# A Continuously Adaptive Equalizer for General-Purpose Communication Channels

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*This paper describes and analyzes a technique for the automatic and continuous minimization of the linear distortion in a communication channel. The channel can be used for the transmission of information in any format while the minimization process continues simultaneously. The equalizer which implements this strategy thus responds adaptively to changes in the channel's frequency-response characteristics.*

*Recent work in the field of automatic equalization has been along two main avenues. The first has led to a series of special-purpose equalizers, each designed to function with a particular information signalling format. This first class of equalizers is readily made adaptive by use of the known signal structure. The second approach is the optimization of the channel's frequency response in the general sense so that its transmission capability is improved for any signal. Equalizers of the latter type have, to date, been of the preset type, requiring adjustment in a dedicated period prior to information transmission.*

*This paper describes a technique which retains the advantage of the general approach yet adds that of adaptive operation. The result is an equalizer which can be used with any information transmission scheme and which continuously strives to compensate for any change which occurs in the channel's characteristics.*

## I. INTRODUCTION

Automatic equalization techniques have done much to alleviate the deleterious effects of linear distortion in communication channels.[1] In particular, equalization schemes such as that suggested in Ref. 2 have permitted significantly increased transmission speeds and better error performance for linear, synchronous data transmission systems. The same improvements can be obtained for arbitrary information transmission systems (not necessarily synchronous or linear) through the use of the generalized equalization techniques described in Ref. 3.

The equalization problem becomes more complicated when the linear distortion in the communication channel becomes time varying. In the time-varying case an adaptive equalizer is needed; such an equalizer for the synchronous, linear information transmission case has been discussed in the literature.[4] This paper describes a continuously adaptive equalizer which can be used with any information transmission system: synchronous or asynchronous, using linear or nonlinear modulation.

The continuously adaptive equalizer, described here, has the advantage over that described in Ref. 3 that the communication channel is under continuous surveillance by the equalizer controller. If the frequency characteristic of the transmission path should change, the equalizer would immediately begin to compensate for that change. Such a correction would be made regardless of the format being used for information transmission in the channel and independent of whether the channel were being used for information transmission.

The operation of the system relies on the transmission of a low-level test signal sent simultaneously with the information-bearing signal. Despite the low energy level of the test signal, powerful correlation detectors extract the necessary information for continuous, adaptive equalization of the channel.

## II. THE TECHNIQUE

The preset version of a general-purpose channel equalizer is shown in Fig. 1. A test signal is transmitted and the channel equalized prior to use of the channel for information transmission purposes.[3] The test signal used for equalization is removed before information transmission begins.

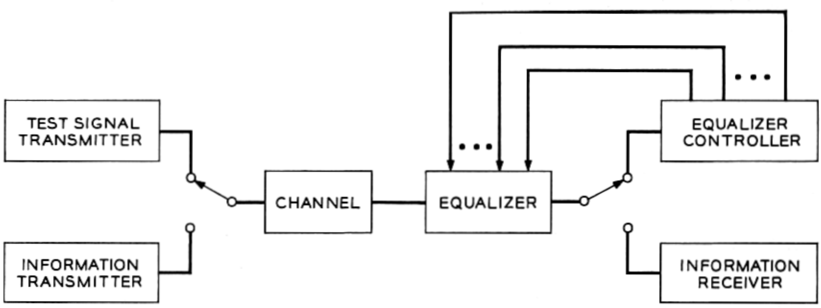


Fig. 1 — Preset mean-square channel equalizer.

The adaptive channel equalizer is shown in Fig. 2. Here the test signal is added to the information signal. The equalizer controller operates continuously so that the channel is always being monitored and the equalizer immediately responds to a change in the channel characteristic.

Because the test signal appears in the high "noise" environment created by the information-bearing signal, steps must be taken to ensure that the test signal and not the "noise" (or information signal) dictates the behavior of the equalizer. The preset equalizer responded to both test signal and noise.[3] This is an advantageous mode of operation for the preset case because if the signaling statistics are known *a priori* the equalizer could be designed to maximize the total received signal-to-noise ratio where the noise consists of both random noise and a component resulting from residual linear distortion. The same approach cannot be used in the adaptive equalizer because the much larger effective noise would dominate in the control of the equalizer. A stunt is used to make the equalizer controller blind to the information signal.

### 2.1 *Review of Preset Equalizer Operation*[3]

In both preset and adaptive mean-square channel equalizers, the desire is to equalize the channel transmission characteristic so that it best resembles an ideal transmission characteristic. The fit is optimized under a mean-squared error criterion. The distortion to be minimized is

$$E_1 = \int_{-\infty}^{\infty} | H(\omega) - G(\omega) |^2 \, d\omega \tag{1}$$

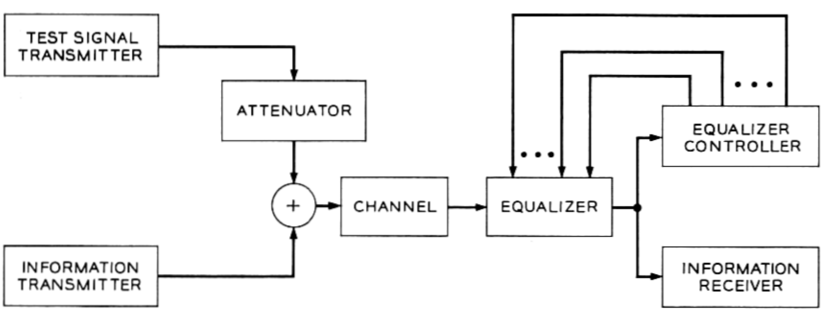


Fig. 2 — Adaptive mean-square channel equalizer.

