A System Approach to Quantization and Transmission Error

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In a system designed to quantize the output of an analog data source and to transmit this information over a digital channel, errors are introduced by the quantization and transmission processes. Quantization resolution can be improved by using all positions available in a data stream to carry information, or transmission accuracy can be improved if some of the positions are used for redundancy with error-correcting codes. The problem is to determine, from a system viewpoint, the proper allocation of the available positions in order to reduce the average system error rather than concentrate exclusively on either the quantization problem or the transmission problem.

We develop a criterion for the performance of data transmission systems based upon the numerical error that occurs between the analog source and the destination. The criterion, termed the average system error, is used to evaluate and compare possible system configurations. Significant-bit packed codes are defined. These codes are useful because their protection can be matched to the numerical significance of the data and their redundancy can be sufficiently small to maintain good quantization resolution. The average system error resulting from representative system designs is numerically evaluated and compared.

I. INTRODUCTION

When designing a system to sample the output of an analog data source and to transmit the samples over a digital channel, the usual approach is to consider the errors introduced by quantization and transmission as separate problems. However, from a system viewpoint, a conflict arises. On the one hand, the quantization resolution can be improved by using all of the available positions in a data stream to carry information. Alternatively, the transmission accuracy can be improved if redundancy and error-correcting codes are introduced by converting some of the information positions into parity check positions. The problem then is to determine the proper allocation of the available symbols in order to reduce the average system error rather than concentrate exclusively on either the quantization problem or the transmission problem.

We consider a data transmission system with uniform quantization. The average absolute error that occurs between the analog source and the destination is used as the criterion of system performance. The criterion, termed the average system error (ASE), is used to evaluate and compare the effectiveness of various systems.

Some work has been done on the design of error-correcting codes which provide different amounts of protection for different positions within a code word. In Ref. 1, the general algebraic properties of these codes, referred to as unequal error protection codes, were investigated. In Ref. 2, significant-bit codes (which turn out to be a subclass of unequal error protection codes) and a criterion for evaluating the performance of codes for the transmission of numerical data were developed.

In this paper, we define packed codes and significant-bit packed codes, we analyze their performance, and we numerically evaluate the average system error resulting from the use of representative quatization resolutions and coding schemes.

II. PRELIMINARIES

We consider a binary symmetric channel in which the errors are independent of the symbols actually transmitted. In the numerical examples, we further assume that the errors occur independently with probability $p = 1 - q$. The error-correcting codes to be discussed are binary block codes in which the code vectors form a group under component by component modulo 2 addition. Let n denote the block length and k denote the number of information positions per code vector. The notation (n, k) is used to denote such a code. A complete discussion of these codes is contained in Ref. 3.

The encoder receives k binary information symbols [called a message and denoted by $(v_k, v_{k-1}, \cdots, v_1)$] as an input and determines from the message $(n - k)$ binary parity check symbols. The decoder operates upon the blocks of n binary symbols coming from the channel in an attempt to correct transmission errors and provides k binary symbols at its output.

Let H denote the parity check matrix for such a code. An n-tuple u is a code vector if and only if

$$u\tilde{H} = 0 \tag{1}$$

where $\tilde{H}$ is the transpose of H. The matrix H can be written in the form

$$H = (C_k, C_{k-1}, \cdots, C_1 I_{n-k})$$

where $C_i (1 \leqq i \leqq k)$ is the column of H in the position corresponding to information position v_i in a code vector and I_{n-k} is the $(n - k) \times (n - k)$ identity matrix.

When the integer s is to be sent, the message used is $B_k(s)$ such that*

$$B_k(s) = (v_k, v_{k-1}, \cdots, v_1)$$

where

$$s = \sum_{i=1}^{k} 2^{i-1} v_i .$$

The parity check symbols $E(s)$ are chosen so that the code vector $C(s) = B_k(s) \mid E(s)$ satisfies equation (1) where the symbol $\mid$ indicates that $C(s)$ can be partitioned into $B_k(s)$ and $E(s)$.

III. PACKED CODES

A model of the data transmission system is shown in Fig. 1. Let us assume that each quantization step is of equal size and that there are 2^l

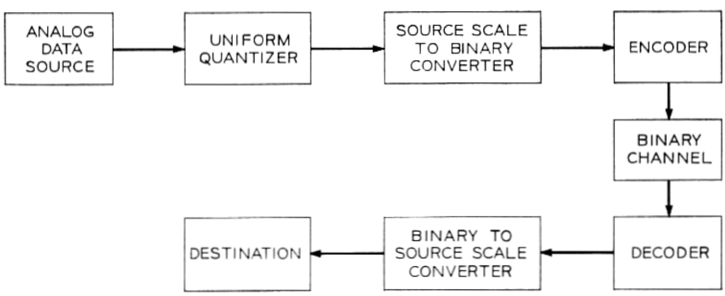

Fig. 1 — System model.

quantization levels. For many applications, the quantizer uses a relatively small l (perhaps 15 or less). In addition, coding schemes must have low redundancy; otherwise so many information positions are converted into check positions that the quantization error becomes too large. These requirements lead us to define "packed" codes in the follow-

* $B_i(j)$ denotes the i-bit binary representation of the integer j where $0 \leqq j \leqq 2^i - 1$.

ing manner. Consider an (n, k) binary group code in which α samples are packed into each code vector. If each sample consists of l bits, then $k = \alpha l$. Let s_m denote the integer that is transmitted for the mth sample in a code vector where $0 \leqq s_m \leqq 2^l - 1$ and $1 \leqq m \leqq \alpha$. Accordingly, the code vector actually transmitted is

$$C(s) = B_l(s_\alpha) \mid B_l(s_{\alpha-1}) \mid \cdots \mid B_l(s_1) \mid E(s)$$

where

$$s = \sum_{m=1}^{\alpha} 2^{(m-1)l} s_m . \tag{2}$$

A packed code vector is shown schematically in Fig. 2.

Two examples are in order. In the first, a (7, 4) perfect single error-correcting code is used to form a packed code with $\alpha = 2$ and $l = 2$.

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & \\ 1 & 1 & 0 & 1 & I_3 \\ \underbrace{1 \;\; 0}_{s_2 \text{ positions}} & & \underbrace{1 \;\; 1}_{s_1 \text{ positions}} & & \end{bmatrix} .$$

In the second example, the idea behind significant-bit codes is applied to packed codes and results in what will be referred to as a significant-bit packed code.[2] Specifically, the basic (7, 4) code can have its protection capabilities arranged to match the numerical significance of the bit positions; that is, to protect the most significant bit of each of four samples ($\alpha = 4$ and $l = 2$).

