Amplitude and Phase Modulations in Resistive Diode Mixers

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This paper presents some new aspects of mixer operation and shows that the behavior of a mixer, using a resistive diode, can in general be represented by means of an equivalent circuit consisting of two transducers connected in cascade. The first of these two transducers transforms the input signals into amplitude modulations; the second is an AM detector which transforms these amplitude modulations into output signals. An important feature of this equivalent circuit is that it gives the dependence of the conversion loss upon certain important mixer parameters.

We also show that extremely low (< 0.3 dB) conversion losses can be achieved from a Schottky barrier diode, if the pump frequency ω_0 is much smaller than the cutoff frequency of the diode. To achieve such low conversion losses the diode must be open-circuited at the harmonics $2\omega_0$, $3\omega_0$, $4\omega_0$, and so on, of the pump frequency.

I. INTRODUCTION AND SUMMARY OF THE PRINCIPAL RESULTS

The process of frequency conversion and its applications are well known and are extensively treated in the literature.[1-12] This paper considers the special case of a resistive diode frequency converter. We assume that the diode is pumped periodically by a strong source, the pump, which generates power at a single frequency ω_0. In most of the analysis we assume that the frequency converter is required to transform small input signals occurring in the vicinity of ω_0 into low-frequency output signals.

In such a frequency converter the large signals occurring at the frequencies 0, ω_0, $2\omega_0$, $3\omega_0$, and so on, are perturbed by small amplitude and phase modulations caused by the input signals. The occurrence of these modulations can be shown by using the amplitude-phase representation (sometimes called AM-PM representation) which describes the small signals occurring in the vicinity of the kth harmonic of ω_0 in terms of the corresponding amplitude and phase modulations. When this representation is used, a convenient way of determining the small-signal terminal behavior of the frequency converter is provided by the incremental method. This method consists of perturbing the large-signals by introducing small and stationary variations in the pump level and in the dc bias. Then, from the relations between these variations and the perturbations they cause in the large signals, the small-signal terminal behavior of the frequency converter can be readily determined. In fact, we show that the connection between the foregoing relations and the desired terminal behavior can be represented by means of a simple equivalent circuit.

This equivalent circuit represents the frequency converter as a cascade connection of two transducers, as shown in Fig. 1. The first of these two transducers transforms the input signals into amplitude modulations with carrier frequency ω_0. The second is an AM detector which transforms these amplitude modulations into the desired low-frequency output signals.

In other words, when a small signal generator at some frequency $\omega_0 + p$, with $p \ll \omega_0$, is connected to the input terminals of the frequency converter, small signals are produced in the diode at $\omega_0 + p$ and $\omega_0 - p$. These signals contain two distinct types of components, the components which are produced by amplitude modulations, and the components which are produced by phase modulations. The second transducer of Fig. 1 shows that there is a one-to-one correspondence between the output signals of the frequency converter and the amplitude modulation components. The first transducer represents the relations between these components and the input signals.

An important feature of the foregoing equivalent circuit is that it shows the dependence of the conversion loss upon certain important parameters of the frequency converter. This follows from the fact that the first transducer can be represented by means of a network consisting of two ideal transformers and two admittances. One of these two admittances is the admittance $y_{\beta 1}$ terminating the diode at the

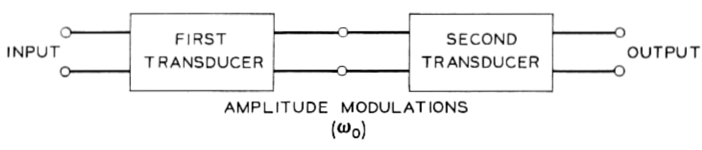

Fig. 1 — Equivalent circuit representing a frequency converter as a cascade connection of two transducers.

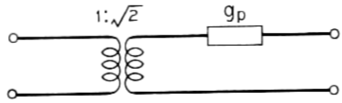

Fig. 2 — First transducer of Fig. 1 for $y_{\beta 1} = 0$.

image frequency; the other is the nonlinear admittance g_p presented by the diode at the pump frequency ω_0. In particular, if $y_{\beta 1} = 0$ then the first transducer can be represented as shown in Fig. 2. An analogous equivalent circuit is obtained when $y_{\beta 1} = \infty$.

Finally, we examine the conversion loss of a Schottky barrier diode frequency converter. Its minimum value can be expressed as

$$L \cong 1 + 4 \exp\left(-\frac{q}{KT}\frac{\Delta V}{4}\right), \tag{1}$$

if the parasitic capacitance of the diode can be neglected. In equation (1) ΔV is the difference between the maximum and minimum values of the large-signal voltage of the diode. In order to achieve this conversion loss the diode must be open-circuited at all frequencies except $\omega = 0$, $\omega = \omega_0$, and the input and output frequencies.

Practical considerations almost always require that the input and output impedances be much smaller than the impedances required for achieving the conversion loss given by equation (1). If these practical considerations are taken into account, one finds that the frequency converter can be represented by means of the equivalent circuit shown in Fig. 3. The conversion loss of this equivalent circuit depends upon: the impedance R of the input signal generator, the diode series resistance R_s, and the dc component I_{c0} of the current flowing through the diode. In fact, the conversion loss can be expressed as

$$L = 1 + \frac{2}{R}\left(R_s + \frac{KT}{qI_{c0}}\right). \tag{2}$$

Notice that the operating frequencies at which this conversion loss

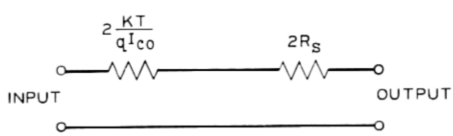

Fig. 3 — Low frequency equivalent circuit of a Schottky barrier diode frequency converter under optimum circuit conditions.

can be achieved are limited by the junction capacitance C_j of the diode. Finally, notice that equation (2) implies that R_s does not set, by itself, any limit to the conversion loss.

II. SMALL-SIGNAL TERMINAL BEHAVIOR OF PUMPED NONLINEAR RESISTORS

Consider a nonlinear element in which the terminal voltage is related to the current by

$$i = f(v) \tag{3}$$

and let

$$g_d(v) = \frac{df(v)}{dv}. \tag{4}$$

Then, if the voltage v is the sum of a large component $v_c(t)$ and a small component $\delta v(t)$, that is,

$$v = v(t) = v_c(t) + \delta v(t), \tag{5}$$

the current can be written to a first approximation as

$$i \cong (t) = i_c(t) + \delta i(t), \tag{6}$$

where

$$i_c(t) = f[v_c(t)] \tag{7}$$

and

$$\delta i(t) = g(t)\ \delta v(t) \tag{8}$$

with

$$g(t) = g_d[v_c(t)]. \tag{9}$$

Equation (8) completely describes the small-signal terminal behavior of a nonlinear resistor pumped by a large-signal voltage $v_c(t)$, in the absence of internal noise sources.

In equation (8) $g(t)$ is the time-varying differential conductance of the nonlinear resistor, as shown by equations (4) and (9). In the following analysis we assume that $v_c(t)$ is periodic with some frequency ω_0. Therefore $i_c(t)$ and $g(t)$ also are periodic. Furthermore, we assume that the circuit connected to the nonlinear resistor is resistive at the harmonics $2\omega_0$, $3\omega_0$, $4\omega_0$, and so on, and at these frequencies, does not contain generators. Under these conditions it is always possible to choose the origin of time in such a way as to make $v_c(t)$, $i_c(t)$, and $g(t)$ even functions of time.[4–6] Thus let

$$v_c(t) = v_c(-t), \qquad i_c(t) = i_c(-t), \qquad g(t) = g(-t). \tag{10}$$

This allows one to write $v_c(t)$, $i_c(t)$, and $g(t)$ in the form

$$v_c(t) = V_{c0} + 2\sum_{k=1}^{\infty} V_{ck}\cos k\omega_0 t \tag{11}$$

$$i_c(t) = I_{c0} + 2\sum_{k=1}^{\infty} I_{ck}\cos k\omega_0 t \tag{12}$$

$$g(t) = g_0 + 2\sum_{k=1}^{\infty} g_k\cos k\omega_0 t. \tag{13}$$

III. AMPLITUDE-PHASE REPRESENTATION OF $\delta v(t)$ AND $\delta i(t)$

Normally, it is convenient to represent the perturbations $\delta v(t)$ and $\delta i(t)$ in terms of the Fourier coefficients of their frequency components. The discussion of certain properties of the behavior of a pumped diode is often simplified by the use of an alternative representation, the so-called amplitude-phase representation. The main feature of this representation is that it emphasizes the amplitude and phase modulations occurring, because of $\delta v(t)$ and $\delta i(t)$, in the various quasisinusoidal harmonic components of $v(t)$ and $i(t)$.

This representation is well known and has already been shown to be particularly useful in simplifying the analysis of certain pumped nonlinear systems.[13–15] In this section a complete derivation of it is given, in a form suitable for the discussion of the behavior of a frequency converter. We regard $\delta v(t)$ and $\delta i(t)$ as representing small perturbations produced on $v(t)$ and $i(t)$, about the condition $v(t) = v_c(t)$ and $i(t) = i_c(t)$. In accordance with this point of view, which is clarified by the following discussion, $\delta v(t)$ and $\delta i(t)$ are often referred to as perturbations.