where $H(\omega)$ is the equalized channel characteristic and $G(\omega)$ is the ideal channel characteristic. The error criterion given in equation (1) can be made more general by adding information concerning the relative importance of errors at various frequencies by the inclusion of a real, nonnegative weighting function $| W(\omega) |^2$ which assigns a relative weight $| W(\omega) |^2$ to the equalization error at each frequency $\omega$, as shown in Fig. 3. The resultant expression for the distortion to be minimized is

$$E = \int_{-\infty}^{\infty} | H(\omega) - G(\omega) |^2 \, | W(\omega) |^2 \, d\omega. \tag{2}$$

The ideal characteristic $G(\omega)$ would normally have flat amplitude and linear phase responses in the band of interest while the error weighting function $| W(\omega) |^2$ would be chosen to resemble the spectral density of the signal most likely to be transmitted. This choice of $| W(\omega) |^2$ would ensure that the best equalization would occur at frequencies where most of the information signal energy occurs.

Parseval's theorem allows equation (2) to be rewritten

$$E = \int_{-\infty}^{\infty} [h(t)*w(t) - g(t)*w(t)]^2 \, dt \tag{3}$$

where $h(t)$, $g(t)$, and $w(t)$ are the time functions corresponding to $H(\omega)$, $G(\omega)$, and $W(\omega)$, respectively, and * denotes convolution. Convergence of the algorithm to be stated shortly can be guaranteed for the case where the equalizer impulse response can be written as a linear,

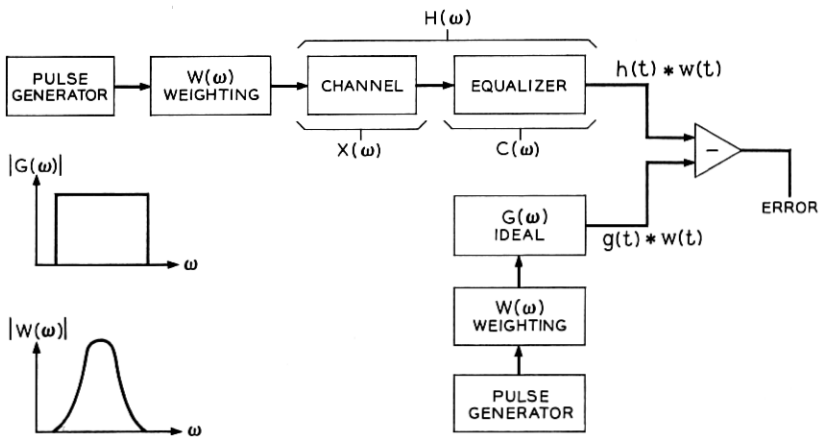


Fig. 3 — Mean-square equalizer.

weighted sum of the form[3,5-7]

$$h(t) = \sum_{n=-N}^{N} c_n y_n(t) * x(t). \tag{4}$$

Here $x(t)$ is the distorting channel's response, and $c_n$ is the weight (which can be either positive or negative) associated with the impulse response $y_n(t)$ of the elemental network $Y_n(\omega)$ shown in Fig. 4. When equation (4) is substituted into equation (3) the resulting distortion measure is

$$E = \int_{-\infty}^{\infty} \left\{ \sum_{n=-N}^{N} [c_n y_n(t) * x(t) * w(t)] - g(t) * w(t) \right\}^2 dt. \tag{5}$$

Partial differentiation of this last relation with respect to the $j$th attenuator, $c_j$, yields

$$\frac{\partial E}{\partial c_j} = 2 \int_{-\infty}^{\infty} \{h(t) * w(t) - g(t) * w(t)\} \{y_j(t) * x(t) * w(t)\} \, dt. \tag{6}$$

Since it can be easily demonstrated that $E$ is a convex function of the weights $c_n$ ($n = -N, N$), the set of $2N + 1$ equations of the form of equation (6) provides the information necessary to obtain the desired optimum adjustment of the weights under the specified mean-squared error criterion. The polarity of equation (6) determines the directions in which the attenuator weights must be incremented to

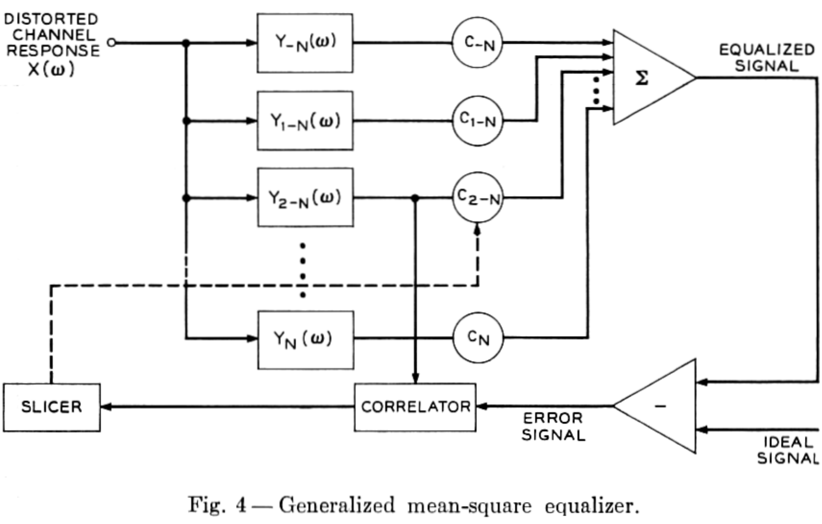


Fig. 4 — Generalized mean-square equalizer.

minimize the distortion $E$; the minimum is achieved when all the partial derivatives are zero.

