$$(\tilde{A})_{ij} = -a_{ij} \exp(-s\tau_{ij}) \qquad \text{for all } i \neq j \text{ and all complex } s \tag{19}$$

(*vii*) $\Delta(s)$ denotes the determinant $\det [s^2 1_n + sC + \tilde{A}]$.

Assumption (*ii*) is satisfied if there exists a path (not necessarily a direct path) from each station to every other station.*

* This proposition is a special case of a known result in probability theory. A simple proof is given in Ref. 3. A "network theoretic" proof is as follows. The matrix A may be interpreted as the indefinite conductance matrix of a nonnegative element n-node resistance network, and $A^{(n,n)}$ is the node-pair conductance matrix of the network obtained by grounding node n. If the original n-node network is connected, then the common-ground network possesses an open-circuit resistance matrix, which means that $\det A^{(n,n)} \neq 0$. But, since the network contains only nonnegative elements, $\det A^{(n,n)} > 0$.

The equations of our system are

$$\dot{f}_i = c_i(f_{0i} - f_i) + \sum_{j \neq i} a_{ij} \int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)]\, d\tau + \sum_{j \neq i} a_{ij} b_{ij}(0), \qquad t \geqq 0$$

for all i. Therefore, with $F = (f_1, f_2, \cdots, f_n)^{tr}$, we have

$$\ddot{F} + C\dot{F} + \underline{A}F = \theta, \qquad t \geqq 0.$$

2.2 *Results*

Theorem 1 is the principal result of this paper:

Theorem 1: *Let $F(\cdot)$ be a twice differentiable n-vector-valued function defined on $[-\bar{\tau}, \infty)$, in which $\bar{\tau} = \max_{i \neq j} \{\tau_{ij}\}$, such that*

$$\ddot{F} + C\dot{F} + \underline{A}F = \theta, \qquad t \geqq 0^*.$$

If

$$c_i > \sum_{j \neq i} a_{ij} \tfrac{1}{2}(\tau_{ij} + \tau_{ji})$$

for all i, then there exist a constant ρ, positive constants β and γ, and an n-vector-valued function $G(\cdot)$ defined for $t \in [0, \infty)$ such that

$$|\, [G(t)]_i \,| \leqq \beta e^{-\gamma t}, \qquad t \geqq 0$$

for all i, and

$$F(t) = G(t) + \rho(1, 1, \cdots, 1)^{tr}$$

for all $t \geqq 0$.

Some information concerning the rate of decay of the complex natural modes of the system, assuming that the stability condition of Theorem 1 is satisfied, is provided by the following theorem.

Theorem 2: *Let $\tau_{ij} = \tau_{ji}$ for all $i \neq j$. If there exists a positive constant δ such that*

$$c_i - \sum_{j \neq i} a_{ij}\tau_{ij} \geqq \delta$$

for all i, and if $\Delta(s) = 0$ with $\mathrm{Im}\,[s] \neq 0$, then $\mathrm{Re}\,[s] \leqq -\alpha_0$ in which α_0 is the solution of

$$-2\alpha_0 + \max_i c_i = [\max_i c_i - \delta] \exp(\alpha_0 \bar{\tau})$$

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* Questions concerning the existence and uniqueness of solutions of equations of this type are discussed in Ref. 7.

where $\bar{\tau} = \max_{i \neq j} \{\tau_{ij}\}$.

An expression for the final frequency ρ of the system is given in Theorem 3:

Theorem 3: *If* $F(\cdot)$ *is differentiable on* $[-\bar{\tau}, \infty)$ *with* $\bar{\tau} = \max_{i \neq j} \{\tau_{ij}\}$ *such that for all* i

$$\dot{f}_i = c_i(f_{0i} - f_i) + \sum_{j \neq i} a_{ij} \int_0^t [f_j(\tau - \tau_{ij}) - f_i(\tau)]\, d\tau + \sum_{j \neq i} a_{ij} b_{ij}(0), \qquad t \geqq 0$$

in which the f_{0i} *and the* $b_{ij}(0)$ *are constants, and if there exists a constant* ρ *such that for all* i: $(f_i - \rho) \to 0$ *as* $t \to \infty$, *and* $(f_i - \rho)$ *is (absolutely integrable on* $[0, \infty)$, *then*

$$\rho = \frac{\sum_i c_i f_{0i} + \sum_i \sum_{j \neq i} a_{ij} b_{ij}(0) + \sum_i \sum_{j \neq i} a_{ij} \int_{-\tau_{ij}}^0 f_j(u)\, du}{\sum_i c_i + \sum_i \sum_{j \neq i} a_{ij} \tau_{ij}}.$$

Our final result is as follows.

Theorem 4: *The statement obtained from the statement of Theorem 1 by replacing the condition that*

$$c_i > \sum_{j \neq i} a_{ij} \tfrac{1}{2}(\tau_{ij} + \tau_{ji}) \qquad \text{for all } i$$

by the condition that

$$c_i \geqq (2 \sum_{j \neq i} a_{ij})^{\frac{1}{2}} \qquad \text{for all } i$$

is a theorem.

III. PROOFS

3.1 *Proof of Theorem 1*

Our proof consists of proving the following four lemmas.

Lemma 1: $\Delta(s)$ *has a simple zero at* $s = 0$.

Lemma 2: If

$$c_i > \sum_{j \neq i} a_{ij} \tfrac{1}{2}(\tau_{ij} + \tau_{ji})$$

for $i = 1, 2, \cdots, n$ *then*

(*i*) $\triangle(s) \neq 0$ *for* Re $[s] > 0$

(*ii*) *there exists a positive constant* α_0 *such that if* $\triangle(s) = 0$ *with* $s = -\alpha + i\omega$ (α, ω *real*; $\alpha \geqq 0$; $\omega \neq 0$), *then* $\alpha \geqq \alpha_0$.