An arbitrary voltage perturbation $\delta v(t)$ can always be expressed in the form

$$\delta v(t) = \delta v_{a0}(t) + \sqrt{2}\sum_{k=0}^{\infty}[\delta v_{ak}(t)\cos k\omega_0 t + \delta v_{pk}(t)\sin k\omega_0 t] \tag{14}$$

where $\delta v_{a0}(t)$, $\delta v_{ak}(t)$, and $\delta v_{pk}(t)$ are low-pass functions limited to the band $|\omega| < \omega_0/2$. Therefore in equation (14) the time function

$$\delta v_k(t) = \sqrt{2}[\delta v_{ak}(t)\cos k\omega_0 t + \delta v_{pk}(t)\sin k\omega_0 t] \tag{15}$$

represents the components of $\delta v(t)$ occurring in the frequency range between $k\omega_0 - \omega_0/2$ and $k\omega_0 + \omega_0/2$ and between $-k\omega_0 - \omega_0/2$ and $-k\omega_0 + \omega_0/2$. In particular, if $\delta v(t)$ contains frequency components at only the side-frequencies $r\omega_0 \pm p(|r| = 0, 1, 2,$ and so on; $0 \leqq p < \omega_0/2)$, then $\delta v_k(t)$ and $\delta v_{a0}(t)$ can be written as

$$\delta v_k(t) = 2(\text{Re})\{V_{\alpha k} \exp [j(p + k\omega_0)t] + V_{\beta k} \exp [j(p - k\omega_0)t]\} \qquad (k = 1, 2, \text{and so on}) \tag{16}$$

$$\delta v_{a0}(t) = 2(\text{Re})(V_{\alpha 0} e^{jpt}) \tag{17}$$

where $V_{\alpha 0}$, $V_{\alpha k}$ and $V_{\beta k}$ are the complex amplitudes of the components of $\delta v(t)$ occurring at p, $p + k\omega_0$, and $p - k\omega_0$, respectively ($k = 1, 2, 3, \cdots$). Equation (16) can be rewritten in the form

$$\delta v_k(t) = 2(\text{Re})[(V_{\alpha k} + V_{\beta k})e^{jpt} \cos k\omega_0 t + j(V_{\alpha k} - V_{\beta k})e^{jpt} \sin k\omega_0 t]. \tag{18}$$

A comparison of this expression and equation (15) shows that the low-pass functions $\delta v_{ak}(t)$ and $\delta v_{pk}(t)$ are sinusoidal of frequency p. That is, if equation (16) is satisfied one can write

$$\delta v_{ak}(t) = 2(\text{Re})(V_{ak} e^{jpt}) \qquad (k = 0, 1, 2, \cdots) \tag{19}$$

$$\delta v_{pk}(t) = 2(\text{Re})(V_{pk} e^{jpt}) \qquad (k = 1, 2, 3, \cdots). \tag{20}$$

Furthermore, by comparing equation (18) and the expressions that one obtains by substituting equation (19) and (20) into equation (15), one obtains the following relations between the Fourier coefficients of $\delta v_{ak}(t)$ and $\delta v_{pk}(t)$ and those of $\delta v(t)$

$$\begin{bmatrix} V_{ak} \\ V_{pk} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} V_{\alpha k} \\ V_{\beta k} \end{bmatrix} \qquad (k = 1, 2, 3, \cdots). \tag{21}$$

Furthermore,

$$V_{a0} = V_{\alpha 0}. \tag{22}$$

The two sets of low-pass functions $\delta v_{a0}(t)$, $\delta v_{a1}(t)$, and so on, and $\delta v_{p1}(t)$, $\delta v_{p2}(t)$, and so on, defined by equation (14) provide a complete representation of the voltage perturbation $\delta v(t)$. Their significance can be readily obtained by noticing that if $v(t)$ is written as

$$v(t) = V_0(t) + \sqrt{2} \sum_{k=1}^{\infty} V_k(t) \cos [k\omega_0 t + \delta\varphi_k(t)], \tag{23}$$

where $V_0(t)$, $V_k(t)$ and $\delta\varphi_k(t)$ are low-pass functions limited to the band $|\omega| < \omega_0/2$, from equations (5), (11), and (14) one obtains

$$V_0(t) = V_{c0} + \delta v_{a0}(t) \tag{24}$$

$$V_k(t) = \sqrt{2}\, V_{ck} + \delta v_{ak}(t) \qquad (k > 0) \tag{25}$$

$$\delta\varphi_k(t) = \frac{-1}{\sqrt{2}\, V_{ck}}\, \delta v_{pk}(t) \qquad (k > 0). \tag{26}$$

That is, if one decomposes the voltage $v(t)$ into a sum of harmonic terms, one finds that, because of the presence of the perturbation $\delta v(t)$, each harmonic component is modulated. The kth harmonic component is amplitude modulated by $\delta v_{ak}(t)$ and phase modulated by $\delta v_{pk}(t)$. Because of this property the representation of $\delta v(t)$ in terms of the low-pass functions $\delta v_{ak}(t)$ and $\delta v_{pk}(t)$ (or of their Fourier coefficients V_{ak} and V_{pk}) is called the amplitude-phase representation of $\delta v(t)$. The direct representation of $\delta v(t)$ by means of the Fourier coefficients $V_{\alpha k}$ and $V_{\beta k}$ is called the α-β representation.[13]

The foregoing discussion can be extended to $\delta i(t)$ by replacing v with i throughout. That is, $\delta i(t)$ can be represented in terms of low-pass functions $\delta i_{ak}(t)$ and $\delta i_{pk}(t)$

$$\delta i(t) = \delta i_{a0}(t) + \sqrt{2} \sum_{k=1}^{\infty} [\delta i_{ak}(t) \cos k\omega_0 t + \delta i_{pk}(t) \sin k\omega_0 t]. \tag{27}$$

Furthermore, in the special case where $\delta i(t)$ of the type

$$\delta i(t) = 2(\mathrm{Re})\left\{ \sum_{k=0}^{\infty} I_{\alpha k} \exp[j(p + k\omega_0)t] + \sum_{k=1}^{\infty} I_{\beta k} \exp[j(p - k\omega_0)t] \right\}, \tag{28}$$

the low-pass functions $\delta i_{ak}(t)$ and $\delta i_{pk}(t)$ are sinusoidal of frequency p and, if I_{ak} and I_{pk} are their Fourier coefficients, one has

$$\begin{bmatrix} I_{ak} \\ I_{pk} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} I_{\alpha k} \\ I_{\beta k} \end{bmatrix} \qquad (k = 1, 2, 3, \cdots) \tag{29}$$

and

$$I_{a0} = I_{\alpha 0}\,. \tag{30}$$

Equations (21) and (29) can be interpreted, because of their particular form, as the relations describing the terminal behavior of a four-terminal-pairs lossless network consisting of ideal transformers, such as the network T shown in Fig. 4.

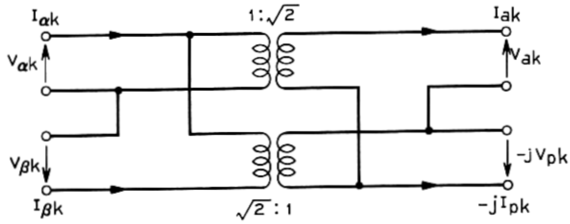

Fig. 4 — Network T describing the relations between the $\alpha - \beta$ signals at $p \pm k\omega_0$ and the corresponding amplitude and phase modulations.

IV. SMALL-SIGNAL TERMINAL BEHAVIOR OF THE DIODE IN THE AMPLITUDE-PHASE REPRESENTATION

According to equations (14) and (27)

$$\delta v(t) = \delta v_a(t) + \delta v_p(t) \tag{31}$$

$$\delta i(t) = \delta i_a(t) + \delta i_p(t) \tag{32}$$

where $\delta v_a(t)$ and $\delta i_a(t)$ are the amplitude modulation components of $\delta v(t)$ and $\delta i(t)$, and $\delta v_p(t)$ and $\delta i_p(t)$ are the phase modulation components. Thus

$$\delta v_a(t) = \delta v_{a0}(t) + \sqrt{2} \sum_{k=1}^{\infty} \delta v_{ak}(t) \cos k\omega_0 t \tag{33}$$

$$\delta v_p(t) = \sqrt{2} \sum_{k=1}^{\infty} \delta v_{pk}(t) \sin k\omega_0 t \tag{34}$$

$$\delta i_a(t) = \delta i_{a0}(t) + \sqrt{2} \sum_{k=1}^{\infty} \delta i_{ak}(t) \cos k\omega_0 t \tag{35}$$

$$\delta i_p(t) = \sqrt{2} \sum_{k=1}^{\infty} \delta i_{pk}(t) \sin k\omega_0 t. \tag{36}$$