Equation (6) is simply the cross-correlation between the error signal and the input signal to the $j$th attenuator. Thus the system can be implemented as shown in Fig. 4 where only one correlator is shown for clarity.

If the general parallel structure of Fig. 4 is replaced by a series tapped delay line structure so that

$$h(t) = \sum_{n=-N}^{N} c_n x(t - n\tau) \tag{7}$$

where $\tau$ is the tap spacing, a relation for the attenuator settings identical to equation (8) of Ref. 3 is obtained.

If noise is introduced, the optimum settings of the attenuator weights are changed. If noise $\eta(t)$ with spectrum $N(\omega)$ is added in the channel shown in Fig. 3, the received signal is

$$y(t) = w(t)^* x(t) + \eta(t). \tag{8}$$

If the measure of distortion in the presence of noise, $E_n$, is again taken as the average mean-square error between the equalized received signal $z(t)$ and the transmitted signal passed through the ideal, noiseless channel $G(\omega)$,

$$E_n = \langle [z(t) - w(t)^* g(t)]^2 \rangle. \tag{9}$$

The brackets $\langle \ \rangle$ denote a time average. The equalized signal $z(t)$ is now given by

$$z(t) = \sum_{n=-N}^{N} c_n[\eta(t - n\tau) + w(t)^* x(t - n\tau)], \tag{10}$$

where the equalizer is of the tapped delay line type. Substitution of equation (10) into (9) and partial differentiation of the result yields:

$$\frac{\partial E_n}{\partial c_j} = 2\langle [z(t) - w(t)^* g(t)][\eta(t - j\tau) + w(t)^* x(t - j\tau)] \rangle. \tag{11}$$

The noise $\eta$ now appears in both the first (or error) term, as a component of $z(t)$, and in the second term so that the noise does in general affect the optimum attenuator settings.

In the preset equalizer this phenomenon can be used to advantage. If the spectral density of the information signal to be transmitted over the equalized channel is known *a priori*, then the $W(\omega)$ error spectral weighting function can be selected such that the equalizer will minimize the total expected mean-squared error including both noise and linear distortion components over the frequency band of interest.

### 2.2 *Operation of the Adaptive Channel Equalizer*

While the dependence of the optimum equalizer attenuator settings on the noise could be used to advantage in the preset equalizer, such is not the case in the adaptive equalizer. In the adaptive equalizer the information-bearing signal is added to the test signal. Insofar as the equalization process is concerned, the information- bearing signal acts as noise and would interfere with the equalization process. Since this "noise" is apt to be larger than the test signal, the equalizer controller must be made insensitive to it.

Because it is necessary to generate a replica of the transmitted test signal at the location of the equalizer controller, it is convenient to use a particular pseudorandom sequence for the test signal. Such a test signal has a periodic autocorrelation function.[3] Over a short period of time the clocked ones and zeroes of the sequence look quite random but seen over a longer period of time it is clear that the sequence is in fact a periodic sequence.

The periodic property of the pseudorandom sequence permits the use of a simple stunt to achieve the desired independence of the attenuator settings from the high-noise environment. In the adaptive version of the channel equalizer the error signal is passed through a delay of $T_D$ seconds before reaching the correlator. The delay $T_D$ is chosen equal to a multiple of the period of the pseudo-random sequence. The fully expanded version of equation (11) is

$$\frac{\partial E_n}{\partial c_j} = 2\left\langle \left\{ \sum_{n=-N}^{N} c_n[\eta(t - n\tau) + w(t)^*x(t - n\tau)] - w(t)^*g(t) \right\} \cdot [\eta(t - j\tau) + w(t)^*x(t - j\tau)] \right\rangle. \tag{12}$$

If the error signal is now passed through the delay $T_D$, the relation becomes

$$\frac{\partial E_n}{\partial c_j} = 2\left\langle \left\{ \sum_{n=-N}^{N} c_n[\eta(t - n\tau - T_D) + w(t)^*x(t - n\tau - T_D)] - w(t)^*g(t - T_D) \right\} [\eta(t - j\tau) + w(t)^*x(t - j\tau)] \right\rangle. \tag{13}$$

Since the test signal and information signal are generated in completely different fashions it is certainly reasonable to assume that

$x(t)$ and $\eta(t)$ are uncorrelated as are $g(t)$ and $\eta(t)$. Also, because $T_D$ is selected large compared with the channel's dispersion (or length of the channel's impulse response), time translations such as the pair $[\eta(t - j\tau), \eta(t - n\tau - T_D)]$ are also independent. Thus, assuming zero mean values for the signals involved, equation (13) may be rewritten

$$\frac{\partial E_n}{\partial c_j} = 2\left\langle \left\{ \sum_{n=-N}^{N} c_n[w(t)*x(t - n\tau - T_D)] - w(t)*g(t - T_D) \right\} \cdot [w(t)*x(t - j\tau)] \right\rangle. \tag{14}$$

Because of the periodicity of the test waveform, $g(t - T_D) = g(t)$ and $x(t - T_D) = x(t)$. Equation (14) then becomes

$$\frac{\partial E_n}{\partial c_j} = 2\langle [h(t)*w(t) - g(t)*w(t)][x(t - j\tau)*w(t)] \rangle \tag{15}$$

which is entirely equivalent to equation (6) for the noise-free case and where the equalizer is of the tapped delay line or transversal filter type. The same arguments can be made for the parallel form of equalizer shown in Fig. 4. Use of delay to achieve the kind of independence needed here has been described elsewhere in the literature.[8]

So far the discussion has dealt with the effect of the information-bearing signal on the test signal. Insofar as the information receiver is concerned, the test signal looks just like noise. If the input to the information receiver is taken as the output of the error differencing amplifier as shown in Fig. 5, the noise component resulting from the test signal can be substantially reduced. The better the equalization, the better the match between desired and equalized test signals and the lower the effective noise for the information-bearing signal. In the laboratory the effective noise of the test signal is reduced by about 20 dB.