Lemma 3: *Let* F$(\cdot)$ *be a twice differentiable n-vector-valued function defined on* $[-\bar{\tau}, \infty)$, *in which* $\bar{\tau} = \max_{i \neq j} \{\tau_{ij}\}$, *such that*

$$\ddot{F} + C\dot{F} + \underline{A}F = \theta, \qquad t \geqq 0.$$

Let $\triangle$(s) *have a simple zero at* $s = 0$, *and with the exception of this zero assume that* $\triangle$(s) $\neq 0$ *for all* Re [s] $\geqq -\alpha_0$ *for some positive constant* α_0 . *Then there exist a constant n-vector* K, *positive constants* β *and* γ, *and an n-vector-valued function* $G(\cdot)$ *defined for* $t \in [0, \infty)$, *such that*

$$| [G(t)]_i | \leqq \beta e^{-\gamma t}, \qquad t \geqq 0$$

for all i, *and* $F(t) = G(t) + K$ *for all* $t \geqq 0$.

Lemma 4: *If* $F(t) = G(t) + K$ *satisfies*

$$\ddot{F} + C\dot{F} + \underline{A}F = \theta, \qquad t \geqq 0$$

with $G(t) \rightarrow \theta$ *as* $t \rightarrow \infty$, *and* K *a constant vector, then, for some constant* ρ,

$$K = \rho(1, 1, \cdots, 1)^{tr}.$$

Proof of Lemma 1: The determinant $\triangle(s)$ is analytic throughout the s-plane. It can be written as some power series

$$\sum_{n=0}^{\infty} \xi_n s^n$$

which converges for all s. Since $\triangle(0) = \det A = 0$, $\xi_0 = 0$. To prove Lemma 1, it suffices to show that

$$\lim_{s \to 0} \frac{\triangle(s)}{s} \neq 0,$$

for then $\xi_1 \neq 0$. We show this as follows.

We write $s^{-1} \det [s^2 1_n + sC + \tilde{A}]$ as $\det M$ in which

$$M = \operatorname{diag}(s^2 + c_1 s, s^2 + c_2 s, \cdots, s^2 + c_{n-1} s, s + c_n) + \tilde{A}'$$

where $\tilde{A}'$ is obtained from $\tilde{A}$ by dividing each element of the nth row of $\tilde{A}$ by s. But, with $M^{(n,n)}$ the submatrix obtained from M by deleting the nth row and column,

$$\det M = (s + c_n) \det M^{(n,n)} + s^{-1} \det P$$

in which

$$P = \tilde{A} + \text{diag}\,(s^2 + c_1 s, s^2 + c_2 s, \cdots, s^2 + c_{n-1}s, 0).$$

Since, by assumption, $\det A^{(n,n)} > 0$, we have

$$0 < \lim_{s\to 0}\,(s + c_n) \det M^{(n,n)}.$$

The determinant of P is an analytic function of s for all s. For $s = 0$, it vanishes. Therefore its power-series expansion at $s = 0$ is of the form

$$\sum_{n=1}^{\infty} p_n s^n.$$

We note that for s real and positive, $\det P > 0$ because for s real and positive P is strongly dominant in its first $(n - 1)$ rows and at least weakly dominant in its last row.* Thus p_1 is not negative, and

$$\lim_{s\to 0} s^{-1} \det P \geqq 0.$$

Therefore

$$\lim_{s\to 0} s^{-1} \det\,[s^2 1_n + sC + \tilde{A}] > 0. \quad \square$$

Proof of Lemma 2: If s is such that $\triangle(s) = 0$, then there exists a nonzero complex n-vector x such that

$$(s^2 1_n + sC + \tilde{A})x = \theta$$

and consequently

$$x^*(s^2 1_n + sC + \tilde{A})x = 0.$$

With $\tilde{A}_H = \frac{1}{2}(\tilde{A} + \tilde{A}^*)$ and $\tilde{A}_S = \frac{1}{2}(\tilde{A} - \tilde{A}^*)$, we have

$$s^2 + s\,\frac{x^*Cx}{\|\,x\,\|^2} + \frac{x^*\tilde{A}_H x}{\|\,x\,\|^2} + \frac{x^*\tilde{A}_S x}{\|\,x\,\|^2} = 0 \qquad (20)$$

in which $\|\,x\,\| = (x^*x)^{\frac{1}{2}}$,

$$(\tilde{A}_H)_{ij} = -\tfrac{1}{2}a_{ij}[\exp\,(-s\tau_{ij}) + \exp\,(-s^*\tau_{ji})], \qquad \text{for all } i \neq j$$

$$(\tilde{A}_H)_{ii} = \sum_{j\neq i} a_{ij}\,, \qquad \text{for all } i$$

and

$$(\tilde{A}_S)_{ij} = -\tfrac{1}{2}a_{ij}[\exp\,(-s\tau_{ij}) - \exp\,(-s^*\tau_{ji})], \qquad \text{for all } i \neq j$$

$$(\tilde{A}_S)_{ii} = 0, \qquad \text{for all } i.$$

* In other words, for such values of s, $\det P > 0$ because (for each such value of s) there exists a diagonal matrix $D = \text{diag}(1, 1, \ldots, k)$ with $k > 1$ such that PD is strongly row-sum dominant.

Notice that both $\tilde{A}_H$ and $(i)^{-1}\tilde{A}_S$ [$i \triangleq (-1)^{\frac{1}{2}}$ when not used as an index] are hermitian matrices, and that

$$\frac{x^*\tilde{A}_H x}{\|x\|^2} \quad \text{and} \quad \frac{x^*\tilde{A}_S x}{\|x\|^2}$$

are real and pure imaginary, respectively.