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & I_3 \\ \underbrace{1 \;\; 0}_{s_4 \text{ positions}} & & \underbrace{0 \;\; 0}_{s_3 \text{ positions}} & & \underbrace{1 \;\; 0}_{s_2 \text{ positions}} & & \underbrace{1 \;\; 0}_{s_1 \text{ positions}} & & \end{bmatrix} .$$

Notice that the significant-bit packed code requires only half as many parity check positions per sample as the packed code.

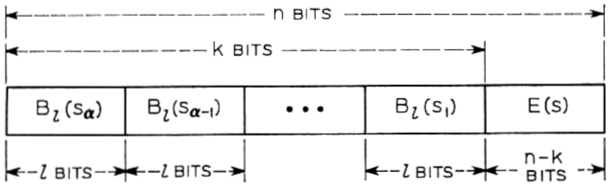

Fig. 2 — Packed code vector.

Many packed codes can be designed to provide desired levels of protection and redundancy. Numerical data concerning the effectiveness of representative packed codes are presented in Sections VI and VII.

IV. FORMULATION OF A CRITERION OF SYSTEM FIDELITY

In this section, we develop a criterion of system fidelity as a function of the number of quantization levels and the capability of the error-correcting code. This is done for packed codes because of their generality.

Let x_m denote the output of the analog source that results in s_m being transmitted. It is assumed that x_m is a random variable that is uniformly distributed on the interval (X_1, X_2). The probability density function for x_m is

$$\begin{aligned} f(x_m) &= \frac{1}{X_2 - X_1} \quad \text{for} \quad X_1 \leqq x_m \leqq X_2 \\ &= 0 \qquad\qquad \text{for} \quad x_m < X_1 \quad \text{or} \quad x_m > X_2 \,. \end{aligned} \tag{3}$$

If

$$X_1 + s_m\left(\frac{X_2 - X_1}{2^l}\right) < x_m < X_1 + (s_m + 1)\left(\frac{X_2 - X_1}{2^l}\right),$$

then the output of the quantizer is

$$X_1 + (s_m + \tfrac{1}{2})\left(\frac{X_2 - X_1}{2^l}\right).$$

The "source scale to binary converter" receives

$$X_1 + (s_m + \tfrac{1}{2})\left(\frac{X_2 - X_1}{2^l}\right)$$

from the quantizer and delivers $B_l(s_m)$ to the encoder. After α samples are received by the encoder, the message

$$B_k(s) = B_l(s_\alpha) \mid B_l(s_{\alpha-1}) \mid \cdots \mid B_l(s_1)$$

is encoded to form the code vector $C(s) = B_k(s) \mid E(s)$ where the value of s is determined from equation (2). At the destination, the decoder attempts to correct errors and provides the message

$$B_k(r) = B_l(r_\alpha) \mid B_l(r_{\alpha-1}) \mid \cdots \mid B_l(r_1)$$

at its output where $0 \leqq r_m \leqq 2^l - 1$ for $1 \leqq m \leqq \alpha$ and

$$r = \sum_{m=1}^{\alpha} 2^{(m-1)l} r_m . \tag{4}$$

The "binary to source scale converter" receives $B_l(r_m)$ and delivers

$$X_1 + (r_m + \tfrac{1}{2})\left(\frac{X_2 - X_1}{2^l}\right)$$

to the destination. Because uniform quantization is used, a useful measure of the numerical error that occurs as a result of the quantization and transmission of x_m is

$$\left| x_m - \left[X_1 + (r_m + \tfrac{1}{2})\left(\frac{X_2 - X_1}{2^l}\right)\right] \right|^\gamma$$

where $\gamma > 0$. The appropriate value of γ will depend upon the nature and use of the signal. For this paper, let $\gamma = 1$.

For the mth sample position in a packed code, let $\Pr_m\{r_m \mid s_m\}$ denote the probability that r_m is received when s_m is sent. Accordingly, the average system error for the mth sample (ASE$_m$) is

$$\text{ASE}_m = \sum_{r_m=0}^{2^l-1} \sum_{s_m=0}^{2^l-1} \int_{X_1+s_m(X_2-X_1)/2^l}^{X_1+(s_m+1)(X_2-X_1)/2^l} \left| x_m - X_1 - (r_m + \tfrac{1}{2})\left(\frac{X_2 - X_1}{2^l}\right)\right| \cdot \Pr_m\{r_m \mid s_m\} f(x_m)\, dx_m . \tag{5}$$

It is desirable to express $\Pr_m\{r_m \mid s_m\}$ in terms of the properties of the error-correcting code. Let $\Pr\{r \mid s\}$ denote the probability that r occurs at the output of the decoder when s is the input to the encoder. As shown in Appendix A, for a channel in which the errors are independent of the symbols actually transmitted,

$$\Pr_m\{r_m \mid s_m\} = \underbrace{\sum_{t_\alpha=0}^{2^l-1} \cdots \sum_{t_1=0}^{2^l-1}}_{\text{excluding } t_m} \Pr\left\{ \sum_{m'=1}^{\alpha} 2^{(m'-1)l} t_{m'} \,\middle|\, 0 \right\}$$

where $B_l(t_m) = B_l(r_m) \oplus B_l(s_m)$.* This expression is interesting because it permits us to compute $\Pr_m\{r_m \mid s_m\}$ from the properties of the code. Specifically, it is necessary to determine the probability that each possible sequence of α samples, in which the mth position equals t_m, is received, given that zero is transmitted for each sample, and then to sum these probabilities.

For the case in which one sample is transmitted per code word (that is, $\alpha = 1$ and $l = k$) and all samples are equally likely to be transmitted, the average numerical error (ANE) that occurs during transmission has been defined as[2]

$$\text{ANE} = \frac{1}{2^k} \sum_{r=0}^{2^k-1} \sum_{s=0}^{2^k-1} |r - s| \Pr\{r \mid s\}.$$

The average numerical error is the average magnitude by which the output of the decoder differs numerically from the input to the encoder and thus provides a measure of the performance of the channel and the code. This concept can be generalized by defining the average numerical error for the mth sample as

$$\text{ANE}_m = \frac{1}{2^l} \sum_{r_m=0}^{2^l-1} \sum_{s_m=0}^{2^l-1} |r_m - s_m| \Pr_m\{r_m \mid s_m\}. \tag{6}$$

By reasoning analogous to that in Theorem 1 of Ref. 2, for a binary group code used with a binary symmetric channel, equation (6) can be reduced to

$$\text{ANE}_m = \sum_{i=1}^{l} 2^{i-1} \sum_{r_m=2^{i-1}}^{2^i-1} \Pr_m\{r_m \mid 0\}.$$

With this definition of ANE_m, the probability density function in equation (3), and the steps shown in Appendix B, the average system error for the mth sample, as given in equation (5), can be expressed as

$$\text{ASE}_m = \left(\frac{X_2 - X_1}{2^l}\right)(\text{ANE}_m + \tfrac{1}{4}\Pr_m\{0 \mid 0\}).$$

One feature of packed codes is that the protection afforded various samples against transmission errors can be unequal. If this occurs, different positions will have different system error. Therefore, in general, the average system error per sample (ASE) is

$$\text{ASE} = \frac{1}{\alpha} \sum_{m=1}^{\alpha} \text{ASE}_m .$$

* The symbol $\oplus$ denotes component by component modulo 2 addition of vectors.