### 2.3 *An Estimate of Settling Time*

In the preset mean-square channel equalizer the settling time (the time required for the equalizer to reach equilibrium) is dependent largely on the nature of the dispersion in the channel. One reason for this is that the preset equalizer functions typically in a low-noise environment. Such is not the case for the adaptive equalizer, where the test signal is effectively buried in the noise of the information signal. In this instance the effective noise determines the response time of the equalizer.

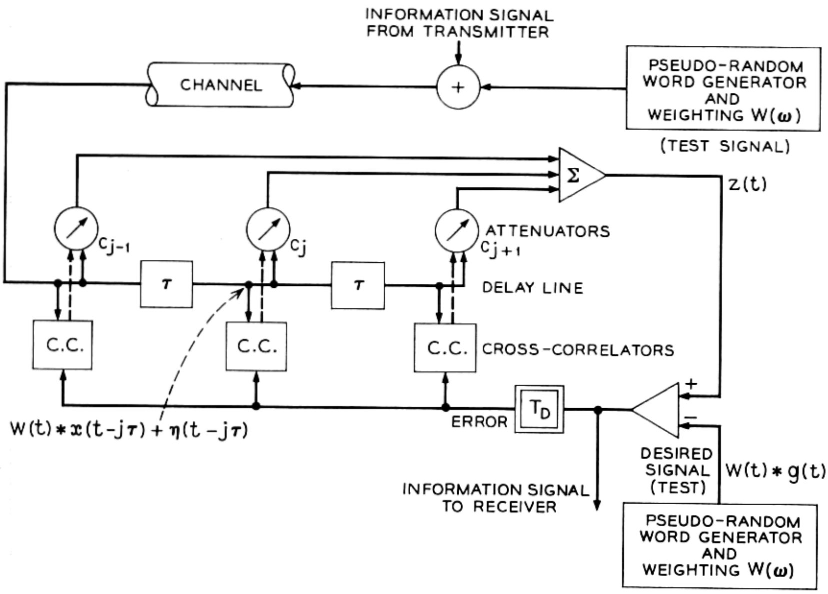


Fig. 5 — Adaptive channel equalizer block diagram.

The implication in many of the equations written thus far is that the integration or averaging necessary for determining the various cross-correlations is carried out over an infinite period of time. In the implementation, however, integration is carried out only for as long as needed to determine cross-correlation to the desired statistical accuracy. A discussion of settling time, then, hinges upon an estimate of just how long an integration time is necessary.

In the implementation of the automatic transversal filter used here (described in greater detail in Ref. 3) the digitally controlled attenuators each can take on $2^M$ values separated by constant increments. $M$ is the number of bits in the attenuator's memory (see Fig. 6). Specifically then, one wants to know how long an integration time is necessary to determine whether or not an undesirable component in the received signal waveform warrants a correction by a $\pm 2/2^M$ portion of the signal from an elemental network. Such a decision must be made with a satisfactorily large probability of being correct.

Thus the problem can be viewed as a problem in signal detection theory. Specifically this question must be answered for the $j$th tap: "Is there or is there not a component of the error signal present which is of the form $2^{1-M}$ $[w(t)^*x(t - j\tau)]$?" This question is

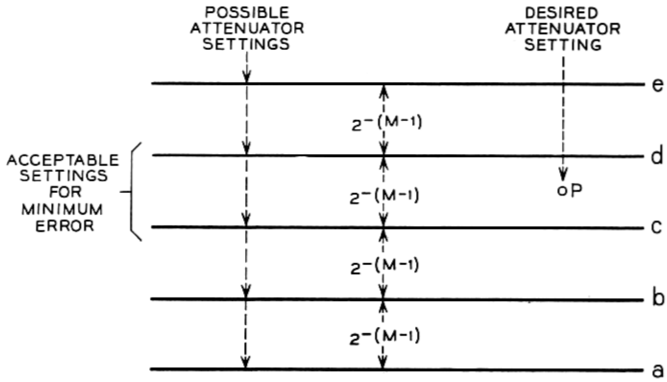


Fig. 6 — Attenuator setting errors.

frequently asked in statistical decision theory.[9] As discussed above in the nonadaptive case the component signal $w(t)^*x(t - j\tau)$ is directly available as the $j$th tap voltage. Thus a maximum likelihood estimator of the component to be measured can be constructed by correlating the tap signal with the error signal. There is a new wrinkle in the adaptive mode, however: although the component signal is still available, it is effectively buried in a very noisy environment (created by the information signal). Thus the problem at hand can be called the detection of a known signal by correlation with a noisy reference signal. The details of the mathematical argument are given in the appendix and essentially follow Helstrom's development but with modifications for the noisy reference case.[9]