Sublemma 1: If $\triangle(s) = 0$ with $s = \alpha + i\omega$ and $\alpha \geqq 0$, then

$$\omega^2 \geqq \alpha^2 + \alpha \min_i c_i \ . \tag{21}$$

Proof: From

$$s^2 + s\frac{x^*Cx}{\|x\|^2} + \frac{x^*\tilde{A}_H x}{\|x\|^2} + \frac{x^*\tilde{A}_S x}{\|x\|^2} = 0 \tag{22}$$

for some nonzero x corresponding to the assumed value of $s = \alpha + i\omega$, we have

$$\alpha^2 - \omega^2 + \alpha\frac{x^*Cx}{\|x\|^2} + \frac{x^*\tilde{A}_H x}{\|x\|^2} = 0.$$

But for $\alpha \geqq 0$, $\tilde{A}$ is both at least weakly row-sum dominant and weakly column-sum dominant. Thus $\tilde{A}_H$ is at least weakly dominant and hence nonnegative definite (that is, $x^*\tilde{A}_H x \geqq 0$). Therefore

$$\omega^2 \geqq \alpha^2 + \alpha\frac{x^*Cx}{\|x\|^2}$$
$$\geqq \alpha^2 + \alpha \min_i c_i \ . \quad \square$$

Sublemma 2: *If*

$$c_i > \sum_{j \neq i} a_{ij} \max(\tau_{ij}, \tau_{ji})$$

for all i, *then* $\triangle(s) \neq 0$ *for* Re $[s] > 0$.

Proof: Sublemma 1 implies that $\triangle(s)$ has no zeros on the positive-real axis. Assume now that $s = \alpha + i\omega$ with $\alpha > 0$ and $\omega \neq 0$. Then, using equation (20),

$$2\alpha i\omega + i\omega\frac{x^*Cx}{\|x\|^2} + \frac{x^*\tilde{A}_S x}{\|x\|^2} = 0$$

or

$$2\alpha + \frac{x^*Cx}{\|x\|^2} + \frac{x^*[(i\omega)^{-1}\tilde{A}_S]x}{\|x\|^2} = 0. \tag{23}$$

Let m_{ij} denote the ijth element of $(i\omega)^{-1}\tilde{A}_S$. Then $m_{ii} = 0$ for all i, and for $i \neq j$:

$$\begin{aligned} m_{ij} &= -\tfrac{1}{2}a_{ij}(i\omega)^{-1}[\exp(-\alpha\tau_{ij})\exp(-i\omega\tau_{ij}) - \exp(-\alpha\tau_{ji})\exp(i\omega\tau_{ji})] \\ &= -\tfrac{1}{2}a_{ij}\exp(-\alpha\tau_{ij})(i\omega)^{-1}[\exp(-i\omega\tau_{ij}) - \exp(i\omega\tau_{ji})] \\ &\quad + \tfrac{1}{2}a_{ij}(i\omega)^{-1}[\exp(-\alpha\tau_{ji})\exp(i\omega\tau_{ji}) - \exp(-\alpha\tau_{ij})\exp(i\omega\tau_{ji})] \\ &= a_{ij}\exp(-\alpha\tau_{ij})\exp[i\omega\tfrac{1}{2}(\tau_{ji} - \tau_{ij})](\omega)^{-1}\sin[\tfrac{1}{2}(\tau_{ij} + \tau_{ji})\omega] \\ &\quad + \tfrac{1}{2}a_{ij}\exp(i\omega\tau_{ji})(i\omega)^{-1}[\exp(-\alpha\tau_{ji}) - \exp(-\alpha\tau_{ij})]. \end{aligned}$$

It follows that

$$| m_{ij} | \leqq a_{ij}\tfrac{1}{2}(\tau_{ij} + \tau_{ji}) + \tfrac{1}{2}a_{ij} \,|\, \omega^{-1}[\exp(-\alpha\tau_{ji}) - \exp(-\alpha\tau_{ij})] \,|.$$

But, from inequality (21), $\omega^2 \geqq \alpha^2$, so that

$$\begin{aligned} &| \omega^{-1}[\exp(-\alpha\tau_{ji}) - \exp(-\alpha\tau_{ij})] | \\ &\qquad\qquad \leqq | \alpha^{-1}[\exp(-\alpha\tau_{ji}) - \exp(-\alpha\tau_{ij})] | \end{aligned}$$

and, as can easily be verified,

$$| \alpha^{-1}[\exp(-\alpha\tau_{ji}) - \exp(-\alpha\tau_{ij})] | \leqq | \tau_{ji} - \tau_{ij} |.$$

Therefore, for $i \neq j$,

$$| m_{ij} | \leqq a_{ij}\tfrac{1}{2}(\tau_{ij} + \tau_{ji}) + a_{ij}\tfrac{1}{2} \,|\, \tau_{ji} - \tau_{ij} \,|\, = a_{ij}\max(\tau_{ij}, \tau_{ji}).$$

However, by the hypothesis of the sublemma,

$$c_i > \sum_{j \neq i} a_{ij}\max(\tau_{ij}, \tau_{ji})$$

for all i, and thus $C + (i\omega)^{-1}\tilde{A}_S$ is a strongly-dominant hermitian matrix. This means that $C + (i\omega)^{-1}\tilde{A}_S$ is positive definite, and hence that α in equation (23) is negative, which is a contradiction. □

Sublemma 3: *If $b_1, b_2, \cdots, b_n$ are positive constants such that*

$$b_i > \sum_{j \neq i} a_{ij}\tfrac{1}{2}(\tau_{ij} + \tau_{ji})$$

for all i, then there exists a positive constant σ with the property that if $c_i \geqq b_i$ for all i, and if $\Delta(s) = 0$ with $s = \alpha + i\omega$ and $\alpha \geqq 0$ and $\omega \neq 0$, then $\alpha > \sigma$.