The range of the analog source is specified by X_1 and X_2. When considering system design, it is convenient to let $X_2 - X_1 = 1$ (or to consider a normalized average system error). Accordingly, in the remainder of this paper, we shall be concerned with the expression in equation (7).

$$\text{ASE} = \frac{1}{\alpha} \sum_{m=1}^{\alpha} \left[\frac{1}{2^l} \left(\text{ANE}_m + \tfrac{1}{4} \Pr_m \{0 \mid 0\} \right) \right]. \tag{7}$$

For a system in which one sample is transmitted per code word (that is, $\alpha = 1$ and $l = k$),

$$\text{ASE} = \frac{1}{2^l} \left(\text{ANE} + \tfrac{1}{4} \Pr \{0 \mid 0\} \right) \tag{8}$$

where ANE and $\Pr\{0 \mid 0\}$ are for the entire code.

For error-free transmission, $\Pr_m\{0 \mid 0\} = 1$ and $\text{ANE}_m = 0$ for all coding schemes including uncoded transmission. In this case, $\text{ASE} = 2^{-(l+2)}$. Thus, the system error is independent of the particular code, is minimized by maximizing l, and cannot be reduced to zero but is bounded by the quantization error.

V. THE AVERAGE SYSTEM ERROR FOR UNCODED TRANSMISSION

Before examining the role that error-correcting codes can play in reducing the average system error, it is advantageous to consider system effectiveness when uncoded transmission is used with a memoryless channel. In the system model, uncoded transmission is characterized by $\alpha = 1$ and $l = k = n$. Let ASN_{UC} denote the average system error for uncoded transmission. From Theorem 2 and the comment following the proof of the theorem in Ref. 2 (these are summarized in Appendix C), the average numerical error for uncoded transmission is

$$\text{ANE}_{\text{UC}} = p \sum_{i=1}^{l} 2^{i-1} q^{l-i} = 2^{l-1} p \frac{1 - \left(\dfrac{q}{2}\right)^l}{1 - \dfrac{q}{2}}.$$

The probability of correct transmission is q^l. Therefore, from equation (8)

$$\text{ASE}_{\text{UC}} = \frac{1}{2^l} \left(p \sum_{i=1}^{l} 2^{i-1} q^{l-i} + \frac{q^l}{4} \right). \tag{9}$$

Figures 3 and 4 present the average system error for uncoded transmission for representative values of l and p.

For each value of l, notice that as $p \to 0$, $\mathrm{ASE}_{\mathrm{UC}} \to 2^{-(l+2)}$ which is the limitation imposed by the quantization error. Also, $\mathrm{ASE}_{\mathrm{UC}}$ increases monotonically with p for $0 < p < \frac{1}{2}$ (see Appendix D). For a given value of l, how large must p become so that $\mathrm{ASE}_{\mathrm{UC}}$ deviates appreciably from $2^{-(l+2)}$ (that is, for what values of p does the transmission error make a significant contribution to the system error?)

For small p, equation (9) yields

$$\mathrm{ASE}_{\mathrm{UC}} \cong \frac{1}{2^l}\left[\left(2^l - 1 - \frac{l}{4}\right)p + \frac{1}{4}\right]. \tag{10}$$

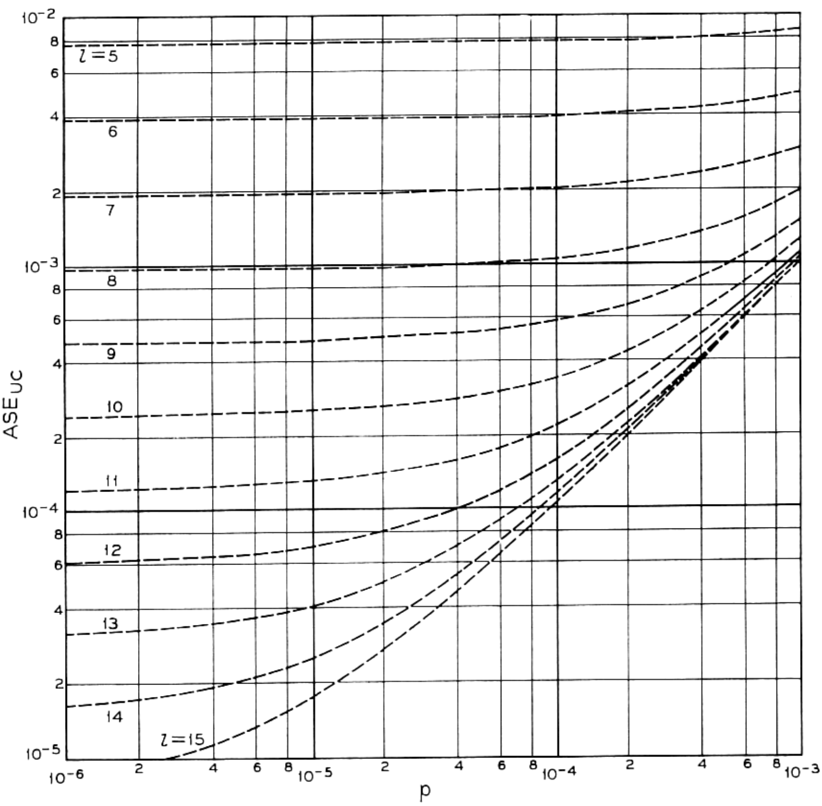

Fig. 3 — Average system error for uncoded transmission ($\mathrm{ASE}_{\mathrm{UC}}$) for various l.