In the appendix an approximate relation is obtained which relates the equalizer's maximum residual signal-to-noise ratio to the correlators' integration time. This residual signal-to-noise ratio considers only the error or noise resulting from improper setting of the attenuators. In the analysis there is one noise component resulting solely from the granularity of the attenuators. For an equalizer of specified length and composed of attenuators with $M$-bit memories this residual noise (called the noise floor) is shown as a number of horizontal lines in Fig. 7. In the appendix the increase in this noise resulting from finite correlator integration time (which causes further inaccuracies in the setting of the attenuators) is determined. The resultant total noise as a function of integration time is displayed in Fig. 7 as curved lines. The results are shown for the case of a 19-tap equalizer with information signal and test signal transmitted at the same power level and a useful channel bandwidth of 2,400 Hz. The increase in noise should be maintained at a small value, for if this is not the case the same residual noise could have been obtained from attenuators with fewer bits accuracy and smaller cost. If the increase in noise is held to, say, ¼ of the value of the noise floor (that is, an increase of 1 dB), the equalizer design is restricted to points on the dashed line in Fig. 7. The probability $P$ that an attenuator is incorrectly set can then be determined from equations (25) and (26) in the appendix by setting the permitted fraction of the noise floor equal to the increase in noise, that is,

$$\frac{1}{4}\frac{(2N+1)}{(2)^{2M}} = \frac{(2N+1)}{(2^{M-1})^2} P. \tag{16}$$

For the case stated the probability of incorrect setting should be 1/16, independent of equalizer length or attenuator granularity. Because of the assumed gaussian statistics, the value of $P$ equal to 1/16 determines the ratio of the mean [equation (21)] to the standard deviation [equation (23)]. An approximate relation for the integra-

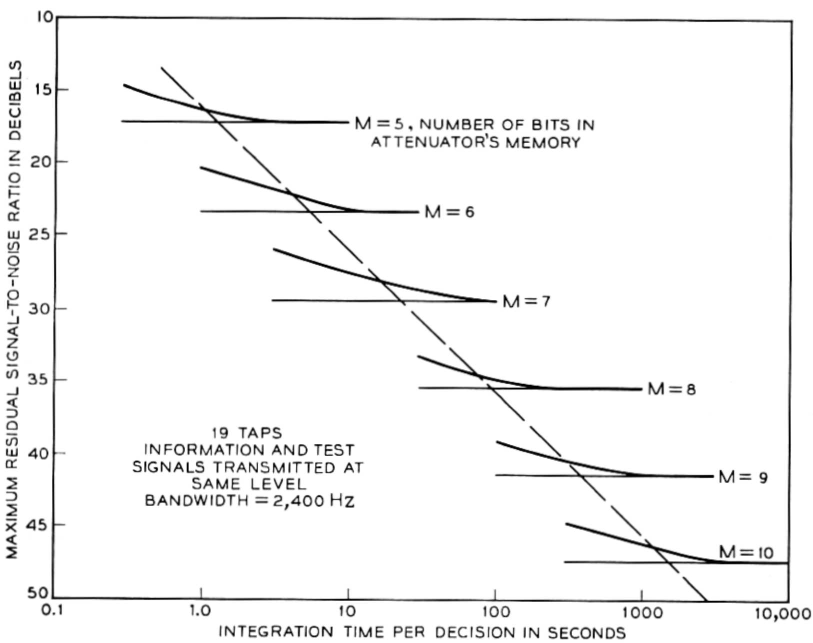


Fig. 7 — Integration time.

tion time $T$ can then be found for the region of interest, that is,

$$2^{-2[M-1]}N_o < V_o .$$

In fact, in this region it follows that

$$T \cong \frac{2^{2M}}{\Omega}\left[\frac{V_0}{S} + \frac{N_o V_o}{S^2}\right] \tag{17}$$

where $N_o/S$ is the received information to test signal power ratio, $V_o/S$ is the equalized information to test signal power ratio, $\Omega$ is the channel bandwidth, and $M$ is the number of bits in each attenuator's memory.

The noise levels shown in Fig. 7 were calculated for received and equalized reference to information signal ratios of unity. An increase in the information signal to reference noise ratio can be obtained by reducing the level of the transmitted reference signal. Thus, more of the power capacity of the channel can be used for the information-bearing signal. The penalty which must be paid is an increase in the necessary integration time, as shown by equation (17).

It may be wise to emphasize that the argument here has dealt with the time, $T$, required for the equalizer to make its finest adjustment reliably. The most abrupt possible change in the transmission channel's characteristics would force at least one of the equalizer's attenuators to move through its entire range of $2^M$ steps. The time required for this is long, in fact $2^M T$ seconds, ignoring the interaction among attenuator settings. This period could easily be reduced by the incorporation of a scheme for accelerated operation of the equalizer when the reference error signal exceeds some threshold. The equalizer could first run quickly to an approximate setting and then slowly to the exact optimum setting.