Proof: With $s = \alpha + i\omega$ and $\alpha \geqq 0$ and $\omega \neq 0$, and proceeding as in the proof of Sublemma 2, with any C such that $c_i \geqq b_i$ for all i we have for some nonzero vector x

$$2\alpha + \frac{x^*Cx}{\|x\|^2} + \frac{x^*Mx}{\|x\|^2} = 0$$

in which M is hermitian with elements $m_{ii} = 0$ for all i and m_{ij} $(i \neq j)$ such that

$$|m_{ij}| \leqq a_{ij}\tfrac{1}{2}(\tau_{ij} + \tau_{ji}) + \tfrac{1}{2}a_{ij}\,|\omega^{-1}[\exp(-\alpha\tau_{ji}) - \exp(-\alpha\tau_{ij})]|.$$

By Sublemma 1,

$$\omega^2 \geqq \alpha^2 + \alpha b$$

in which $b = \min_i b_i$. Thus $|\omega|^{-1} \leqq (\alpha^2 + \alpha b)^{-\frac{1}{2}}$. Since (for all $\alpha \geqq 0$)

$$|\exp(-\alpha\tau_{ji}) - \exp(-\alpha\tau_{ij})| \leqq \alpha\,|\tau_{ji} - \tau_{ij}|,$$

we have (for all $i \neq j$)

$$|m_{ij}| \leqq a_{ij}\tfrac{1}{2}(\tau_{ij} + \tau_{ji}) + \tfrac{1}{2}a_{ij}\alpha(\alpha^2 + \alpha b)^{-\frac{1}{2}}\,|\tau_{ji} - \tau_{ij}|.$$

Let

$$\epsilon = \min_i \{b_i - \sum_{j \neq i} a_{ij}\tfrac{1}{2}(\tau_{ij} + \tau_{ji})\}.$$

Of course $\epsilon > 0$. Since $b > 0$, it is clear that there exists a positive constant σ such that (for all $i \neq j$)

$$(2n)^{-1}\,\epsilon \geqq \tfrac{1}{2}a_{ij}\alpha(\alpha^2 + \alpha b)^{-\frac{1}{2}}\,|\tau_{ji} - \tau_{ij}|$$

for all $0 \leqq \alpha \leqq \sigma$. It follows that $(C + M)$ is positive definite for all α such that $0 \leqq \alpha \leqq \sigma$. This means that if $\alpha \geqq 0$ satisfies equation (23), then $\alpha > \sigma$. □

Sublemma 4: There exists a positive constant k, independent of C, such that if $\triangle(s) = 0$ *with* $s = \alpha + i\omega$ *and* $\alpha \geqq 0$, *then*

$$|s| < \max_i c_i + k.$$

Proof: The elements of $\tilde{A}$ are uniformly bounded on Re $[s] \geqq 0$. Let k be any positive constant greater than

$$\sup\left\{\left|\frac{x^*Ax}{\|x\|^2}\right|^{\frac{1}{2}} : \text{Re}\,[s] \geqq 0, \quad x \neq \theta\right\}.$$

If $\triangle(s) = 0$ with $s = \alpha + i\omega$ and $\alpha \geqq 0$, then for some $x \neq \theta$

$$s^2 + s\frac{x^*Cx}{\|x\|^2} + \frac{x^*\tilde{A}x}{\|x\|^2} = 0$$

and

$$2s = -\frac{x^*Cx}{\| x \|^2} \pm \left[\left(\frac{x^*Cx}{\| x \|^2}\right)^2 - 4\,\frac{x^*\tilde{A}x}{\| x \|^2}\right)^{\frac{1}{2}}.$$

Thus, since

$$\left| \left[\left(\frac{x^*Cx}{\| x \|^2}\right)^2 - 4\,\frac{x^*Ax}{\| x \|^2}\right]^{\frac{1}{2}} \right| \leqq \frac{x^*Cx}{\| x \|^2} + 2\left| \frac{x^*\tilde{A}x}{\| x \|^2} \right|^{\frac{1}{2}},$$

it follows that

$$\begin{aligned} | s | &\leqq \frac{x^*Cx}{\| x \|^2} + \left| \frac{x^*\tilde{A}x}{\| x \|^2} \right|^{\frac{1}{2}} \\ &\leqq \max_i c_i + k. \quad \square \end{aligned}$$

By Sublemmas 1 and 3, if

$$c_i > \sum_{j \neq i} a_{ij}\tfrac{1}{2}(\tau_{ij} + \tau_{ji}) \tag{24}$$

for all i, then $\triangle(s)$ has no zeros on the positive-real axis of the s plane, and $\triangle(i\omega) \neq 0$ for all real $\omega \neq 0$. We are now in a position to show that if condition (24) is satisfied for all i, then $\triangle(s) \neq 0$ for Re $[s] > 0$.

Let $\underline{c}_i$ for $i = 1, 2, \cdots, n$ be any set of positive constants such that

$$\underline{c}_i > \sum_{j \neq i} a_{ij}\tfrac{1}{2}(\tau_{ij} + \tau_{ji})$$

for all i, and let $\bar{c}_i$ for $i = 1, 2, \cdots, n$ be any set of positive constants such that $\bar{c}_i \geqq \underline{c}_i$ and

$$\bar{c}_i > \sum_{j \neq i} a_{ij} \max (\tau_{ij}, \tau_{ji})$$

for all i. Let $c_i \in [\underline{c}_i, \bar{c}_i]$ for all i. Then, by Sublemmas 3 and 4 there exists a "half-moon-shaped" finite region $\mathfrak{R}$ in the strict right-half plane bounded by a line Re $[s] = \sigma$ and a semicircular arc $| s | \leqq K$ such that if $c_i \in [\underline{c}_i, \bar{c}_i]$ for all i and $\triangle(s) = 0$ with Re $[s] > 0$, then s lies *inside* $\mathfrak{R}$. By Sublemma 2, for $c_i = \bar{c}_i$ for all i, $\triangle(s) \neq 0$ for Re $[s] > 0$. On the boundary of $\mathfrak{R}$, $\triangle(s) \neq 0$ for all $c_i \in [\underline{c}_i, \bar{c}_i]$.