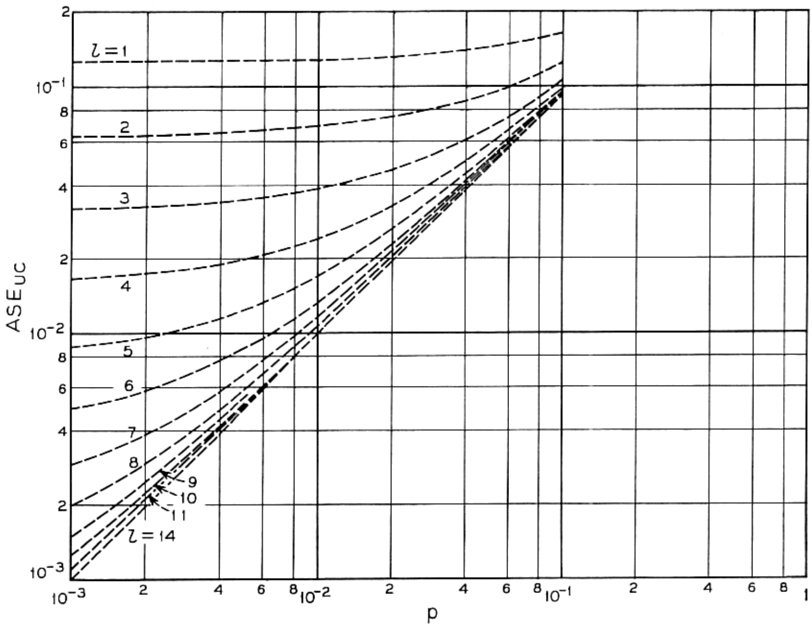

Fig. 4 — Average system error for uncoded transmission (ASE_{UC}) for various l.

This expression can be broken into two components; the term

$$\left[\frac{2^l - 1 - \frac{l}{4}}{2^l}\right]p$$

and the term $2^{-(l+2)}$. These components are shown in Fig. 5 for $l = 15$. In Fig. 5, the terms intersect at a probability of error denoted by p_c where

$$p_c = \frac{1}{4\left(2^l - 1 - \frac{l}{4}\right)}.$$

Notice that p_c is the value of p for which the transmission error equals the quantization error [within the approximations leading to equation (10)]. Accordingly, for $p = p_c$, $ASE_{UC} \cong 2^{-(l+1)}$. In Fig. 6, p_c is given for various l. From p_c, it is possible to obtain an estimate of the general region in which ASE_{UC} begins to deviate from $2^{-(l+2)}$ because of transmission errors.

An additional feature of Figs. 3 and 4 is that, for a given value of l and for p greater than the appropriate p_c, ASE_{UC} is approximately equal to p. This causes the converging of the curves as p increases and implies that systems with different l will have essentially the same performance. Let us consider qualitatively the cause of this phenomenon.

For $p > p_c$, the transmission error is significantly greater than the quantization error and, thus, the average system error is largely determined by the transmission error. If a single error occurs in a sample and if it occurs in the most significant position, on the average, a numerical error of ½ will result for any l. For values of p that are of practical interest, the probability that this occurs is essentially independent of l and equal to p. Similar reasoning can be applied to the less significant positions although the numerical error that results will, of course, be less than ½. The point is that the probability that

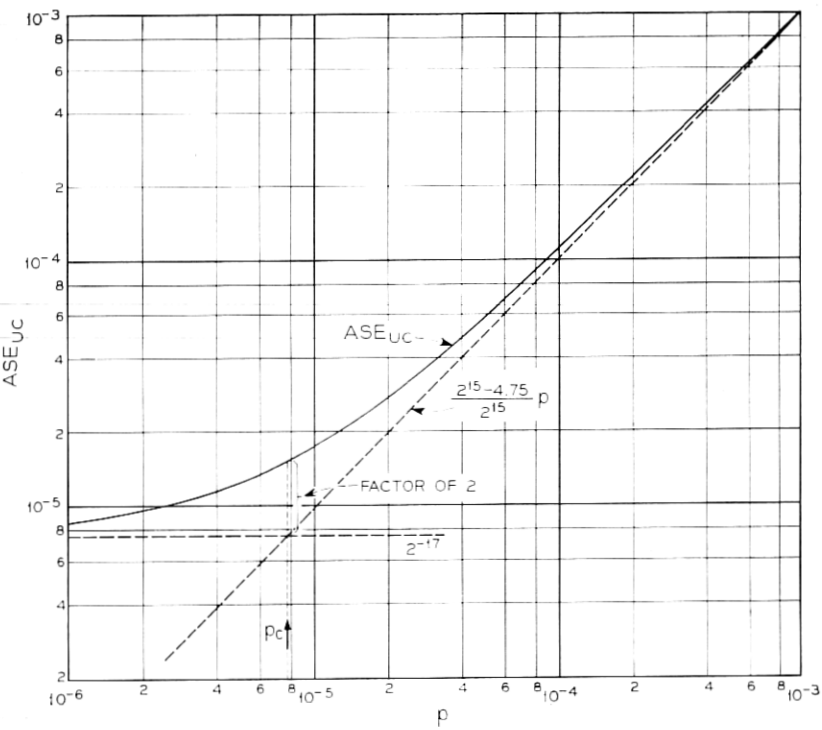

Fig. 5 — Average system error for uncoded transmission (ASE_{UC}) for $l = 15$.

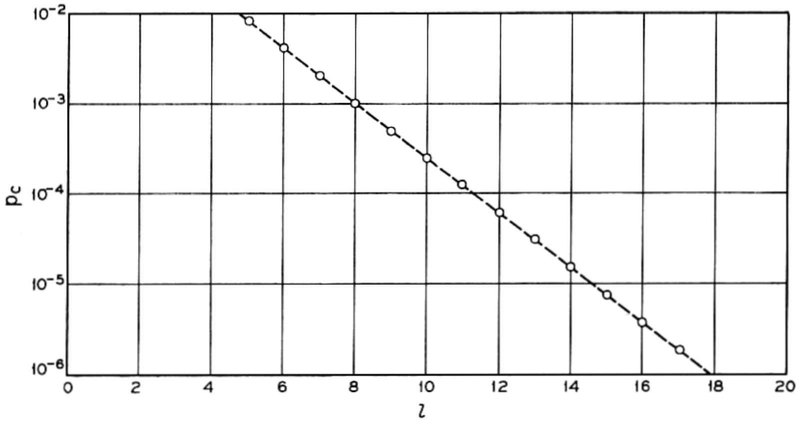

Fig. 6 — p_c for various l.

these single errors occur and the numerical errors that result are essentially independent of l. This implies that the transmission error (and thus the average system error) will be relatively insensitive to l.

Notice that p_c decreases as l increases. The reason is that the quantization error decreases as l increases whereas the transmission error is approximately independent of l. Thus, the value of p where the transmission error becomes a significant portion of the system error decreases.

From equation (10), no system using uncoded transmission can have an average system error significantly less than p no matter how large l becomes. This leads to the problem of how to make the average system error less than p.