## III. EXPERIMENTAL IMPLEMENTATION AND PERFORMANCE

In order to see what difficulties might arise in the implementation of the adaptive equalizer described here, such a device was constructed in the laboratory. The heart of the implementation is the automatic transversal filter as shown in Fig. 8. This has been described in considerable detail in Ref. 3 as are the necessary timing and carrier recovery circuits. The function of the timing circuitry is to establish the synchronization of the two pseudorandom sequences while the carrier recovery circuitry compensates for any net carrier offset which may have occurred during transmission. A Bell System *Data-Phone*® data communications set 201B was used as the information transmitter

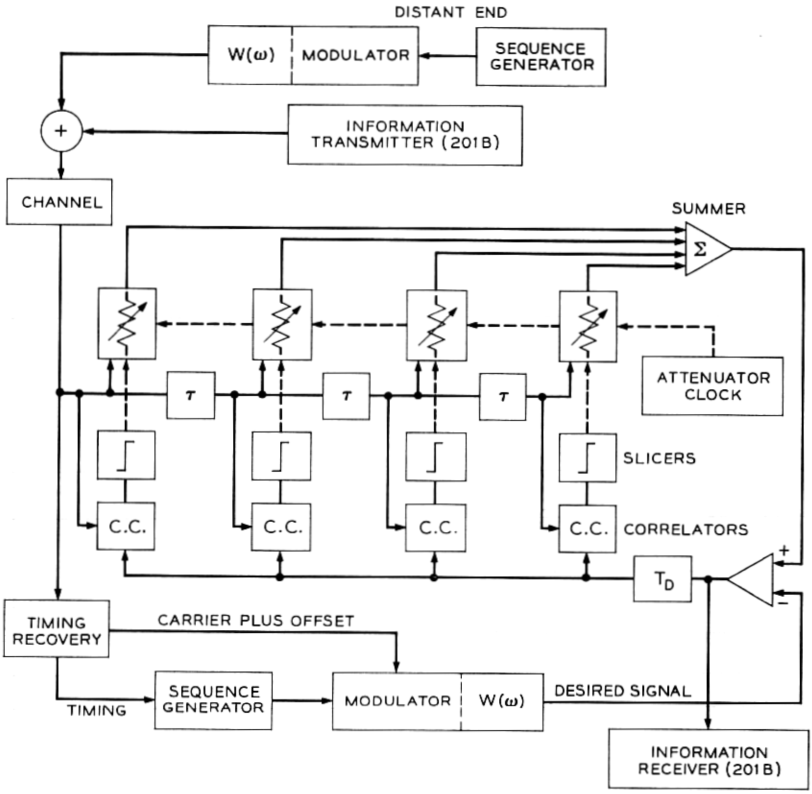


Fig. 8 — Adaptive equalizer block diagram.

and receiver. This modem happens to be of the four-phase, differentially detected type and is capable of 2400 bits per second transmission on well-equalized (C2 conditioned) private lines. It should be emphasized that the design and utility of the adaptive equalizer is independent of the information transmission format.

In the laboratory experiment the channel was simulated and had a frequency response given by $X(\omega) = 1 - ae^{-i\omega 2\tau}$, where $\tau$ is the tap spacing of the equalizer and is equal to 150 microseconds. The performance of the equalizer on this simulated channel (with the constant $a$ assuming the value of about 0.5) is shown in the series of eye patterns in Fig. 9. Figure 9a shows the closed eye-pattern for the received, distorted signal in the absence of the equalizer. Figure 9b shows the improved eye pattern resulting from the adaptively equalized signal.

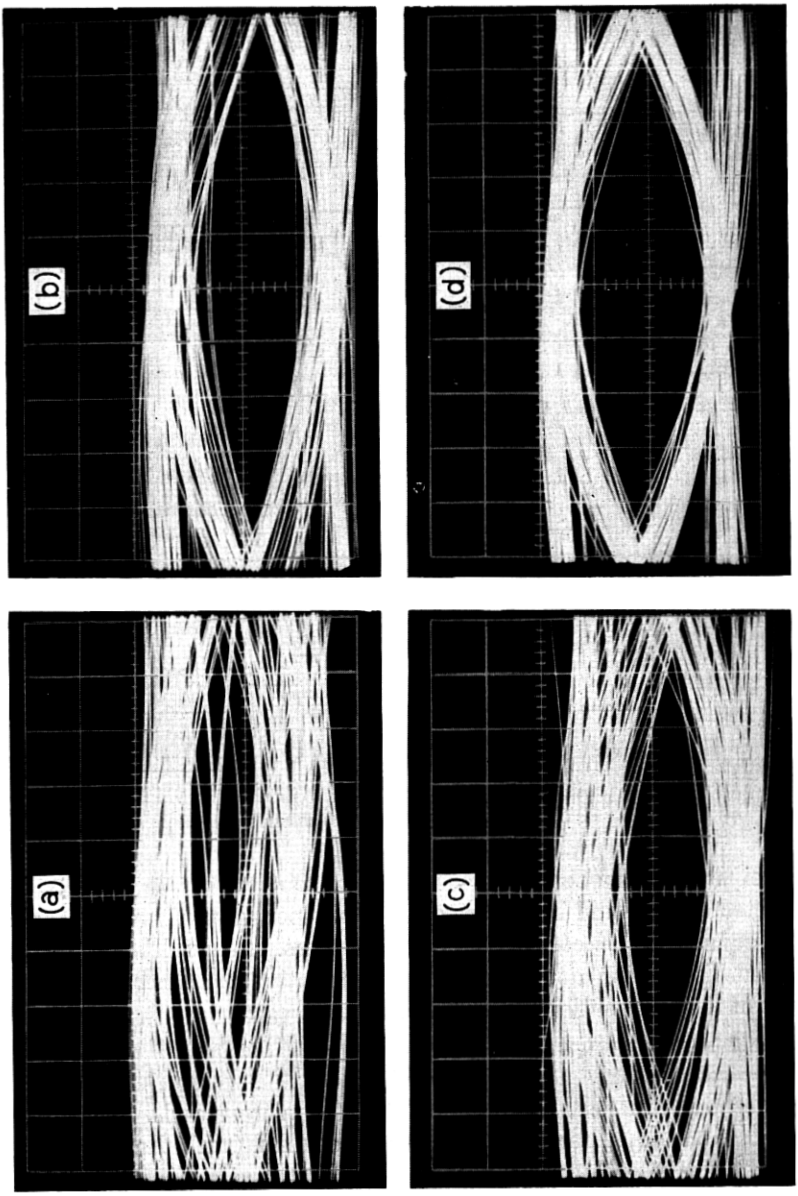


Fig. 9 — Eye patterns: (a) received, distorted signal, (b) adaptively equalized signal, test signals removed, (c) adaptively equalized signal, (d) preset-equalized