For each i, we can write $\triangle(s)$ as $A_i(s) + c_iB_i(s)$ in which A_i and B_i are entire functions which are independent of c_i. For $c_i = \bar{c}_i$ for $i = 2, 3, \cdots, n$

$$A_1(s) + c_1B_1(s) \neq 0$$

on the boundary of $\mathfrak{R}$ for all $c_1 \in [\underline{c}_1, \bar{c}_1]$. Thus $\triangle(s)$ has no zeros inside

$\mathcal{R}$ for $c_1 = \underline{c}_1$ and $c_i = \bar{c}_i$ for $i = 2, 3, \cdots, n$.* Similarly, for $c_1 = \underline{c}_1$ and $c_i = \bar{c}_i$ for $i = 3, 4, \cdots, n$

$$A_2(s) + c_2 B_2(s) \neq 0$$

on the boundary of $\mathcal{R}$ for all $c_2 \varepsilon [\underline{c}_2, \bar{c}_2]$. Therefore $\Delta(s)$ has no zeros inside $\mathcal{R}$ for $c_1 = \underline{c}_1$, $c_2 = \underline{c}_2$ and $c_i = \bar{c}_i$ for $i = 3, 4, \cdots, n$. By continuing in this manner we find that $\Delta(s)$ has no zeros inside $\mathcal{R}$ when $c_i = \underline{c}_i$ for all i, which shows that if condition (24) is satisfied for all i, then $\Delta(s) \neq 0$ for Re $[s] > 0$.

The following argument completes the proof of Lemma 2.

If $\Delta(s) = 0$ with $s = -\alpha + i\omega$, then for some $x \neq \theta$

$$\alpha^2 - \omega^2 - \alpha \frac{x^*Cx}{\| x \|^2} + \frac{x^* \tilde{A}_H x}{\| x \|^2} = 0.$$

Since $\Delta(s)$ is an entire function of s, in any circle in the s plane $\Delta(s)$ has at most a finite number of zeros. Thus, either there exists a positive constant α_0 such that if $\Delta(s) = 0$ with $s = -\alpha + i\omega$ and $\alpha > 0$, then $\alpha \geqq \alpha_0$, or there exists an infinite sequence $s_1, s_2, \cdots$, such that $\Delta(s_n) = 0$ for all n and, with $s_n = -\alpha_n + i\omega_n$, $\omega_n \to \infty$ and $\alpha_n \to 0+$ as $n \to \infty$. In the later case, since for $n = 1, 2, \cdots$

$$\omega_n^2 = \alpha_n^2 - \alpha_n \frac{x_n^* C x_n}{\| x_n \|^2} + \frac{x_n^* \tilde{A}_H x_n}{\| x_n \|^2}$$

with $x_n \neq \theta$ an associate of s_n, we would have

$$\alpha_n^2 - \alpha_n \frac{x_n^* C x}{\| x \|^2} + \frac{x_n^* A_H x_n}{\| x \|^2} \to \infty$$

as $\alpha_n \to 0+$ which is impossible since for any real $\beta < 0$

$$\frac{x_n^* \tilde{A}_H x_n}{\| x_n \|^2}$$

is bounded on the strip $\beta \leqq$ Re $[s] \leqq 0$ uniformly in x_n. □

Proof of Lemma 3: We have

$$\ddot{F} + C\dot{F} + \underline{A}F = \theta, \qquad t \geqq 0. \tag{25}$$

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* Here we use the following known result.[8] Let $\mathcal{R}$ be a closed region in the s plane, the boundary of which consists of a finite number of regular arcs; let the functions $f(s)$ and $h(s)$ be regular on $\mathcal{R}$. Assume that for no value of the real parameter c, running through the interval $a \leq c \leq b$, does the function $f(s) + ch(s)$ become equal to 0 on the boundary of $\mathcal{R}$. Then the number $N(c)$ of the zeros of $f(s) + ch(s)$ inside $\mathcal{R}$ is independent of c for $a \leq c \leq b$.

Using the Bellman–Gronwall Lemma (see p. 31 of Ref. 7), we can prove that there are constants k_1, $k_2 > 0$ such that for all i

$$|F_i(t)| \leqq k_1 \exp(k_2 t), \qquad t \geqq 0.$$

Thus $F(\cdot)$ possesses a Laplace transform

$$\int_0^\infty F(t)e^{-st}\, dt \triangleq \hat{F}(s)$$

which is well defined for all s such that Re $[s] > k_2$. Therefore

$$s^2\hat{F}(s) + sC\hat{F}(s) + \tilde{A}\hat{F}(s) = sV_1 + V_2 + W(s)$$

in which the constant vectors V_1 and V_2 take into account the values of $\dot{F}(0)$ and $F(0)$, and for all i

$$[W(s)]_i = \sum_{j \neq i} a_{ij} \exp(-s\tau_{ij}) \int_{-\tau_{ij}}^0 F_j(t)e^{-st}\, dt.$$

Thus

$$\hat{F}(s) = [s^2 1_n + sC + \tilde{A}]^{-1}[sV_1 + V_2 + W(s)].$$

Of course

$$F(t) = (2\pi i)^{-1} \int_{(l+)} \hat{F}(s)e^{st}\, ds, \qquad t \geqq 0$$

in which $(l+)$ is some line parallel to and to the right of the imaginary axis of the s plane. Let $(l-)$ denote the line indicated in Fig. 1. Then

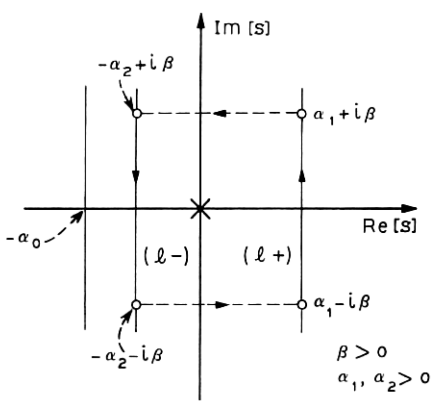

Fig. 1 — Relation between $(l-)$ and $(l+)$.