Suppose that the σ most significant positions per sample are protected by coding and that the remaining $(l - \sigma)$ positions are not protected. Further, assume that sufficient protection is provided so that the probability of error in the protected positions is substantially less than p. Under these conditions, the transmission error is determined primarily by errors in the least significant positions and we can consider the protected positions to be free of errors. Then, from Theorem 2 of Ref. 2 (summarized in Appendix C),

$$\text{ASE} = \frac{1}{2^l}\left(p \sum_{i=1}^{l-\sigma} 2^{i-1} q^{l-\sigma-i} + \tfrac{1}{4} q^{l-\sigma}\right).$$

For values of p that are of practical interest,

$$\text{ASE} \cong \frac{1}{2^l}\left[\left(2^{l-\sigma} - 1 - \frac{l-\sigma}{4}\right)p + \frac{1}{4}\right]. \tag{11}$$

Accordingly, for p in the range where transmission is the major source of system error, the average system error can be reduced by a factor of approximately $2^{-\sigma}$ from the average system error for uncoded transmission. This implies that we should seek codes that can both protect the significant positions of each sample and maintain quantization resolution by requiring small redundancy. The above requirements provide the motivation for significant-bit packed codes.

VI. SOME EXAMPLES OF SYSTEM PERFORMANCE WITH CODING

In this section we assume that a predetermined number of positions (denoted by ξ) are available to transmit each sample. By numerical evaluation, the average system error that results from the use of representative coding schemes (for $\xi = 7$ and $\xi = 15$*) is determined for various values of p. The examples illustrate that system performance depends upon p and upon the manner in which the ξ positions are allocated between information bits and redundancy for error control.

Let ASE_{UC} denote uncoded transmission. First, consider codes in which one code vector is used per sample ($\alpha = 1$). Listed below is a brief description of each code. The codes are indexed by the notation used for their average system error in Fig. 7 ($\xi = 7$) and Fig. 8 ($\xi = 15$).

$\text{ASE}_{(3,1)}$: A (3, 1) perfect single error-correcting code is used to protect the most significant position.

$$\xi = 7: \qquad \alpha = 1 \qquad l = 5$$

$$\xi = 15: \qquad \alpha = 1 \qquad l = 13$$

$\text{ASE}_{(3,1),(3,1)}$: Independent (3, 1) perfect single error-correcting codes are used to protect the two most significant positions.

$$\xi = 7: \qquad \alpha = 1 \qquad l = 3$$

$$\xi = 15: \qquad \alpha = 1 \qquad l = 11$$

* These values were selected because in each case it is possible to construct a perfect single error-correcting code and thus to compare uniform protection with protection that is heavily weighted in favor of the most significant bit per sample.

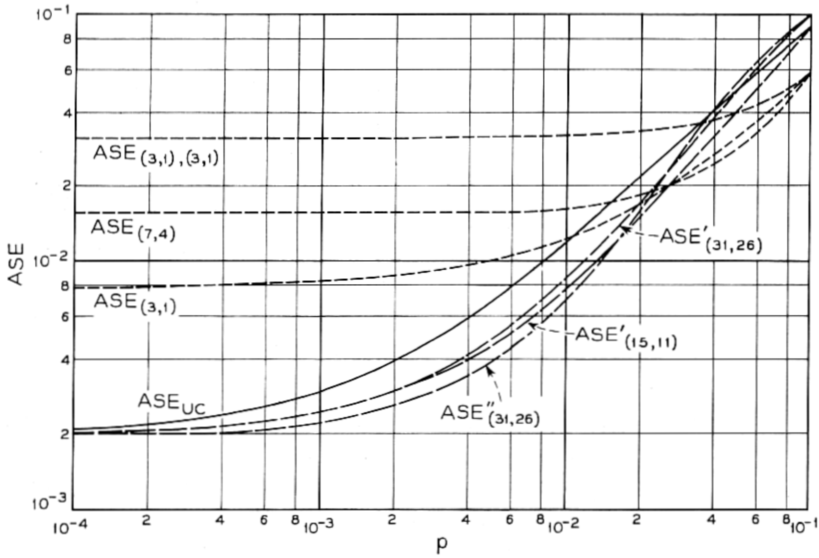

Fig. 7 — Average system error (ASE) with representative codes; 7 positions per sample ($\xi = 7$).

$\text{ASE}_{(7,4)}$: A (7, 4) perfect single error-correcting code is used to protect the four most significant positions.

$$\xi = 7: \qquad \alpha = 1 \qquad l = 4$$

$$\xi = 15: \qquad \alpha = 1 \qquad l = 12$$

$\text{ASE}_{(15,11)}$: A (15, 11) perfect single error-correcting code is used to protect all 11 positions.

$$\xi = 15: \qquad \alpha = 1 \qquad l = 11$$

Although many significant-bit packed codes can be constructed, we consider only three examples. They were selected because the codes should protect the most significant positions of each sample and because a small number of parity check positions per sample should be used so that we can reasonably consider 2^l quantization levels. The codes illustrate the general capabilities of significant-bit packed codes and are easy to implement. One prime is used in the average system error notation to indicate that the most significant position of each sample is protected and two primes to indicate that the two most significant positions of each sample are protected. Let ρ denote the number of parity check positions per sample where $\rho = (n-k)/\alpha$. Let R denote the code rate where $R = k/n$.

$\text{ASE}'_{(15,11)}$: A (15, 11) perfect single error-correcting code is used in a significant-bit packed code to protect the most significant position of each sample.

$\xi = 7$: $\alpha = 11$ $l = 7$ $\rho = 0.36$ $R = 0.950$

$\xi = 15$: $\alpha = 11$ $l = 15$ $\rho = 0.36$ $R = 0.976$

$\text{ASE}'_{(31,26)}$: A (31, 26) perfect single error-correcting code is used in a significant-bit packed code to protect the most significant position of each sample.