Since from equations (13), (35), and (33) one has that $\delta v_a(t) g(t)$ only contains amplitude modulation components and $\delta v_p(t) g(t)$ only contains phase modulation components, by substituting equations (31) and (32) into equation (8) one obtains

$$\delta i_a(t) = g(t)\ \delta v_a(t) \tag{37}$$

$$\delta i_p(t) = g(t)\ \delta v_p(t). \tag{38}$$

The relations between the low-pass functions $\delta i_{a0}(t)$, $\delta i_{a1}(t)$, and so on and $\delta v_{a0}(t)$, $\delta v_{a1}(t)$, δv_{a2}, . . . may now be obtained by substituting equations (13), (33), and (35) into equation (37). One obtains

$$\delta i_{a0}(t) + \sqrt{2} \sum_{k=1}^{\infty} \delta i_{ak}(t) \cos k\omega_0 t = g_0 \, \delta v_{a0}(t) + \sqrt{2} \sum_{r=1}^{\infty} g_r \, \delta v_{ar}(t) + \sum_{k=1}^{\infty} \left[2g_k \, \delta v_{a0}(t) + \sqrt{2} \sum_{r=1}^{\infty} (g_{r+k} + g_{|r-k|}) \, \delta v_{ar}(t) \right] \cos k\omega_0 t, \tag{39}$$

which gives

$$\begin{bmatrix} \delta i_{a0}(t) \\ \delta i_{a1}(t) \\ \delta i_{a2}(t) \\ \delta i_{a3}(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} g_0, & \sqrt{2}\, g_1, & \sqrt{2}\, g_2, & \sqrt{2}\, g_3, & \cdots \\ \sqrt{2}\, g_1, & g_0 + g_2, & g_1 + g_3, & g_2 + g_4, & \cdots \\ \sqrt{2}\, g_2, & g_1 + g_3, & g_0 + g_4, & g_1 + g_5, & \cdots \\ \sqrt{2}\, g_3, & g_2 + g_4, & g_1 + g_5, & g_0 + g_6, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \delta v_{a0}(t) \\ \delta v_{a1}(t) \\ \delta v_{a2}(t) \\ \delta v_{a3}(t) \\ \vdots \end{bmatrix}. \tag{40}$$

In a completely similar way, from equations (13), (34), (36) and (38) one obtains

$$\begin{bmatrix} \delta i_{p1}(t) \\ \delta i_{p2}(t) \\ \delta i_{p3}(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} g_0 - g_2, & g_1 - g_3, & g_2 - g_4, & \cdots \\ g_1 - g_3, & g_0 - g_4, & g_1 - g_5, & \cdots \\ g_2 - g_4, & g_1 - g_5, & g_0 - g_6, & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \delta v_{p1}(t) \\ \delta v_{p2}(t) \\ \delta v_{p3}(t) \\ \vdots \end{bmatrix}. \tag{41}$$

Equations (40) and (41) provide the amplitude-phase admittance-matrix representation of the small-signal terminal behavior of the diode, in the absence of internal noise sources. If the various low-pass functions $\delta v_{ak}(t)$ and $\delta v_{pk}(t)$ are sinusoidal of frequency p, then equations (40) and (41) give

$$\begin{aligned} I_a] &= [G_a] V_a] \\ I_p] &= [G_p] V_p] \end{aligned} \tag{42}$$

where the matrix notation is defined as

$$[G_a] = \begin{bmatrix} g_0 , & \sqrt{2}\, g_1 , & \sqrt{2}\, g_2 , & \sqrt{2}\, g_3 , & \cdots \\ \sqrt{2}\, g_1 , & g_0 + g_2 , & g_1 + g_3 , & g_2 + g_4 , & \cdots \\ \sqrt{2}\, g_2 , & g_1 + g_3 , & g_0 + g_4 , & g_1 + g_5 , & \cdots \\ \sqrt{2}\, g_3 , & g_2 + g_4 , & g_1 + g_5 , & g_0 + g_6 , & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{43}$$

$$[G_p] = \begin{bmatrix} g_0 - g_2 , & g_1 - g_3 , & g_2 - g_4 , & \cdots \\ g_1 - g_3 , & g_0 - g_4 , & g_1 - g_5 , & \cdots \\ g_2 - g_4 , & g_1 - g_5 , & g_0 - g_6 , & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{44}$$

and

$$I_a] = \begin{bmatrix} I_{a0} \\ I_{a1} \\ \vdots \end{bmatrix}, \qquad V_a] = \begin{bmatrix} V_{a0} \\ V_{a1} \\ \vdots \end{bmatrix}, \qquad I_p] = \begin{bmatrix} I_{p1} \\ I_{p2} \\ \vdots \end{bmatrix}, \qquad V_p] = \begin{bmatrix} V_{p1} \\ V_{p2} \\ \vdots \end{bmatrix}. \tag{45}$$

Notice that I_{ak}, V_{ak}, I_{pr}, and V_{pr} are the Fourier coefficients of $\delta i_{ak}(t)$, $\delta v_{ak}(t)$, $\delta i_{pr}(t)$, and $\delta v_{pr}(t)$, as shown by equations (19) and (20).

Equation (42) shows that the terminal behavior of the diode has the following special property: the diode does not produce any coupling between the amplitude modulation components and the phase modulation components of the perturbations $\delta v(t)$ and $\delta i(t)$; or equivalently, the diode does not produce amplitude $\rightleftarrows$ phase conversion. This property can be useful in simplifying the discussion of the terminal behavior of the diode because it allows the two types of signals (amplitude and phase) to be treated separately.

According to equation (42) the small-signal terminal behavior of the diode can be represented by two separate linear and time-invariant networks D_a and D_p, as shown in Fig. 5. In these two equivalent networks the terminal voltages and currents occur at the same frequency (p) and their Fourier coefficients are equal to those of the various low-pass functions of $\delta v(t)$ and $\delta i(t)$. Notice that from equations (43) and (44) one has that $[G_a]$ and $[G_p]$ are real symmetric matrices and

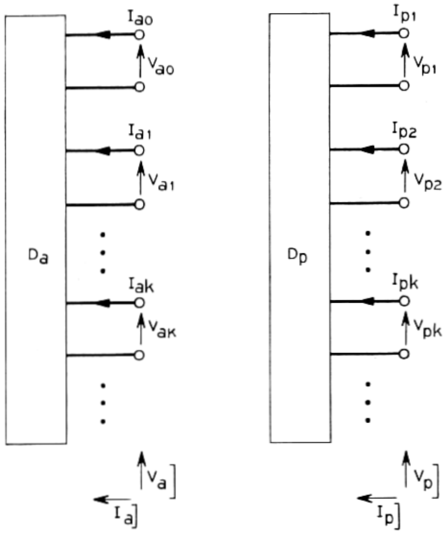

Fig. 5 — Time-invariant equivalent networks D_a and D_p providing the amplitude-phase representation of the terminal behavior of a resistive pumped diode.

therefore the two networks D_a and D_p can be realized by means of ordinary resistive networks.

V. TERMINAL BEHAVIOR OF A FREQUENCY CONVERTER IN THE AMPLITUDE-PHASE REPRESENTATION

In the following part of this paper the results obtained in the preceding sections are applied to the study of a very common type of frequency converter, a frequency converter which is required to transform input signals occurring at some frequency $\omega_0 + p$, with $p \ll \omega_0$, into output signals occurring at $\omega = p$. We assume that $\delta v(t)$ and $\delta i(t)$ contain components at only the pairs of side-frequencies $k\omega_0 + p$ and $k\omega_0 - p(|k| = 0, 1, 2,$ and so on; $2p < \omega_0)$.

In this section attention is focused on the signals occurring at p and $p \pm \omega_0$. At set of relations among these signals is derived, from the equations describing the small-signal terminal behavior of the diode, by taking into account the constraints imposed by the external circuit on the signals occurring at the side-frequencies of $2\omega_0$, $3\omega_0$, $4\omega_0$, and so on. We show that these relations establish the existence of a one-to-one correspondence between the output signals (V_{a0} and I_{a0}) and the amplitude modulation coefficients (V_{a1} and I_{a1}) of the signals occurring at $p \pm \omega_0$. That is, they do not impose any relation between the output signals and the phase modulation coefficients (V_{p1} and I_{p1}) of the signals occurring at $p \pm \omega_0$.

Let y_k, $y_{\alpha k}$ and $y_{\beta k}$ $(k > 0)$ denote the admittances terminating the diode at $k\omega_0$, $p + k\omega_0$, and $p - k\omega_0$, respectively. We assume that

$$y_k = y_{\alpha k} = y_{\beta k}, \qquad k > 1, \tag{46}$$

a condition satisfied in many practical applications. Notice that this condition is satisfied for $p \to 0$ because y_k is real.

Consider the equivalent network T shown in Fig. 4. One can verify that if one terminates the two terminal pairs relative to the (α, β) signals with two admittances equal to y_k then its behavior at the remaining two terminal pairs is described by the relations

$$I_{ak} = -y_k V_{ak}, \qquad I_{pk} = -y_k V_{pk} \qquad (k > 1), \tag{47}$$

which show that, if condition (46) is satisfied, then the external circuit does not produce amplitude $\rightleftarrows$ phase conversion. That is, if $y_{\alpha k} = y_{\beta k}$, then in the amplitude-phase representation the behavior of the external circuit at $p \pm k\omega_0$ can be represented by means of two separate one-terminal-pair networks, Furthermore, the admittances of these two networks are equal to the admittance presented by the external circuit at $p \pm k\omega_0$.