(see Fig. 1)

$$\int_{(l+)} = \lim_{\beta\to\infty} \int_{\alpha_1 - i\beta}^{\alpha_1 + i\beta},$$

and for $t \geqq 0$

$$\int_{\alpha_1 - i\beta}^{\alpha_1 + i\beta} + \int_{\alpha_1 + i\beta}^{-\alpha_2 + i\beta} + \int_{-\alpha_2 + i\beta}^{-\alpha_2 - i\beta} + \int_{-\alpha_2 - i\beta}^{\alpha_1 - i\beta} = K$$

in which K is a constant n-vector (since the contour contains a single pole at the origin). Since the integrand $\to \theta$ uniformly in $-\alpha_2 \leqq \mathrm{Re}\,[s] \leqq \alpha_1$ as $|\,\mathrm{Im}\,[s]\,| \to \infty$, the integrals over the horizontal pieces $\to \theta$ as $\beta \to \infty$, and therefore for $t \geqq 0$

$$\lim_{\beta\to\infty} \int_{\alpha_1 - i\beta}^{\alpha_1 + i\beta} = K - \lim_{\beta\to\infty} \int_{-\alpha_2 + i\beta}^{-\alpha_2 - i\beta}.$$

We now show that

$$\lim_{\beta\to\infty} \int_{-\alpha_2 + i\beta}^{-\alpha_2 - i\beta} \to \theta$$

exponentially as $t \to \infty$.

On $(l-)$ each element of $W(s)$ is bounded. Thus on $(l-)$ each element of

$$Q(s) \triangleq [1_n s^2 + sC + \tilde{A}]^{-1}[V_2 + W(s)]$$

(we consider the part involving V_1 later) is bounded and of order at most $|\,s\,|^{-2}$ as $|\,s\,| \to \infty$. Therefore

$$\int_{(l-)} Q(s)e^{st}\,ds = \exp\,(-\alpha_2 t) \lim_{\beta\to\infty} \int_{i\beta}^{-i\beta} Q(-\alpha_2 + i\omega)e^{i\omega t}\,d(i\omega)$$

But $\int_{i\beta}^{-i\beta}$ is uniformly bounded on the t-set $[0, \infty)$, and hence

$$\int_{(l-)} Q(s)e^{st}\,ds$$

approaches zero exponentially.

Consider now the integral

$$\int_{(l-)} [1_n s^2 + sC + \tilde{A}]^{-1} s V_1 e^{st}\,ds.$$

We cannot directly apply the same argument as for the integration of Q because here it need not be true that on $(l-)$ each component of the integrand is of order at most $|s|^{-2}$ as $|s| \to \infty$. However, if on $(l-)$ some component $R_i(s)$ of $[1_n s^2 + sC + \tilde{A}]^{-1} s V_1$ is not of order at most $|s|^{-2}$ as $|s| \to \infty$, then it can be written (to within a constant multplier) as

$$R_i(s) = \frac{s^{2n-1}\Sigma^{(2n-1)} + s^{2n-2}\Sigma^{(2n-2)} + \cdots + \Sigma^{(0)}}{s^{2n} + \xi_D(s)}$$

in which $s^{-2n}\xi_D(s) \to 0$ as $|s| \to \infty$ on $(l-)$, and $\sum^{(2n-1)}$, $\sum^{(2n-2)}, \cdots, \sum^{(0)}$ denote sums of exponentials (some sums may be constants). Thus the sum of $R_i(s)$ and

$$-\frac{\Sigma^{(2n-1)}}{s+\sigma} \qquad (\sigma > \alpha_0)$$

is of order at most $|s|^{-2}$ as $|s| \to \infty$ on $(l-)$. Since the inverse Laplace transform of

$$\frac{\Sigma^{(2n-1)}}{s+\sigma}$$

vanishes faster than some damped exponential, we see that

$$\int_{(l-)} [1_n s^2 + sC + \tilde{A}]^{-1} s V_1 e^{st}\, ds, \qquad t \geqq 0$$

can be written as the sum of two integrals, the components of both of which approach zero at least as fast as some damped exponential. □

Proof of Lemma 4: Since $F(\cdot)$ satisfies

$$\ddot{F} + C\dot{F} + \underline{A}F = \theta, \qquad t \geqq 0$$

we have

$$\dot{F}(t) = e^{-Ct}\dot{F}(0) - \int_0^t e^{-C(t-\tau)}(\underline{A}F)(\tau)\, d\tau, \qquad t \geqq 0.$$

But $F(t) \to K$ as $t \to \infty$. Therefore as $t \to \infty$,

$$-\dot{F}(t) \to \lim_{t\to\infty} \int_0^t e^{-C(t-\tau)} AK\, d\tau = C^{-1}AK.$$

In addition, since

$$F(t) = \int_0^t \dot{F}(t)\, dt + F(0), \qquad t \geqq 0$$

we see that $-C^{-1}AK$ [that is, the limit of $\dot{F}(t)$ as $t \to \infty$] must be the zero vector, for otherwise $F(t)$ could not approach a constant vector as