$\xi = 7$: $\alpha = 26$ $l = 7$ $\rho = 0.19$ $R = 0.974$

$\xi = 15$: $\alpha = 26$ $l = 15$ $\rho = 0.19$ $R = 0.987$

$\text{ASE}''_{(31,26)}$: A (31, 26) perfect single error-correcting code is used in a significant-bit packed code to protect the two most significant

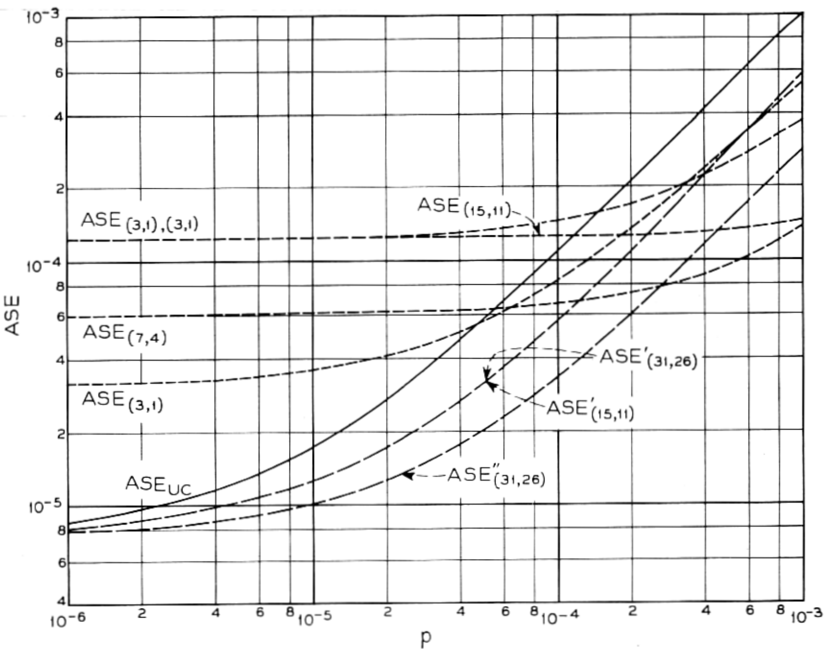

Fig. 8 — Average system error (ASE) with representative codes; 15 positions per sample ($\xi = 15$).

positions of each sample.

$$\xi = 7: \quad \alpha = 13 \quad l = 7 \quad \rho = 0.38 \quad R = 0.948$$

$$\xi = 15: \quad \alpha = 13 \quad l = 15 \quad \rho = 0.38 \quad R = 0.975$$

We can make the following observations concerning system performance when codes are used. In all cases, as $p \to 0$, ASE $\to 2^{-(l+2)}$ which is the limitation on system performance because of quantization. As l increases, the quantization error decreases. Thus, the value of p for which the transmission error becomes a significant portion of the system error decreases. In other words, if you design for good quantization resolution, then you need a good channel. This implies that, as the number of positions per sample increases, codes are useful for smaller values of p in order to bring the channel up to the required quality.

Because all $\alpha = 1$ codes necessitate a sizable reduction in l to allow for redundancy, they are only attractive for larger p where considerable coding capability is required. For these p, we have demonstrated that system performance can be improved (by an appreciable amount in some cases) by sacrificing quantization resolution for an improvement in transmission fidelity. However, because significant-bit packed codes provide protection for the most significant positions without the large penalty in quantization resolution required by the $\alpha = 1$ codes, significant-bit packed codes are effective for considerably smaller values of p than are the $\alpha = 1$ codes.

Notice that $\text{ASE}'_{(31,26)}$ and $\text{ASE}'_{(15,11)}$ are nearly equal. The reason is that although the significant-bit packed code using the (31, 26) code provides less error protection than the significant-bit packed code based on the (15, 11) code, in each case the protection provided for the most significant position is "sufficient" and, thus, the errors that hurt are coming in the less significant positions.

On the other hand, $\text{ASE}''_{(31,26)}$ is less than either $\text{ASE}'_{(31,26)}$ or $\text{ASE}'_{(15,11)}$ for the values of p where significant-bit packed codes are preferable. The reason is that errors are now nearly eliminated in the two most significant positions in each sample. Further reductions in system error could be achieved by using significant-bit packed codes which protect three or more positions per sample. However, we must be careful not to go too far or we should begin to charge the redundancy against quantization resolution.

Significant-bit packed codes achieve an effect similar to interleaving. Thus, although the computations herein have been for independent errors, significant-bit packed codes could prove useful for a channel with clustered errors.

VII. SIGNIFICANT-BIT PACKED CODES FOR DIFFERENT l

Several interesting points are illustrated in Fig. 9. Indexed on the left are the four values of l considered. For $l = 15$, ASE_{UC} is shown. For $l = 15$, 14, 13, and 12, $\text{ASE}'_{(31,26)}$ and $\text{ASE}''_{(31,26)}$ are given.

The following observations concerning Fig. 9 can be made. For small p, the $l = 15$ schemes are best. This is to be expected because quantization is the major source of system error for small p.

However, for larger p, the significant-bit packed codes with $l < 15$ have less system error than uncoded transmission for $l = 15$. This is particularly interesting because, in these significant-bit packed codes, more positions are saved by reducing l than are added by the parity check positions. For example, in the $l = 13$ system that results in $\text{ASE}'_{(31,26)}$, $\alpha = 26$ and $n = 343$. If uncoded transmission with $l = 15$ is

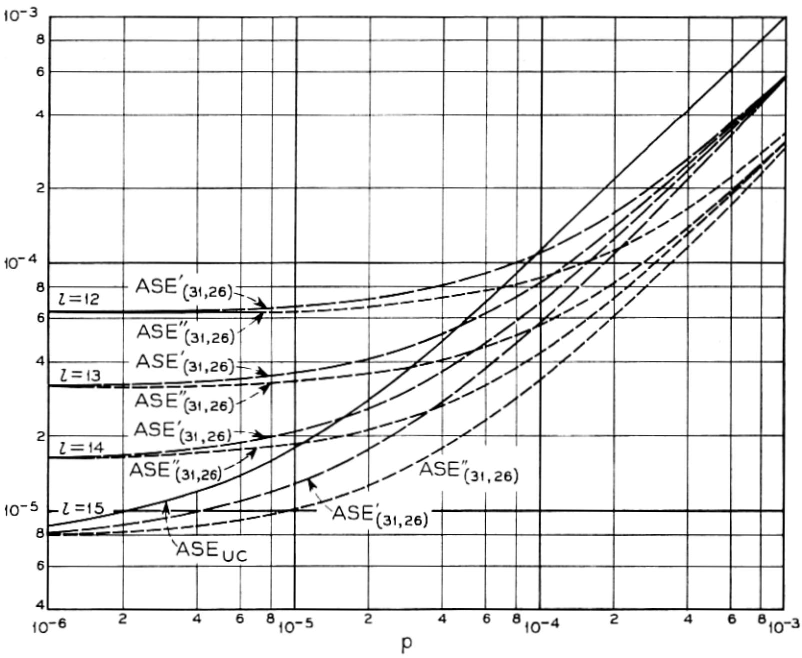

Fig. 9 — Average system error (ASE) with significant-bit packed codes.

used to send these 26 samples, 390 positions are required. Thus, for $p > 4.5 \cdot 10^{-5}$, this significant-bit packed code reduces system error and saves 47 positions every 26 samples. Similar behavior can be noted for other significant-bit packed codes considered in Fig. 9.