Now, consider the relations among the signals occurring at $p \pm \omega_0$ and p when the remaining signals are constrained by the conditions imposed by the external circuit at $p \pm 2\omega_0$, $p \pm 3\omega_0$, and so on. If the two networks D_a and D_p shown in Fig. 5 are terminated as specified by equations (47) one obtains the two networks A and P shown in Figs. 6 and 7. The relations between the terminal voltages and currents of these two networks can be written in the form

$$I_{p1} = g_p V_{p1} \tag{48}$$

$$\begin{bmatrix} I_{a0} \\ I_{a1} \end{bmatrix} = [G_a'] \begin{bmatrix} V_{a0} \\ V_{a1} \end{bmatrix} \tag{49}$$

where g_p and $[G_a']$ can be derived from equations (42) and (47). For example, in the special case where $y_k = \infty$ for $k > 1$, from equations (42) and (47) one obtains

$$[G_a'] = \begin{bmatrix} g_0 & \sqrt{2}\, g_1 \\ \sqrt{2}\, g_1 & g_0 + g_2 \end{bmatrix}, \qquad g_p = g_0 - g_2 . \tag{50}$$

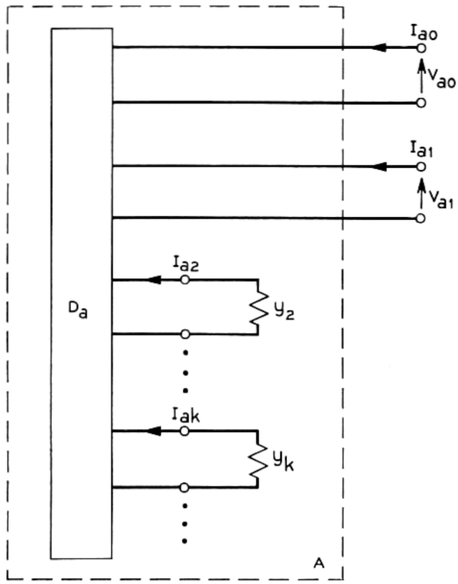

Fig. 6 — Network A representing the relations in a frequency converter between the output signals and the amplitude modulation signals at $p \pm \omega_0$.

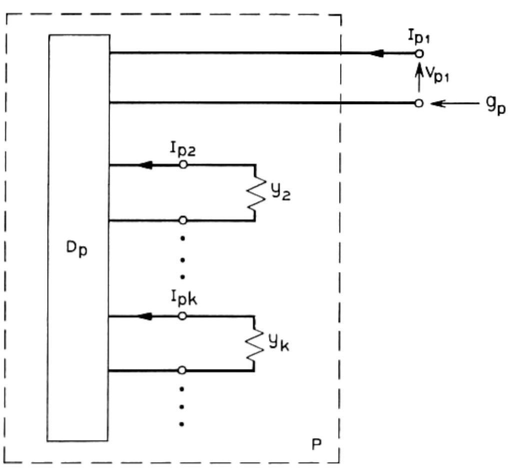

Fig. 7 — Network P representing the behavior of the diode at $p \pm \omega_0$ with respect to phase modulation.

Notice that equation (49) establishes the existence, in a frequency converter satisfying equation (47), of a one-to-one correspondence between the output signals V_{a0} and I_{a0} and the amplitude modulation signals V_{a1} and I_{a1} .

The significance of the two networks A and P is best understood by considering the following alternative method of deriving g_p and $[G_a']$. Suppose, for the moment, that $\delta v(t) = \delta i(t) = 0$ and that the level of the pump and the dc bias applied to the diode are variable. That is, assume that the amplitudes of the various harmonic components of $v(t)$ and $i(t)$ can be varied, by changing the level of the pump and the dc bias, while the terminations y_2 , y_3 , y_4 , $\cdots$ presented by the external circuit to the diode are kept constant. The functions $v_c(t)$ and $i_c(t)$ must satisfy the constraint $i_c(t) = f[v_c(t)]$ imposed by the diode. Furthermore, the harmonic components of $v_c(t)$ and $i_c(t)$ must satisfy the constraints $I_{ck} = -y_k V_{ck}$ $(k > 1)$ imposed by the external circuit. Clearly, from these constraints two independent nonlinear relations can be derived, among the four variables V_{c0} , V_{c1} , I_{c0} and I_{c1} . That is, if one considers the variables $\eta_0 = I_{c0}$, $\eta_1 = \sqrt{2} I_{c1}$, $x_0 = V_{c0}$ and $x_1 = \sqrt{2} V_{c1}$, one can write

$$\begin{aligned} \eta_0 &= F_0(x_0 , x_1) \\ \eta_1 &= F_1(x_0 , x_1) \end{aligned} \tag{51}$$

where the two nonlinear functions $F_0(x_0 , x_1)$ and $F_1(x_0 , x_1)$ are determined by the particular function $f(v)$ describing the diode nonlinearity and by the parameters y_2 , y_3 , y_4 , $\cdots$. Notice that the variables η_0 , η_1 , x_0 and x_1 represent the root mean square values of the harmonic components of order zero and one of $i(t)$ and $v(t)$.

Now, suppose $\delta v(t)$ and $\delta i(t)$ differ from zero and are produced by small generators occurring in the external circuit at the frequencies $p \pm \omega_0$ and p. We want to derive from equation (51) the relations among the coefficients $V_{\alpha 1}$, $I_{\alpha 1}$, $V_{\alpha 0}$, $I_{\alpha 0}$, V_{p1} and I_{p1} of $\delta v(t)$ and $\delta i(t)$. Since the terminal behavior of the two networks A and P (shown in Figs. 6 and 7) is frequency independent, the relations in question are independent of p and therefore one can set $p = 0$ without loss of generality. Thus, let $\delta v(t)$ and $\delta i(t)$ be produced by small-signal generators occurring at $\omega = \omega_0$ and $\omega = 0$. These small-signal generators can be interpreted as follows. A small-signal amplitude modulation generator occurring at ω_0 can be regarded as representing a small change of the level of the pump; a small-signal phase modulation generator occurring at ω_0 can be interpreted as a small change $\delta\varphi$ of the phase of the pump.

Finally, a small-signal generator occurring at dc can be interpreted as a small change produced in the dc bias circuit. When small changes of the above three types occur in the external circuit, the components of $\delta v(t)$ and $\delta i(t)$ at $\omega = 0$ and $\omega = \omega_0$ can be evaluated as follows.

First, consider the amplitude modulation components $\delta v_a(t)$ and $\delta i_a(t)$ of $\delta v(t)$ and $\delta i(t)$. Clearly, the harmonic components of order zero and one of $v_c(t) + v_a(t)$ and $i_c(t) + i_a(t)$ must satisfy equation (51). From equations (11), (12), (33), and (35) and from the fact that $\delta v_{ak}(t) = 2V_{ak}$ and $\delta i_{ak}(t) = 2I_{ak}$ because $p = 0$, the root mean square values of the harmonic components in question are $x_0 = V_{c0} + 2V_{a0}$, $x_1 = \sqrt{2}V_{c1} + 2V_{a1}$, $\eta_0 = I_{c0} + 2I_{a0}$ and $\eta_1 = \sqrt{2}I_{c1} + 2I_{a1}$. Therefore, from equation (51),

$$\begin{aligned} I_{c0} + 2I_{a0} &= F_0(\sqrt{2}\, V_{c1} + 2V_{a1}, V_{c0} + 2V_{a0}) \\ \sqrt{2}\, I_{c1} + 2I_{a1} &= F_1(\sqrt{2}\, V_{c1} + 2V_{a1}, V_{c0} + 2V_{a0}), \end{aligned} \tag{52}$$

and since $V_{ak} \ll V_{ck}$, from equations (49) and (52) one obtains

$$[G_a'] = \begin{bmatrix} \left(\dfrac{\partial F_0}{\partial x_0}\right)_c & \left(\dfrac{\partial F_0}{\partial x_1}\right)_c \\ \left(\dfrac{\partial F_1}{\partial x_0}\right)_c & \left(\dfrac{\partial F_1}{\partial x_1}\right)_c \end{bmatrix}, \tag{53}$$

where $(\ )_c$ indicates that the partial derivatives are calculated for

$$x_0 = V_{c0}, \qquad x_1 = \sqrt{2}V_{c1}. \tag{54}$$

Next, consider the phase modulation components $\delta v_p(t)$ and $\delta i_p(t)$ of $\delta v(t)$ and $\delta i(t)$. They are produced by a small change $\delta\varphi$ of the phase of the pump. Such a change is equivalent to a small change $\delta\varphi/\omega_0$ of the origin of time. Therefore

$$v_c(t) + \delta v_p(t) = v_c\left(t + \frac{\delta\varphi}{\omega_0}\right) \tag{55}$$

$$i_c(t) + \delta i_p(t) = i_c\left(t + \frac{\delta\varphi}{\omega_0}\right). \tag{56}$$

From these relations and from equations (11), (12), (34), and (36) one can calculate the phase modulation coefficients V_{p1} and I_{p1} of $\delta v_p(t)$ and $\delta i_p(t)$. One obtains

$$V_{p1} = -\delta\varphi V_{c1}/\sqrt{2}, \qquad I_{p1} = -\delta\varphi I_{c1}/\sqrt{2}.$$

Therefore, from these relations and equation (48),

$$g_p = \frac{I_{c1}}{V_{c1}} = \left[\frac{F_1(x_0, x_1)}{x_1}\right]_c . \tag{57}$$

In words, g_p is the nonlinear conductance presented at ω_0 by the diode to the external circuit under normal operating conditions.