The correlators used in the adaptive equalizer were designed for use in the preset equalizer. These correlators imposed stringent restrictions on the relative levels of the test and data signals for optimum performance. Specifically the test signal had to be transmitted at a higher level than would be necessary, given correlators with a greater dynamic range. Comparison of the second and third eye patterns shown in Figs. 9b and c shows the resulting noise. The two patterns are identical except that the second includes the effect of the imperfectly matched equalized and locally generated test signals. The final eye pattern (Fig. 9d) shows the results which would be expected from an adaptive equalizer using correlators with greater dynamic range than those now available.

## IV. OBSERVATIONS AND CONCLUSION

The experimental performance of the adaptive channel equalizer encourages further development. At the same time, however, the experimental work has indicated several problems which require special attention.

First, the correlators used to establish the attenuator settings in the adaptive channel equalizer perform in a very hostile environment. They must search out small components of a signal which itself is buried in noise; this requires tremendous dynamic range. The observation times involved are long; this imposes constraints on the integrators necessary for performing cross-correlation.

Second, the delay line of length $T_D$ used to prevent domination of the equalizer by the information signal also poses a problem. The operation of the equalizer can be severely impaired by any irregularities in this long delay line. The correlators have no means of compensating for distortion which occurs in this delay, and even an amount of distortion which would be considered small in many applications can make the equalizer virtually useless.

These two observations indicate that considerable advantage might well be obtained in an all-digital implementation. In such a device a small penalty would initially be suffered in analog-to-digital conversion, but this is the only source of distortion. Delay of arbitrary length can easily be obtained by inexpensive shift registers, and correlators can be constructed to arbitrary accuracy.

Third, a substantial component of the noise seen by the information signal receiver is the very small difference between the nearly identical desired and equalized transmitted test signals. Even a small difference or jitter in timing or carrier phase is thus capable of causing a very large increase in the difference signal or noise seen by the information receiver. Therefore the carrier and timing recovery circuits must be designed with care and the reference signal should be transmitted at as low a level as is practicable. It should also be stated that this technique does nothing to eliminate the effects of nonlinear distortion in the channel.

In conclusion, there are problems which need further attention but the indications are that this technique holds great promise for the continuous adaptive equalization of communication channels where the designer has no knowledge of the information format used on the channel.

## V. ACKNOWLEDGMENT

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## APPENDIX

### *Calculation of Correlator Integration Time*

Let the signal, which is the input to the $j$th attenuator, be broken down into two parts: the signal $s_j(t)$ resulting from the test signal, and the noise $\eta_j(t)$ resulting from the information signal. As the equalization process nears completion, the correctable components of the error signal become more and more difficult to detect. Finally the $j$th correlator must determine whether or not a component of the error signal of the form $[2/2^M]\ [s_j(t - T_D)]$ is present. This signal is also buried in noise $\nu(t - T_D)$ which is a result almost exclusively of the information signal. Thus the other waveform delivered to the correlator is $a[2/2^M]\ [s_j(t - T_D)] + \nu(t - T_D)$ where the variable $a$ is either zero (signal not present) or one (signal present). The $j$th correlator thus computes

$$\rho_i = \left\langle \left\{ \frac{2a}{2^M} [s_i(t - T_D)] + \nu(t - T_D) \right\} \{s_i(t) + \eta_i(t)\} \right\rangle \qquad (18)$$

to ascertain whether or not the signal is present, that is, whether $a$ is zero or one.

Following Helstrom's outline for the noiseless reference case, it is necessary to calculate the mean and variance of the statistic $\rho_j$ since in the physical system integration is carried out only over a range of 0 to $T$ seconds.[9] The mean and variance of $\rho_j$ suffice because we assume at this point in the argument that the noise created by the information signaling system is gaussian. For convenience we also assume that the noise is white. This is clearly not the case but the search here is only for an estimate of the settling time and an analysis based on the exact statistics of the noise would yield little for the greatly increased effort necessary.

The mean of the various statistics $\rho_j$ is

$$\langle\rho\rangle_{\text{av}} = \int_0^T \left\langle \left\{ \frac{2a}{2^M} [s_j(t - T_D)] + \nu(t - T_D) \right\} \{s_j(t) + \eta_j(t)\} \right\rangle_{\text{av}} dt \quad (19)$$

where $\langle\ \rangle_{\text{av}}$ indicates an ensemble average. Because the noise and signal sources are independent and have zero mean values and further because $T_D$ is chosen large enough to make $\nu(t - T_D)$ and $\eta_j(t)$ independent,

$$\langle\rho\rangle_{\text{av}} = \frac{2a}{2^M} \int_0^T \langle s_j(t - T_D) s_j(t)\rangle_{\text{av}}\, dt. \quad (20)$$

However, $s_j(t - T_D) = s_j(t)$ so that

$$\langle\rho\rangle_{\text{av}} = \frac{2a}{2^M} \int_0^T \langle s_j^2(t)\rangle_{\text{av}}\, dt = \frac{2a}{2^M} TS \quad (21)$$

where $T$ is the integration time and $S$ is the power level of the test signal.