For $p = 10^{-3}$, the three systems with $\sigma = 1$ converge to approximately 2^{-1} ASE_{UC} and the three systems with $\sigma = 2$ converge to approximately 2^{-2} ASE_{UC} even though the systems use different quantization resolutions. However, for $p = 10^{-6}$, the convergence is determined by l. This clearly demonstrates the two extreme cases in system behavior: limitation by transmission error and limitation by quantization error.

VIII. THE SYNTHESIS PROBLEM—AN EXAMPLE

Suppose that the probability of error and the maximum allowable average system error are specified. Let these be denoted by p_s and ASE_s respectively. From equation (11), σ and l should be chosen to satisfy the relation

$$\text{ASE}_s > 2^{-\sigma} p_s + 2^{-(l+2)} \tag{12}$$

where σ represents the number of protected positions per sample. Because equation (11) is an approximation, values of l and σ that satisfy equation (12) cannot be guaranteed to provide a system with an $\text{ASE} \leqq \text{ASE}_s$. However, as σ decreases compared with l, equation (12) becomes increasingly reliable.*

Notice that l and σ appear as negative exponents in equation (12). Therefore, for a given p_s, a wide range of values for the ASE_s can be achieved by varying l and σ. Also, equation (12) frequently can be satisfied by several pairs of values for l and σ. For each pair, there may be several possible coding schemes. The system designer must then choose the final system configuration from these candidates on the basis of such items as the cost of implementation or the number of positions in the data stream per sample.

As an example of system design, consider a telemetry channel in planetary space missions. This channel can often be modeled satisfactorily by the memoryless binary symmetric channel and typically

* A major assumption leading to equation (11) is that all of the average numerical error comes from the unprotected positions. However, if σ is large, then errors in the protected positions result in a much larger numerical error than errors in the unprotected positions. Therefore, even though errors in the protected positions are less likely, a significant portion of the average numerical error can come from these positions.

has a bit error rate of $5 \cdot 10^{-3}$. Thus, equation (12) becomes

$$\text{ASE}_s > 5 \cdot 10^{-3} \cdot 2^{-\sigma} + 2^{-(l+2)}. \tag{13}$$

If uncoded transmission is a system requirement, then $\sigma = 0$ and

$$\text{ASE}_s > 5 \cdot 10^{-3} + 2^{-(l+2)}.$$

Notice that successive increases in l result in successively smaller reductions in the average system error and that the average system error can never be less than $5 \cdot 10^{-3}$. From Fig. 4, all systems with $l \geqq 8$ have essentially the same average system error and, thus, little is gained by using $l > 8$.

A more interesting situation exists if the system designer is permitted to choose l and the coding scheme. If $\text{ASE}_s > 5 \cdot 10^{-3}$, it is possible to design a system using uncoded transmission although coding could prove effective as ASE_s approaches $5 \cdot 10^{-3}$. However, if $\text{ASE}_s < 5 \cdot 10^{-3}$, some form of coding is mandatory. Conversely, from equation (13), if coding is used, the system error can be made small by choosing appropriate values of l and σ. In Table I, the approximate average system error is given for representative l and σ. The information in Table I was computed by using equation (11) and, thus, is subject to the assumptions and approximations leading to equation (11). However, from Table I, the improvements in system performance that can be achieved by coding are evident.

Table I—Approximate Average System Error (ASE) for Representative l and σ; $p = 5 \cdot 10^{-3}$

l	σ	Approximate ASE
7	1	$4.4 \cdot 10^{-3}$
	2	$3.2 \cdot 10^{-3}$
	3	$2.5 \cdot 10^{-3}$
8	1	$3.5 \cdot 10^{-3}$
	2	$2.2 \cdot 10^{-3}$
	3	$1.6 \cdot 10^{-3}$
9	1	$3.0 \cdot 10^{-3}$
	2	$1.7 \cdot 10^{-3}$
	3	$1.1 \cdot 10^{-3}$
10	1	$2.7 \cdot 10^{-3}$
	2	$1.5 \cdot 10^{-3}$
	3	$8.7 \cdot 10^{-4}$

Consider the following specific example which illustrates certain alternatives in code selection without requiring extensive computational effort. Suppose $\text{ASE}_{s} = 4 \cdot 10^{-3}$. From equation (13) or Table I, we can use $\sigma = 1$ and $l \geqq 8$ or $\sigma = 2$ and $l \geqq 7$. The minimum values of l will be used. Several coding schemes are possible in each case. The codes, indexed below by the notation used for their average system error in Fig. 10, follow the ideas in Section VI. Thus, the parity check matrices are not presented.

For $\sigma = 1, \quad l = 8$:

$\text{ASE}_{(3,1)}$: A (3, 1) perfect single error-correcting code is used to protect the most significant position.

$$\alpha = 1 \qquad l = 8$$

$\text{ASE}'_{(15,11)}$: A (15, 11) perfect single error-correcting code is used in a significant-bit packed code to protect the most significant position of each sample.

$$\alpha = 11 \qquad l = 8$$

$\text{ASE}'_{(31,26)}$: A (31, 26) perfect single error-correcting code is used in a significant-bit packed code to protect the most significant position of each sample.

$$\alpha = 26 \qquad l = 8$$

For $\sigma = 2, \quad l = 7$:

$\text{ASE}_{(3,1),(3,1)}$: Independent (3, 1) perfect single error-correcting codes are used to protect the two most significant positions.

$$\alpha = 1 \qquad l = 7$$

$\text{ASE}''_{(31,26)}$: A (31, 26) perfect single error-correcting code is used in a significant-bit packed code to protect the two most significant positions of each sample.

$$\alpha = 13 \qquad l = 7$$

The design objective, denoted by an asterisk in Fig. 10, is satisfied by each system although the systems vary somewhat in performance for other p. Notice that the systems differ in the coding equipment and quantization resolution required for implementation. Also, notice that the systems vary in the number of positions per sample in the data stream [from a low of 7.4 for $\text{ASE}''_{(31,26)}$ to a high of 11 for $\text{ASE}_{(3,1),(3,1)}$]. Which system would actually be selected would thus depend upon the details of the specific application.