The foregoing discussion presents two alternative methods of evaluating the terminal behavior of the two networks A and P. One method consists of analyzing these two networks by using equations (42) and (47), as suggested by the equivalent circuits of Figs. 6 and 7. Therefore, it requires that the small-signal terminal behavior of the diode be determined. The other method consists of deriving $[G_a]$ and g_p directly from the large-signal terminal behavior, at $\omega = 0$ and $\omega = \omega_0$, of the nonlinear network consisting of the diode terminated with $y_2, y_3, y_4, \cdots$, at $2\omega_0, 3\omega_0, 4\omega_0, \cdots$. If these admittances are nonzero and finite, considerable analytical difficulties often arise when one tries to analyze the equivalent circuits A and P by using the representation shown in Figs. 6 and 7. In these cases the second method may be used. Furthermore, it is important to point out that a well known technique of measuring frequency converters is based on the second method, which is often called the incremental method.[4–6]

In the special case where $y_k = \infty$ for $k > 1$, one can readily verify that

$$F_0(x_0, x_1) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + \sqrt{2}\, x_1 \cos \omega_0 t)\, dt,$$

and

$$F_1(x_0, x_1) = \frac{1}{\sqrt{2}\,\pi} \int_0^{2\pi} f(x_0 + \sqrt{2}\, x_1 \cos \omega_0 t) \cos \omega_0 t\, dt, \tag{58}$$

and by using equations (53) and (57) one obtains equation (50).

VI. COMPLETE EQUIVALENT CIRCUIT OF A FREQUENCY CONVERTER

In Section V a one-to-one correspondence between the output signals of a frequency converter and the amplitude modulation coefficients V_{a1} and I_{a1} was derived. In this section the terminal behavior of the frequency converter is completely determined by deriving the relations between these coefficients and the input signals. The relations in question can be obtained with the help of the equivalent circuits T and P shown in Figs. 4 and 7. In fact, they are given by the terminal behavior of the network shown in Fig. 8.

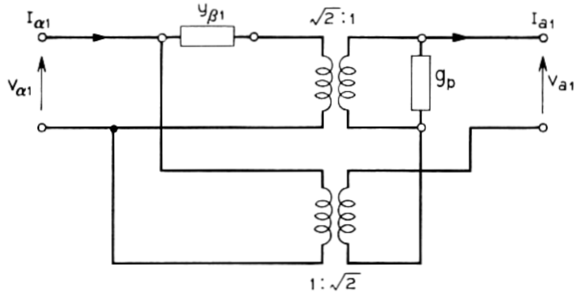

Fig. 8 — Network T' describing the transformation of the input signals of a frequency converter into amplitude modulation signals at $p \pm \omega_0$.

Now, by connecting the two networks of Figs. 6 and 8 in cascade, as shown in Fig. 9, one obtains a two-terminal-pairs network, which provides a complete representation of the terminal behavior of the frequency converter. This network corresponds to the equivalent circuit shown in Fig. 1. Notice that in Fig. 9 the input terminals of the frequency converter are connected to a small-signal generator with short-circuit terminal current I_s and with internal admittance $y_{\alpha 1}$, and that the output terminals are connected to a load $y_{\alpha 0}$.

From equation (15) the signals occurring at the terminals of the diode at $p \pm \omega_0$ can be expressed as

$$\begin{aligned} \delta v_1(t) &= \sqrt{2}\ \delta v_{a1}(t) \cos \omega_0 t + \sqrt{2}\ \delta v_{p1}(t) \sin \omega_0 t \\ \delta i_1(t) &= \sqrt{2}\ \delta i_{a1}(t) \cos \omega_0 t + \sqrt{2}\ \delta i_{p1}(t) \sin \omega_0 t. \end{aligned} \tag{59}$$

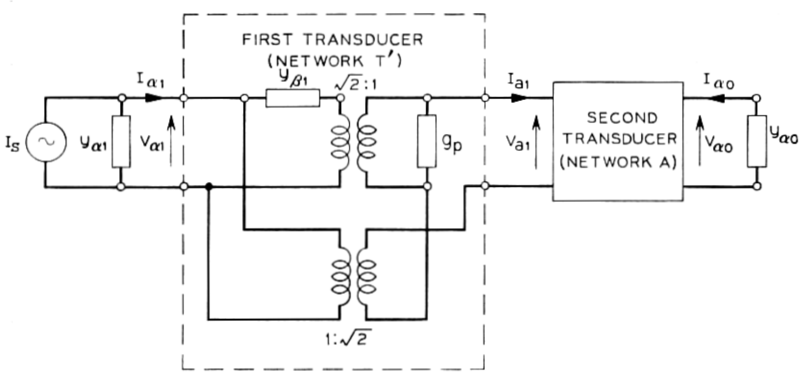

Fig. 9 — Equivalent circuit representing a frequency converter as a cascade connection of two transducers.

From these relations one can verify that

$$P_1 = P_{a1} + P_{p1} , \tag{60}$$

where

$$P_1 = \langle \delta v_1(t) \ \delta i_1(t) \rangle_{\text{av}} \tag{61}$$

$$\begin{aligned} P_{a1} &= 2\langle \delta v_{a1}(t) \ \delta i_{a1}(t) \cos^2 \omega_0 t \rangle_{\text{av}} = \langle \delta v_{a1}(t) \ \delta i_{a1}(t) \rangle_{\text{av}} \\ P_{p1} &= 2\langle \delta v_{p1}(t) \ \delta i_{p1}(t) \sin^2 \omega_0 t \rangle_{\text{av}} = \langle \delta v_{p1}(t) \ \delta i_{p1}(t) \rangle_{\text{av}} \end{aligned} \tag{62}$$

and $\langle \ \rangle_{\text{av}}$ indicates the time average. Equation (59) shows that $\delta v_1(t)$ and $\delta i_1(t)$ contain two types of signals: the amplitude modulation components $\sqrt{2} \ \delta v_{a1}(t) \cos \omega_0 t$ and $\sqrt{2} \ \delta i_{a1}(t) \cos \omega_0 t$, and the phase modulation components $\sqrt{2} \ \delta v_{p1}(t) \sin \omega_0 t$ and $\sqrt{2} \ \delta i_{p1}(t) \sin \omega_0 t$. Equations (59) through (62) show that the total power P_1 flowing into the diode at $p \pm \omega_0$ is the sum of the individual powers P_{a1} and P_{p1} carried by these two types of components. Furthermore, P_{a1} and P_{p1} can be calculated directly from the modulating functions $\delta v_{a1}(t)$ and $\delta i_{a1}(t)$ as shown by equations (62).

Now, consider the equivalent circuit shown in Fig. 9. The network T' can be regarded as a transducer which transforms the input signals of the frequency converter into amplitude modulations of the fundamental harmonic components of $v_c(t)$ and $i_c(t)$. These amplitude modulations are then transformed by the network A into the output signals of the frequency converter. The network T' is dissipative because it contains the two admittances $y_{\beta 1}$ and g_p . The power dissipated in g_p , P_{p1} , represents the power dissipated in the frequency converter because of the generation of phase modulation signals. The power flowing into the network A, at its left terminals, is P_{a1} . Figure 9 clearly shows that $P_{a1} + P_{p1}$, the total power flowing into the diode at $p \pm \omega_0$, is in general less than the power $P_{\alpha 1}$ flowing into the input terminals of the frequency converter. In fact the difference $P_{\alpha 1} - (P_{a1} + P_{p1})$ between these two powers is lost in the admittance $y_{\beta 1}$ of Fig. 9. This is the power flowing from the diode into the external circuit at $p - \omega_0$. In general, only if either one of the two conditions

$$y_{\beta 1} = \infty \tag{63}$$

$$y_{\beta 1} = 0 \tag{64}$$

is satisfied, is this power equal to zero. These two conditions are examined in Section VIII.

In the following section the special and very important case of a frequency converter in which

$$y_{\beta 1} = y_{\alpha 1} \tag{65}$$

is considered. The discussion of the terminal behavior of this frequency converter will also provide the background needed in Section VIII for the discussion of the two cases given by equations (63) and (64).

VII. FREQUENCY CONVERTER WITH EQUAL TERMINATIONS AT $\omega_0 + p$ AND $p - \omega_0$

One can verify that if $y_{\beta 1} = y_{\alpha 1}$, then the Thevenin representation of the one-terminal-pair network connected to the left terminals of the network A of Fig. 9 has a short-circuit current $I_s/\sqrt{2}$ and an internal admittance $y_{\alpha 1}$, as shown in Fig. 10a. One can also verify that the one-terminal-pair network connected to the terminals of g_p, in Fig. 9, has the Thevenin representation shown in Fig. 10b. Notice that the only significant difference between the two equivalent generators of Fig. 10 and the generator connected to the input terminals of the frequency converter (Fig. 9) is that the available power of this generator is twice the power available from each of the two equivalent generators of Fig. 10. Notice, furthermore, that Fig. 10 shows that if $y_{\alpha 1} = y_{\beta 1}$, in the amplitude-phase representation the terminal behavior of the external circuit at $p \pm \omega_0$ can be represented by means of two separate one-terminal-pair networks, the two equivalent generators shown in Fig. 10. This property is in accordance with the remarks in Section VI about the significance of Equation (47).