The variance of the statistics $\rho_j$ can be shown to be essentially independent of $a$ under the assumptions made and is given by

$$\sigma_\rho^2 = \int_0^T \int_0^T \left\langle \left\{ \frac{2a}{2^M} [s_j(t_1 - T_D)] + \nu(t_1 - T_D) \right\} \cdot \left\{ \frac{2a}{2^M} [s_j(t_2 - T_D)] + \nu(t_2 - T_D) \right\} \right\rangle_{\text{av}} \cdot \langle [s_j(t_1) + \eta_j(t_1)][s_j(t_2) + \eta_j(t_2)]\rangle_{\text{av}}\, dt_1\, dt_2\,. \quad (22)$$

After some manipulation the nonzero terms are

$$\sigma_\rho^2 = \left[\frac{2a}{2^M}\right]^2 \int_0^T \int_0^T \langle s_j(t_1) s_j(t_2) \eta_j(t_1) \eta_j(t_2)\rangle_{\text{av}}\, dt_1\, dt_2$$

$$+ \int_0^T \int_0^T \langle \nu(t_1)\nu(t_2)s_j(t_1)s_j(t_2)\rangle_{av}\, dt_1\, dt_2$$

$$+ \int_0^T \int_0^T \langle \nu(t_1)\nu(t_2)\eta_j(t_1)\eta_j(t_2)\rangle_{av}\, dt_1\, dt_2 .$$

$$\sigma_\rho^2 = \left[\frac{2a}{2^M}\right]^2 \int_0^T \int_0^T \langle s_j(t_1)s_j(t_2)\rangle_{av} R_\eta(t_1 - t_2)\, dt_1\, dt_2$$

$$+ \int_0^T \int_0^T \langle s_j(t_1)s_j(t_2)\rangle_{av} R_\nu(t_1 - t_2)\, dt_1\, dt_2$$

$$+ \int_0^T \int_0^T R_\nu(t_1 - t_2) R_\eta(t_1 - t_2)\, dt_1\, dt_2 .$$

$R_\nu$ and $R_\eta$ are autocorrelation functions. After evaluation the relation for $\sigma_\rho^2$ is of the following form:

$$\sigma_\rho^2 \cong \left\{\left[\frac{2a}{2^M}\right]^2 \frac{2N_{j0}}{\Omega} + \frac{2V_0}{\Omega}\right\} TS + \frac{2N_{j0}V_0T}{\Omega}. \tag{23}$$

$N_{j0}$ is the power level of the signal $\eta_j(t)$ and $V_0$ is the power level of the signal $\nu(t)$; the effective channel bandwidth is $\Omega$. Had the reference not been noisy, only the variance of the statistic $\rho_j$ would be changed and would in fact be

$$\sigma_\rho^2 = \frac{2V_0}{\Omega}(TS) \tag{24}$$

which agrees with the standard result.[9]

With the assumptions mentioned above, equations (21) and (23) completely specify the statistics necessary to calculate the probability of making a correct decision.

Figure 6 helps explain the operation of the system. The range of the attenuator $(-1, +1)$ is divided into $2^M$ equal increments of width $2^{-(M-1)}$. A worst case assumption that the desired attenuator setting is midway between two possible settings is shown in Fig. 6 and leads to a minimum attainable signal-to-noise ratio (resulting from the granularity of the attenuator) of

$$\frac{S_\eta}{N_S} = 10 \log_{10} \frac{(2)^{2M}}{(2N + 1)}\, \text{dB} \tag{25}$$

where $(2N + 1)$ is the number of attenuators. Relation (25) was obtained in Ref. 3 and is the ratio of the information signal power to the test-signal-generated noise, assuming that the two signals are transmitted at the same level. This is approximately the signal-to-noise ratio seen by the information receiver.

Incorrect setting of the attenuator decreases the effective signal-to-noise ratio from this value. Referring to Fig. 6, assume that the attenuator is set at level $c$. At the end of the correlator integration period, the correlator output, $\rho_j$, is sampled. If $\rho_j$ is positive, the attenuator setting increases to level $d$; if $\rho_j$ is negative, the setting decreases to level $b$. Assume that the optimum attenuator value corresponds to level $P$, which is not achievable because of the attenuator granularity. Then the expected value of $\rho_j$, $\langle \rho_j \rangle_{\text{av}}$, is positive by an amount proportional to the difference between levels $P$ and $c$. If the actual attenuator settings fluctuate between levels $c$ and $d$, the distortion is that given by equation (25); if, however, the attenuator setting assumes the value of level $b$, the result is an increase in the distortion power of the amount $V_o/(2^{M-1})^2$. The probability that the attenuator setting goes to level $b$ is then the probability that $\rho_j$ is less than zero. Since $\langle \rho_j \rangle_{\text{av}}$ is positive in this example, and assuming gaussian statistics as above, this probability of incorrect attenuator setting is given by the relation

$$\text{P} = \text{erfc}\,(\langle \rho_j \rangle_{\text{av}}/\sigma_P).$$

Further, assuming independence of the attenuator weights, the increase in noise to the information signal is

$$(2N + 1)(P)\,\frac{V_0}{(2^{M-1})^2}. \tag{26}$$

The results of the evaluation of equation (26) are plotted in Fig. 7 in various regions of interest. For Fig. 7 a channel bandwidth of 2,400 Hz is assumed.

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