$t \to \infty$. Thus $AK = \theta$. We note that $A(1, 1, \cdots, 1)^{tr} = \theta$, and that A is of rank $(n - 1)$. Therefore $K = \rho(1, 1, \cdots, 1)^{tr}$ for some constant ρ. □

3.2 *Proof of Theorem 2*

By Lemma 2 of the proof of Theorem 1, $\Delta(s) \neq 0$ for Re $[s] \geqq 0$ and $s \neq 0$. Let $\Delta(s) = 0$ with $s = -\alpha + i\omega$, $\alpha > 0$, and $\omega \neq 0$. Then for some $x \neq \theta$

$$-2\alpha i\omega + i\omega \frac{x^*Cx}{\| x \|^2} + \frac{x^*\tilde{A}_S x}{\| x \|^2} = 0$$

or

$$-2\alpha + \frac{x^*Cx}{\| x \|^2} + \frac{x^*(i\omega)^{-1}\tilde{A}_S x}{\| x \|^2} = 0.$$

Let $\bar{\tau} = \max_{i \neq j} \{\tau_{ij}\}$. Then $(i\omega)^{-1}\tilde{A}_S = e^{\alpha\bar{\tau}}B_S$ with

$$\begin{aligned}(B_S)_{ij} &= 0, \qquad \text{for all } i = j\\ &= a_{ij}\tau_{ij} \exp\,[\alpha(\tau_{ij} - \bar{\tau})] \frac{\sin \omega\tau_{ij}}{\omega\tau_{ij}}, \qquad \text{for all } i \neq j.\end{aligned}$$

Thus since $| (B_S)_{ij} | \leqq a_{ij}\tau_{ij}$ for all $i \neq j$, all $\alpha > 0$, and all $\omega \neq 0$; and

$$c_i - \sum_{j \neq i} a_{ij}\tau_{ij} \geqq \delta$$

for all i, it follows that

$$\frac{x^*Cx}{\| x \|^2} \pm \frac{x^*B_S x}{\| x \|^2} \geqq \delta$$

for all $\alpha > 0$ and all $\omega \neq 0$. Therefore with

$$c = \frac{x^*Cx}{\| x \|^2} \qquad \text{and} \qquad b = \frac{x^*B_S x}{\| x \|^2},$$

we have

$$-2\alpha + c + be^{\alpha\bar{\tau}} = 0$$

with $| b | \leqq c - \delta$ and $c \leqq \max_i c_i$. But

$$-2\alpha + c \leqq | -2\alpha + c | = | b | e^{\alpha\bar{\tau}} \leqq (c - \delta)e^{\alpha\bar{\tau}}$$

and hence

$$-2\alpha + c \leqq (c - \delta)e^{\alpha\bar{\tau}}, \qquad c \leqq \max_i c_i .$$

Therefore α must not be less than α_0, the unique solution of

$$-2\alpha_0 + \max_i c_i = [\max_i c_i - \delta]e^{\alpha_0 \bar{\tau}}. \quad \square$$

3.3 *Proof of Theorem 3*

We have for all i

$$\dot{f}_i = c_i(f_{0i} - f_i) + \sum_{j \neq i} a_{ij} b_{ij}, \qquad t \geqq 0$$

with

$$\dot{b}_{ij}(t) = [f_j(t - \tau_{ij}) - f_i(t)].$$

Observe that

$$\begin{aligned}
\int_0^\tau f_j(t - \tau_{ij})\, dt
&= \int_{-\tau_{ij}}^{\tau - \tau_{ij}} f_j(u)\, du \\
&= \int_{-\tau_{ij}}^{0} f_j(u)\, du + \int_0^{\tau - \tau_{ij}} [f_j(u) - \rho]\, du + (\tau - \tau_{ij})\rho \\
&= (\tau - \tau_{ij})\rho + \int_{-\tau_{ij}}^{0} f_j(u)\, du + \int_0^\infty [f_j(u) - \rho]\, du \\
&\qquad - \int_{\tau - \tau_{ij}}^{\infty} [f_j(u) - \rho]\, du.
\end{aligned}$$

Since

$$f_i = \exp(-c_i t) f_i(0) + \int_0^t \exp[-c_i(t - \tau)] \cdot [c_i f_{0i} + \sum_{j \neq i} a_{ij} b_{ij}(\tau)]\, d\tau, \qquad t \geqq 0$$

for all i,

$$\begin{aligned}
f_i = \exp(-c_i t) f_i(0) + \int_0^t \exp[-c_i(t - \tau)] \Big\{ & c_i f_{0i} + \sum_{j \neq i} a_{ij} b_{ij}(0) \\
& - \sum_{j \neq i} a_{ij} \tau_{ij} \rho + \sum_{j \neq i} a_{ij} \int_{-\tau_{ij}}^{0} f_j(u)\, du + \sum_{j \neq i} a_{ij}(p_j - p_i) \\
& - \sum_{j \neq i} a_{ij}[q_{ij}(\tau) - r_i(\tau)] \Big\}\, d\tau, \qquad t \geqq 0 \qquad (26)
\end{aligned}$$

for all i, in which

$$p_i = \int_0^\infty [f_i(u) - \rho]\, du$$

$$q_{ij}(\tau) = \int_{\tau-\tau_{ij}}^\infty [f_i(u) - \rho]\, du$$

$$r_i(\tau) = \int_\tau^\infty [f_i(u) - \rho]\, du.$$

The functions $q_{ij}(\tau)$ and $r_i(\tau)$ approach zero as $\tau \to \infty$. In order that the asymptotic values (as $t \to \infty$) of both sides of equation (26) agree, we must have