Thus, if $y_{\alpha 1} = y_{\beta 1}$, then only half of the power available from the input generator shown in Fig. 9 is used by the frequency converter to

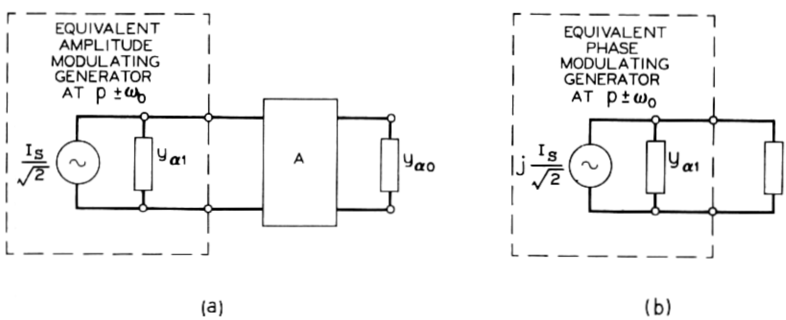

Fig. 10 — Equivalent generators in the amplitude-phase representation for the external circuit at $p \pm \omega_0$, when $y_{\beta 1} = y_{\alpha 1}$.

produce output signals. The remaining half is lost because of the generation of phase modulation signals, as shown by Fig. 10b. This explains why the minimum conversion loss of a frequency converter of the type considered here is always greater than two and can be expressed as

$$L = 2L_a , \tag{66}$$

where L_a is the minimum conversion loss of the network A shown in Fig. 10a. The conversion loss given by equation (66) is achieved when

$$y_{\alpha 1} = Y_{a1} , \qquad y_{\alpha 0} = Y_{a0} , \tag{67}$$

where Y_{ak} $(k = 0, 1)$ are the image admittances of the network A.

A frequency converter of the type considered in this section is characterized by the following special behavior. If one minimizes its conversion loss, the frequency converter reflects, at its input terminals, a part of the power incident from the input generator. The reflected power P_ρ can be calculated with the help of the equivalent circuit shown in Fig. 8, by terminating with Y_{a1} the terminals relative to the signals V_{a1} and I_{a1}, connecting the remaining terminals to the input generator shown in Fig. 9, and setting $y_{\alpha 1} = y_{\beta 1} = Y_{a1}$. One finds that P_ρ is equal to the power dissipated in $y_{\beta 1}$ and

$$P_\rho = \frac{1}{4}\left(\frac{g_p - y_{\alpha 1}}{g_p + y_{\alpha 1}}\right)^2 P_s = \frac{1}{4}\left(\frac{g_p - Y_{a1}}{g_p + Y_{a1}}\right)^2 P_s \tag{68}$$

where P_s is the available power of the input generator. Therefore, the following conclusion can be drawn: in a frequency converter of the type considered in this section, minimum conversion loss is achieved only when the power reflected at the input terminals of the frequency converter is equal to the power dissipated in the admittance terminating the diode at $p - \omega_0$ and is given by equation (68).

This result can be given the following interpretation. The available power of the equivalent generator shown in Fig. 10a represents the amplitude modulation power available from the external circuit at $p \pm \omega_0$. This power is entirely absorbed by the diode, when equations (67) are satisfied. Only a part of the power available from the generator of Fig. 10b, on the other hand, is generally absorbed by the diode, because normally $g_p \neq Y_{\alpha 1} = y_{\alpha 1}$. From Fig. 10b, the power reflected at the terminals of the generator in question is

$$\frac{P_s}{2}\left(\frac{g_p - y_{\alpha 1}}{g_p + y_{\alpha 1}}\right)^2 = \frac{P_s}{2}\left(\frac{g_p - Y_{a1}}{g_p + Y_{a1}}\right)^2 .$$

Since half of this power is reflected at $p + \omega_0$ and the remaining half is reflected at $p - \omega_0$, one obtains equation (68) and the conclusion following it.

Notice that, in the special case where $y_k = \infty$ for $k > 1$, from equations (50) one obtains the well known expressions:[4]

$$L_a = \frac{1 - [1 - 2g_1^2/g_0(g_0 + g_2)]}{1 + [1 - 2g_1^2/g_0(g_0 + g_2)]^{\frac{1}{2}}} \tag{69}$$

$$Y_{a1} = \frac{g_0 + g_2}{g_0}, \qquad Y_{a0} = g_0 + g_2\left[1 - \frac{2g_1^2}{g_0(g_0 + g_2)}\right]^{\frac{1}{2}}. \tag{70}$$

VIII. FREQUENCY CONVERTER WITH EITHER $y_{\beta 1} = \infty$ OR $y_{\beta 1} = 0$

One can readily verify that in the two cases corresponding to conditions (64) and (63), the equivalent circuit of Fig. 9 reduces to the two equivalent circuits shown in Figs. 11 and 12, respectively. In both cases the minimum conversion loss can be expressed as

$$L = 4L_a\{[(L_a + 1)^2 + \gamma(L_a^2 - 1)]^{\frac{1}{2}} - [(L_a - 1)^2 + \gamma(L_a^2 - 1)]^{\frac{1}{2}}\}^{-2} \tag{71}$$

where

$$\gamma = \frac{g_p}{Y_{a1}} \quad \text{if} \quad y_{\beta 1} = \infty \tag{72}$$

$$\gamma = \frac{Y_{a1}}{g_p} \quad \text{if} \quad Y_{\beta 1} = 0. \tag{73}$$

The values of $y_{\alpha 1}$ and $y_{\alpha 0}$ required for achieving this conversion loss are given in the appendix.

Of great practical importance is the problem of determining which

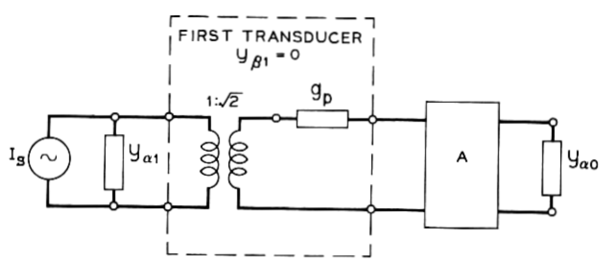

Fig. 11 — Equivalent circuit of a frequency converter when $y_{\beta 1} = 0$.

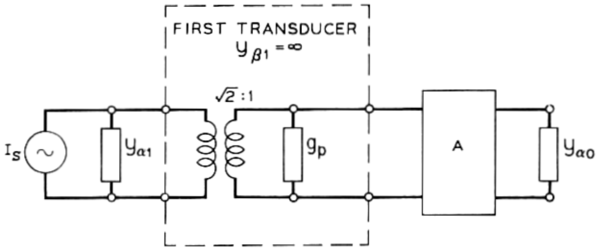

Fig. 12 — Equivalent circuit of a frequency converter when $y_{\beta 1} = \infty$.

of the two conditions $y_{\beta 1} = \infty$ and $y_{\beta 1} = 0$ yields the lowest conversion loss. Since equation (71) shows that L decreases with decreasing γ, for $\gamma \geqq 0$, one has that of the two foregoing conditions the one which gives $\gamma < 1$ yields the lowest L. Therefore one obtains the fundamental result: Of the two conditions $y_{\beta 1} = 0$ and $y_{\beta 1} = \infty$ the one which yields the lowest conversion loss is $y_{\beta 1} = 0$ if $g_p/Y_{a1} > 1$ and is $y_{\beta 1} = \infty$ if $g_p/Y_{a1} < 1$; the two conditions are equivalent if $g_p = Y_{a1}$.

Therefore, if $y_{\beta 1}$ is so chosen as to minimize the conversion loss, then $\gamma \leqq 1$ and from equation (71) one obtains

$$L_a \leqq L \leqq \frac{1}{L_a - (L_a^2 - 1)^{\frac{1}{2}}} \tag{74}$$

where the two equal signs occur when $\gamma = 0$ and $\gamma = 1$, respectively. The first inequality is a direct consequence of the fact that in both equivalent circuits of Figs. 11 and 12 the conversion loss is always greater than L_a, the minimum conversion loss of the network A. The second inequality shows that in a frequency converter of the type considered here the optimum conversion loss is always less than $2L_a$, the lowet conversion loss obtainable when $y_{a1} = y_{\beta 1}$. In fact

$$1 \leqq \frac{1}{L_a - (L_a^2 - 1)^{\frac{1}{2}}} \leqq 2L_a \tag{75}$$

where the equal signs occur when $L_a = 1$ and $L_a = \infty$.

Often practical considerations require that the admittance y_{a1} of the input generator be equal to a prescribed value G, not necessarily equal to the value required for optimum performance. In such a case it is important to point out that one can readily determine, with the help of the two equivalent circuits of Figs. 11 and 12, which one of the two conditions $y_{\beta 1} = 0$ and $y_{\beta 1} = \infty$ is to be chosen, in order to minimize the conversion loss. Clearly the choice depends upon the values of G, Y_{a1} and L_a.

IX. SCHOTTKY BARRIER DIODE

Among the various resistive diodes presently available, the Schottky barrier diode offers the most suitable electrical characteristics for microwave frequency conversion.[9,16–18] The main features of this diode are that its low-frequency terminal behavior is described by the equation

$$i = i_s \left\{ \exp \left[\frac{q}{KT} (v - iR_s) \right] - 1 \right\} \tag{76}$$

over a very wide range of voltages, and that its frequency response is not limited by minority carrier lifetime. In equation (76) i_s is the saturation current, q the electronic charge, K the Boltzmann constant, T the absolute temperature, and R_s the series resistance. Therefore,

$$\frac{q}{KT} \cong 40 \quad \text{for} \quad T = 290°\text{K}. \tag{77}$$

The useful frequency range of the diode is limited primarily by the junction capacitance C_j. In some instances the lead inductance and the case capacitance also have to be considered. However, in Section 9.1 consideration is restricted to the range of frequencies over which the foregoing parasitics can be neglected.