$$\rho c_i = c_i f_{0i} + \sum_{j \neq i} a_{ij} b_{ij}(0) - \sum_{j \neq i} a_{ij} \tau_{ij} \rho + \sum_{j \neq i} a_{ij} \int_{-\tau_{ij}}^0 f_j(u)\, du + \sum_{j \neq i} a_{ij}(p_j - p_i)$$

for all i. But (using the assumption that $a_{ij} = a_{ji}$ for all $i \neq j$)

$$\sum_i \sum_{j \neq i} a_{ij}(p_j - p_i) = 0.$$

Therefore

$$\rho \sum_i c_i = \sum_i c_i f_{0i} + \sum_i \sum_{j \neq i} a_{ij} b_{ij}(0) - \rho \sum_i \sum_{j \neq i} a_{ij} \tau_{ij} + \sum_i \sum_{j \neq i} a_{ij} \int_{-\tau_{ij}}^0 f_j(u)\, du. \quad \square^*$$

3.4 *Proof of Theorem 4*

The following lemma and Lemmas 1, 3, and 4 of the proof of Theorem 1 prove Theorem 4.

Lemma 2′: *If*

$$c_i \geqq (2 \sum_{j \neq i} a_{ij})^{\frac{1}{2}} \qquad \text{for all } i,$$

then there exists a positive constant α_0 *such that* $\Delta(s) = 0$ *and* $s \neq 0$ *imply that* $Re\,[s] \leqq -\alpha_0$.

Proof of Lemma 2′: With $D_0 = \text{diag}\,\{\sum_{j \neq i} a_{ij}\}$ and $\tilde{A}' = \tilde{A} - D_0$,

* The last part of this proof, which involves the observation that the double sum is zero, is similar to an argument used by Brilliant (see the appendix of Ref. 5).

$$\begin{aligned}\Delta(s) &= \det [1_n s^2 + sC + \tilde{A}] \\ &= \det [1_n s^2 + sC + D_0 + \tilde{A}'] \\ &= \det [1_n s^2 + sC + D_0]\cdot\det [1_n + (1_n s^2 + sC + D_0)^{-1}\tilde{A}'].\end{aligned}$$

The diagonal elements of D_0 are positive. It is therefore clear that there is a positive constant α_0' such that if $\det [1_n s^2 + sC + D_0] = 0$, then $\text{Re}\,[s] \leqq -\alpha_0'$. It is also clear that $\det [1_n + (1_n s^2 + sC + D_0)^{-1}\tilde{A}']$ is not zero for all s such that $\text{Re}\,[s] \geqq 0$ and

$$\left| (s^2 + sc_i + \sum_{j\neq i} a_{ij})^{-1} \sum_{j\neq i} a_{ij} \right| < 1 \qquad \text{for all } i, \tag{27}$$

since for those values of s the matrix $[1_n + (1_n s^2 + sC + D_0)^{-1}\tilde{A}']$ is strongly row-sum dominant.

By assumption,

$$c_i \geqq (2 \sum_{j\neq i} a_{ij})^{\frac{1}{2}} \qquad \text{for all } i.$$

It is a simple matter to verify that this assumption implies that condition (27) is satisfied for all $s = i\omega$ with ω real and $\omega \neq 0$. Thus $\Delta(i\omega) \neq 0$ for all $\omega \neq 0$. But for all i

$$(s^2 + sc_i + \sum_{j\neq i} a_{ij})^{-1} \sum_{j\neq i} a_{ij}$$

is analytic throughout the closed right-half s plane, and

$$\left| (s^2 + sc_i + \sum_{j\neq i} a_{ij})^{-1} \sum_{j\neq i} a_{ij} \right| \leqq 1$$

for all i and for all $s = i\omega$. By the Maximum Modulus Theorem, condition (27) is satisfied for all s such that $\text{Re}\,[s] > 0$.* Therefore $\Delta(s) \neq 0$ for all $s \neq 0$ such that $\text{Re}\,[s] \geqq 0$. Finally, the argument used to prove the last part of Lemma 2 shows that there exists a positive constant α_0 such that if $\Delta(s) = 0$ with $s \neq 0$, then $\text{Re}\,[s] \leqq -\alpha_0$. □

APPENDIX

Left-Half-Plane Zeros of det $[I - B(s)]$

We wish to show here that all of the left-half-plane zeros of $\det [I - B(s)]$, in which $I - B(s)$ is as defined in Ref. 3, lie to the left of some line which is parallel to and lies to the left of the imaginary axis of the complex s plane, provided that each $H_i(s)$, which enters into the definition of $B(s)$, is a meromorphic function of s such that there

* This type of argument is also used in Ref. 3.

exist positive constants σ_i and K_i with the property that $| H_i(s) | \leqq K_i$ for all s with $-\sigma_i \leqq \text{Re}\,[s] < 0$.

Assume that what we wish to prove is false. Then, as in the proof of the last part of Lemma 2, there would exist a sequence $\mathcal{S} = \{s_k\}_0^\infty$ such that $\text{Re}\,[s_k] < 0$ for all k, $\text{Re}\,[s_k] \to 0$ as $k \to \infty$, $\text{Im}\,[s_k] \to \infty$ as $k \to \infty$, and $\det\,[I - B(s_k)] = 0$ for all $k \geqq 0$. But the complex numbers $\tilde{a}_{ij}$ of Ref. 3 are bounded on $\mathcal{S}$ and $H_i(s_k)[s_k + H_i(s_k)]^{-1} \to 0$ as $k \to \infty$. This means (see Ref. 3) that there is a positive number k' such that the matrix $[I - B(s_k)]$ is strongly dominant for all $k \geqq k'$, which contradicts the assumption that $\det\,[I - B(s_k)] = 0$ for all $k \geqq 0$, and proves that our assertion is true.

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