9.1 *Analysis*

Assume that

$$i = i_s \exp \left[\frac{q}{KT} (v - R_s i) \right] \tag{78}$$

and that

$$y_k = 0 \qquad \text{for} \qquad k > 1. \tag{79}$$

Then from equation (78) one obtains

$$\frac{dv}{di} = \frac{KT}{q} \frac{1}{i} + R_s \tag{80}$$

and, from equations (12) and (79),

$$i_c(t) = I_{c0} + 2I_{c1} \cos \omega_0 t. \tag{81}$$

Therefore, by setting $i = i_c(t)$ in equation (80), the differential resistance of the diode can be expressed as

$$r(t) = R_s + \frac{KT}{2qI_{c1}} \frac{1}{\xi + \cos \omega_0 t} \tag{82}$$

where

$$\xi = \frac{I_{c0}}{2I_{c1}}. \tag{83}$$

From equation (82) one can readily calculate the Fourier coefficients of $r(t)$. One obtains

$$r_n = \delta(n)R_s + \frac{KT}{2qI_{c1}} \frac{1}{\pi} \int_0^{\pi} \frac{\cos n\theta}{\xi + \cos\theta}\, d\theta = \delta(n)R_s + \frac{KT}{2qI_{c1}} (-1)^n$$

$$\cdot \left(\frac{\xi+1}{\xi-1}\right)^{\frac{1}{2}} (1+\xi)^{n-1} \left[\frac{\xi}{\xi+1} - \left(\frac{\xi-1}{\xi+1}\right)^{\frac{1}{2}}\right]^n \tag{84}$$

where

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}. \tag{85}$$

Because of the formal analogy between the impedance-matrix representation and the admittance-matrix representation, from equation (50) one has that the impedance matrix $[R_a']$ of the network A is

$$[R_a'] = \begin{bmatrix} r_0 & \sqrt{2}\, r_1 \\ \sqrt{2}\, r_1 & r_0 + r_2 \end{bmatrix} \tag{86}$$

and the impedance r_p presented at ω_0 by the diode to the pump is

$$r_p = r_0 - r_2 . \tag{87}$$

Therefore, from equations (84), (86), and (87) one obtains

$$[R_a'] = \frac{KT}{2qI_{c1}} \begin{bmatrix} (\xi^2-1)^{-\frac{1}{2}} & -\sqrt{2}\,[\xi(\xi^2-1)^{-\frac{1}{2}} - 1] \\ -\sqrt{2}\,[\xi(\xi^2-1)^{-\frac{1}{2}} - 1] & 2\xi[\xi(\xi^2-1)^{-\frac{1}{2}} - 1] \end{bmatrix} + \begin{bmatrix} R_s & 0 \\ 0 & R_s \end{bmatrix} \tag{88}$$

and

$$r_p = \frac{KT}{qI_{c1}} [\xi - (\xi^2-1)^{\frac{1}{2}}]. \tag{89}$$

Equation (88) shows that the network A can be represented by means of the equivalent circuit shown in Fig. 13, with

$$R_1 = \frac{KT}{qI_{c0}} \tag{90}$$

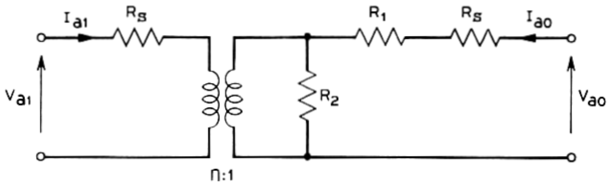

Fig. 13 — Network A for a Schottky barrier diode frequency converter with $y_k = 0$ for $k > 1$.

$$R_2 = \frac{KT}{qI_{c0}} \frac{\xi - (\xi^2 - 1)^{\frac{1}{2}}}{(\xi^2 - 1)^{\frac{1}{2}}} \tag{91}$$

$$n = \sqrt{2}\,\xi. \tag{92}$$

Now, let v_M and v_m denote the maximum and minimum values, respectively, of $v_c(t) - R_s i_c(t)$ and let

$$\Delta V = v_M - v_m\,. \tag{93}$$

Then from equations (78), (81), and (93) one obtains

$$I_{c0} + 2I_{c1} = (I_{c0} - 2I_{c1}) \exp\left(\frac{q}{KT} \Delta V\right)$$

which, by making use of equation (83), gives

$$\xi = 1 + 2\left[\exp\left(\frac{q}{KT} \Delta V\right) - 1\right]^{-1}. \tag{94}$$

At this point the assumption is made that

$$\Delta V > 0.5 \text{ volts}, \tag{95}$$

a condition satisfied in most cases of practical interest. Then, if one substitutes equation (94) into equations (89), (91), (92) and examines the behavior of n, R_2 and r_p for large $q\,\Delta V/KT$, one finds that

$$n \cong \sqrt{2} \tag{96}$$

$$r_p \cong 2\frac{KT}{qI_{c0}} + R_s \tag{97}$$

$$R_2 \cong 2\frac{KT}{qI_{c0}} \exp\left(\frac{q}{KT}\frac{\Delta V}{2}\right). \tag{98}$$

Furthermore, one can verify that the fractional errors in these approximate expressions decrease exponentially with ΔV and are less than 0.01 per cent, because of condition (95). Therefore the equivalent circuit of Fig. 13 can be replaced by that shown in Fig. 14.

Notice that

$$R_s \, , \, R_1 \ll R_2 \, , \tag{99}$$

because of condition (95) and of the fact that typically $R_s < 10\Omega$ and

$$I_{c0} \ll 100 \text{ mA}. \tag{100}$$

Then, with the help of the equivalent circuit of Fig. 14 and by making use of condition (95), from standard network theory one obtains

$$Z_{a1} \cong Z_{a0} \cong \left(2 + \frac{2R_s q I_{c0}}{KT}\right)^{\frac{1}{2}} \frac{KT}{qI_{c0}} \exp\left(\frac{q}{KT}\frac{\Delta V}{4}\right) \tag{101}$$

$$L_a \cong 1 + 2\left(2 + \frac{3R_s q I_{c0}}{KT}\right)^{\frac{1}{2}} \exp\left(-\frac{q}{KT}\frac{\Delta V}{4}\right) \tag{102}$$

where Z_{a0} and Z_{a1} are the image impedances of the network A (that is, $Z_{a0} = 1/Y_{a0}$ and $Z_{a1} = 1/Y_{a1}$).

Equations (97) and (101) show that $r_p \ll Z_{a1}$. Therefore, in accordance with the discussion in Section VIII, condition (64) yields better performance than condition (63). Thus let $y_{\beta 1} = 0$. Then, from the equivalent circuits shown in Figs. 11 and 14 and from equation (97) one obtains the equivalent circuit shown in Fig. 15.

9.2 *Discussion of the Equivalent Circuit*

From the equivalent circuit shown in Fig. 15 one obtains the following expression for the minimum conversion loss of a Schottky barrier diode

$$L \cong 1 + 4\left(1 + \frac{R_s q I_{c0}}{KT}\right)^{\frac{1}{2}} \exp\left(-\frac{q}{KT}\frac{\Delta V}{4}\right) \tag{103}$$

which is achieved when

$$z_{\alpha 1} = z_{\alpha 0} \cong \left(1 + \frac{R_s q I_{c0}}{KT}\right)^{\frac{1}{2}} \frac{KT}{qI_{c0}} \exp\left(\frac{q}{KT}\frac{\Delta V}{4}\right) \tag{104}$$

where $z_{\alpha k} = 1/y_{\alpha k}$. From equation (103), by selecting I_{c0} in such a way that

$$\frac{R_s q I_{c0}}{KT} \ll 1, \tag{105}$$

one obtains equation (1).

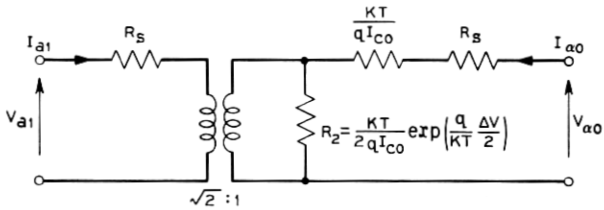

Fig. 14 — Equivalent circuit of Fig. 13 when $\triangle V > 0.5$ volts.

Notice that it is always desirable that ΔV be as large as possible, because this increases the value of the resistance R_2 shown by Fig. 15. On the other hand, equations (104) and (105) show that it $\triangle V$ is too large, then very high input and output impedances are required in order to achieve minimum conversion loss. For instance, if $\triangle V > 0.85$ (which is a condition that can be readily satisfied in most practical cases since the breakdown voltage of the diode typically is greater than 1 volt), $I_{c0} = 10$ mA and $R_s = 2\Omega$, then from equations (103) and (104) one obtains

$$L < 1.0015, \qquad \text{but} \qquad z_{\alpha 1} = z_{\alpha 0} > 17000\Omega. \tag{106}$$

Since normally it is required that the input and output impedances be much smaller than this value, one concludes that practical considerations almost always require that

$$z_{\alpha 1} = z_{\alpha 0} \ll \frac{KT}{qI_{c0}} \exp\left(\frac{q}{KT}\frac{\triangle V}{4}\right). \tag{107}$$

Because of this restriction, the equivalent circuit of Fig. 15 reduces to that of Fig. 16 which shows that, under practical conditions, the terminal behavior of the frequency converter only depends upon the two parameters I_{c0} and R_s. Furthermore, Fig. 16 clearly shows that low conversion losses require high dc currents (I_{c0}); that is, they require

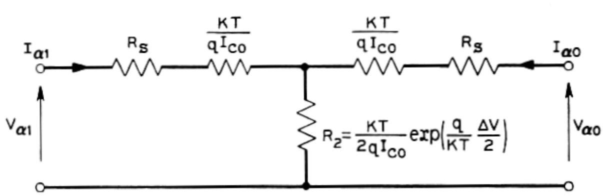

Fig. 15 — Equivalent circuit of a Schottky barrier diode frequency converter with $y_k = 0$ for $k > 1$, $y_{\beta 1} = 0$, and $\triangle V > 0.5$ volts.

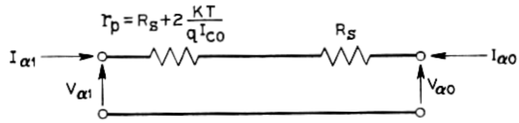

Fig. 16 — Equivalent circuit of a Schottky barrier diode frequency converter under certain optimum circuit conditions.

that the impedance r_p presented by the diode at ω_0 be small. Notice that from the equivalent circuit of Fig. 16 one readily obtains equation (2).

The pump power P_p absorbed by the diode at ω_0 is

$$P_p = 2I_{c1}^2 r_p . \tag{108}$$

Therefore, by using equations (77) and (97) and the fact that $I_{c0} \cong 2I_{c1}$ one obtains

$$P_p \cong \frac{I_{c0}}{40} + \frac{R_s I_{c0}^2}{2000} \qquad \text{(mW)}, \tag{109}$$

where R_s and I_{c0} are measured in ohms and milliamperes, respectively.

9.3 *Optimum Terminations at $2\omega_0$, $3\omega_0$, $4\omega_0$, $\cdots$*

The preceding analysis assumes that the diode is open-circuited at the frequencies $p \pm k\omega_0$ and $k\omega_0$, $k > 1$. In this section we show that the conversion losses obtainable when the diode is short-circuited at these frequencies are appreciably higher than those given by equation (1).

Assume that the diode is short-circuited at $p \pm k\omega_0$ and $k\omega_0 (k > 1)$. Reference 10 shows that then the optimum termination at $p - \omega_0$ is*

$$y_{\beta 1} = 0. \tag{110}$$

With this optimum termination the minimum conversion loss can be expressed as follows[4–10]

$$L' = \left\{ 1 + \left[\frac{1 + \frac{g_2}{g_0} - 2\left(\frac{g_1}{g_0}\right)^2}{\left[1 - \left(\frac{g_1}{g_0}\right)^2\right]\left(1 + \frac{g_2}{g_0}\right)} \right]^{\frac{1}{2}} \right\}^2 \frac{\left[1 - \left(\frac{g_1}{g_0}\right)^2\right]\left(1 + \frac{g_2}{g_0}\right)}{\left(\frac{g_1}{g_0}\right)^2\left(1 - \frac{g_2}{g_0}\right)}, \tag{111}$$

* The analysis of Ref. 10 is valid only if the diode is short-circuited at $p \pm k\omega_0$, $k > 1$. Therefore, even though the analysis covers all cases where such an assumption can be made, it cannot be applied to the case $(y_k = y_{\alpha k} = y_{\beta k} = 0)$.

where

$$\frac{g_2}{g_0} = \frac{J_2\left(\frac{q}{KT}\frac{\Delta V}{2}\right)}{J_0\left(\frac{q}{KT}\frac{\Delta V}{2}\right)}, \tag{112}$$

$$\frac{g_1}{g_0} = \frac{J_1\left(\frac{q}{KT}\frac{\Delta V}{2}\right)}{J_0\left(\frac{q}{KT}\frac{\Delta V}{2}\right)}, \tag{113}$$

provided R_s can be neglected.[10] From these equations, by making use of well known asymptotic expansions of the modified Bessel functions J_0, J_1, and J_2, one obtains

$$L' = 1 + 2\left(\frac{KT}{q\,\Delta V}\right)^{\frac{1}{2}} + 2\left(\frac{KT}{q\,\Delta V}\right) + (5 + \sqrt{2})\left(\frac{KT}{q\,\Delta V}\right)^{\frac{3}{2}} + \cdots . \tag{114}$$

Thus, if one compares L' with the conversion loss L of equation (1) one finds

$$\frac{L' - 1}{L - 1} \to \infty \quad \text{as} \quad \frac{KT}{q\,\Delta V} \to 0.$$

If for example $\Delta V = 1$, then equations (1) and (114) give

$$L = 1.00018 \tag{115}$$

and

$$L' = 1.39, \tag{116}$$

respectively. Clearly, condition (79) gives better performance than the condition $Y_k = \infty$ for $k > 1$.

X. CONCLUSIONS

It has been shown that, under certain circuit conditions, a resistive diode frequency converter can be represented by the equivalent circuit shown in Fig. 9. The general case where the diode cannot be regarded as a nonlinear resistor and the terminations y_2, y_3, y_4, $\cdots$ are not resistive has not been considered, in order to prevent the mathematical difficulties inherent in it from obscuring the significance of the results.

However, the equivalent circuit of Fig. 9 is valid under conditions much more general than those considered in the preceding sections.

In fact, in this section we show that it is valid also in the general case of a frequency converter using an arbitrary nonlinear element which is pumped at ω_0 and is arbitrarily terminated at $2\omega_0$, $3\omega_0$, $4\omega_0$, and so on, provided that three conditions are satisfied. (*i*) $p \ll \omega_0$, (*ii*) small and slow changes of the level of the pump and of the dc bias do not cause phase variations in the fundamental harmonic components of $v_c(t)$ and $i_c(t)$, and (*iii*) the nonlinear admittance (g_p) seen by the pump be real.

Assume first that *iii* is satisfied. Then, the fundamental harmonic components of $v_c(t)$ and $i_c(t)$ have the same phase φ_1. Let the origin of time be chosen in such a way that $\varphi_1 = 0$. If one now defines the amplitude modulation and phase modulation components of $\delta v_1(t)$ and $\delta i_1(t)$ by means equation (59), the relations between V_{a1}, I_{a1}, V_{p1}, I_{p1} and $V_{\alpha 1}$, $I_{\alpha 1}$, $V_{\beta 1}$, $I_{\beta 1}$ are provided by the terminal behavior of the network T shown in Fig. 4.

Next, assume that $p \ll \omega_0$ and let the incremental method be used to derive the relations between the output signals V_{a0} and I_{a0} and the coefficients $V_{\alpha 1}$, $I_{\alpha 1}$, V_{p1} and I_{p1}. By using equations (51), which are valid even if the aforementioned three conditions are not satisfied, one finds that there is a one-to-one correspondence between the output signals of the frequency converter and the amplitude modulation coefficients $V_{\alpha 1}$ and $I_{\alpha 1}$. More precisely,

$$\begin{bmatrix} I_{a0} \\ I_{a1} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial F_0}{\partial x_0}\right)_c & \left(\dfrac{\partial F_0}{\partial x_1}\right)_c \\ \left(\dfrac{\partial F_1}{\partial x_0}\right)_c & \left(\dfrac{\partial F_1}{\partial x_1}\right)_c \end{bmatrix} \cdot \begin{bmatrix} V_{a0} \\ V_{a1} \end{bmatrix} \tag{117}$$

where $F_0(x_0, x_1)$ and $F_1(x_0, x_1)$ have the significance given in Section V.

Finally, assume that all of the three conditions are satisfied. Then, by using the incremental method one finds that I_{p1}/V_{p1} is independent of I_{a1}, V_{a1}, I_{a0} and V_{a0}, because of *ii*. The ratio I_{p1}/V_{p1} can therefore be determined by applying a small variation $\delta\varphi$ to the phase of the pump; and one finds

$$I_{p1} = g_p V_{p1} . \tag{118}$$

Now, from equations (117) and (118) and from the network T of Fig. 4 one obtains the equivalent circuit of Fig. 9, where the terminal behavior of the network A is now specified by equation (117).

Notice that the validity of the equivalent circuit shown in Fig. 9

can be extended to the general case where only the condition $p \ll \omega_0$ is satisfied, by properly modifying the first transducer. This follows from the fact that $p \ll \omega_0$ is sufficient to guarantee the validity of equation (117).

The special case of a Schottky varrier diode is examined in Section IX, which shows that the optimum terminations at the image frequency and at the harmonics $2\omega_0$, $3\omega_0$, $4\omega_0$, $\cdots$, are open-circuits. Furthermore, at low frequency and under optimum circuit conditions, the terminal behavior of a Schottky barrier diode frequency converter is quite simple. It can be adequately represented by means of the equivalent circuit shown in Fig. 16.

Finally, the fact that consideration has been restricted to the frequency range where the Schottky barrier diode can be regarded as a nonlinear resistor should not be interpreted as an indication that C_j can generally be neglected. However, the simple results obtained for the low frequency case serves as a guide for treating the high frequency case, which is reserved to a future article.

APPENDIX

The conversion loss given by equation (71) is obtained when

$$\hat{y}_{\alpha 1} = Y_{a1}\left(1 + \gamma \frac{L_a + 1}{L_a - 1}\right)^s \left(1 + \gamma \frac{L_a - 1}{L_a + 1}\right)^s \tag{119}$$

$$\hat{y}_{\alpha 0} = 2Y_{a0}\left(1 + \gamma \frac{L_a + 1}{L_a - 1}\right)^s \left(1 + \gamma \frac{L_a - 1}{L_a + 1}\right)^s \tag{120}$$

where

$$\gamma = g_p/Y_{a1}, \quad s = 1 \quad \text{if} \quad y_{\alpha 1} = \infty$$

$$\gamma = Y_{a1}/g_p, \quad s = -1 \quad \text{if} \quad y_{\beta 1} = 0.$$

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