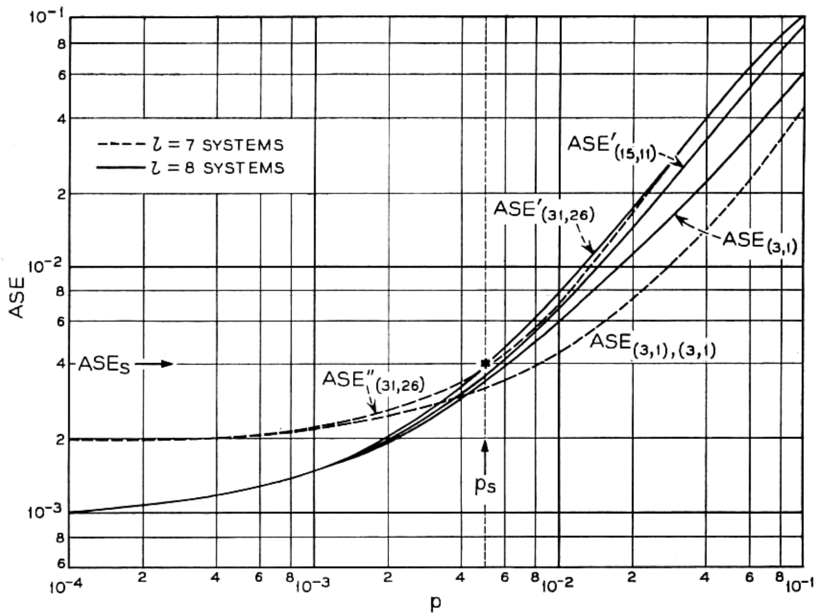

Fig. 10 — Systems for space telemetry channel.

IX. CONCLUSIONS

A general formulation of the error introduced by quantization and transmission has been developed for the data transmission system shown in Fig. 1. It has been shown that system performance is influenced by both the quantization resolution and the channel error characteristics, that certain levels of performance cannot be achieved without the use of coding no matter how fine the quantization, and that performance can, in some cases, be improved by sacrificing quantization for redundancy and error control. In general, when coding is used, it is beneficial to use codes that match their protection to the numerical significance of the information positions. Significant-bit packed codes are particularly useful because they provide protection for the most significant positions without incurring a large penalty in quantization resolution. The problem of determining the coding capability and the number of quantization levels required to achieve a specified average system error has been considered.

The specific results are based upon the choice of $\gamma = 1$ in Section IV. However, varying γ simply changes the "cost" assigned to the numerical errors and, thus, the general ideas presented here are applicable for any $\gamma > 0$: for example, the desirability of the system approach to quantization and transmission error, the possibility of improving system performance by sacrificing quantization resolution for redundancy, and the use of codes that concentrate protection on the numerically most significant positions. Actually, it appears that as γ increases, the desirability of protection for the most significant positions also increases.

Because of the unit distance properties of Gray codes, it is natural to inquire whether Gray codes could prove useful in the system discussed in this paper. It can be shown (for $\gamma = 1$) that a Gray code with 2^l levels gives exactly the same average numerical error and average system error as the natural binary numbering with 2^l levels even when error-correcting codes are used.

APPENDIX A

Derivation of an Expression for $Pr_m\{r_m \mid s_m\}$

Let $\Pr_m \{r_m \mid s_m\}$ denote the probability of receiving r_m when s_m is transmitted using a packed code. Let $\Pr \{s_i\}$ $(1 \leqq i \leqq \alpha)$ denote the probability that s_i is transmitted. Then

$$\Pr_m \{r_m \mid s_m\} = \underbrace{\sum_{r_\alpha=0}^{2^l-1} \cdots \sum_{r_1=0}^{2^l-1} \sum_{s_\alpha=0}^{2^l-1} \cdots \sum_{s_1=0}^{2^l-1}}_{\text{excluding } r_m \text{ and } s_m} \Pr \{r \mid s\} \underbrace{\Pr \{s_\alpha\} \cdots \Pr \{s_1\}}_{\text{excluding } \Pr \{s_m\}}$$

where the values of r and s are determined from equations (4) and (2), respectively. However, $\Pr \{s_i\} = 2^{-l}$ for $1 \leqq i \leqq \alpha$, $i \neq m$. Thus,

$$\Pr_m \{r_m \mid s_m\} = \frac{1}{2^{(\alpha-1)l}} \underbrace{\sum_{r_\alpha=0}^{2^l-1} \cdots \sum_{r_1=0}^{2^l-1} \sum_{s_\alpha=0}^{2^l-1} \cdots \sum_{s_1=0}^{2^l-1}}_{\text{excluding } r_m \text{ and } s_m} \Pr \{r \mid s\}. \tag{14}$$

The expression in equation (14) can be simplified. From equations (2) and (4), equation (14) can be written as

$$\Pr_m \{r_m \mid s_m\} = \frac{1}{2^{(\alpha-1)l}} \underbrace{\sum_{s_\alpha=0}^{2^l-1} \cdots \sum_{s_1=0}^{2^l-1} \sum_{r_\alpha=0}^{2^l-1} \cdots \sum_{r_1=0}^{2^l-1}}_{\text{excluding } r_m \text{ and } s_m}$$
$$\cdot \Pr \left\{ \sum_{m'=1}^{\alpha} 2^{(m'-1)l} r_{m'} \;\middle|\; \sum_{m'=1}^{\alpha} 2^{(m'-1)l} s_{m'} \right\}. \tag{15}$$

By Lemma 1 of Ref. 2, for a binary group code used with a binary symmetric channel in which the errors are independent of the symbols actually transmitted,

$$\Pr\left\{\sum_{m'=1}^{\alpha} 2^{(m'-1)l} t_{m'} \,\middle|\, 0\right\} = \Pr\left\{\sum_{m'=1}^{\alpha} 2^{(m'-1)l} r_{m'} \,\middle|\, \sum_{m'=1}^{\alpha} 2^{(m'-1)l} s_{m'}\right\}$$

where $B_l(t_{m'}) = B_l(r_{m'}) \oplus B_l(s_{m'})$. By Lemma 2 of Ref. 2, equation (15) can be written as

$$\Pr_m \{r_m \mid s_m\} = \frac{1}{2^{(\alpha-1)l}} \underbrace{\sum_{s_\alpha=0}^{2^l-1} \cdots \sum_{s_1=0}^{2^l-1} \sum_{t_\alpha=0}^{2^l-1} \cdots \sum_{t_1=0}^{2^l-1}}_{\text{excluding } s_m \text{ and } t_m} \Pr\left\{\sum_{m'=1}^{\alpha} 2^{(m'-1)l} t_{m'} \,\middle|\, 0\right\}$$

which reduces to

$$\Pr_m \{r_m \mid s_m\} = \underbrace{\sum_{t_\alpha=0}^{2^l-1} \cdots \sum_{t_1=0}^{2^l-1}}_{\text{excluding } t_m} \Pr\left\{\sum_{m'=1}^{\alpha} 2^{(m'-1)l} t_{m'} \,\middle|\, 0\right\}.$$

APPENDIX B

Reduction of the Expression for the Average System Error

By substituting equation (3) into (5) and rewriting,

$$\mathrm{ASE}_m = \frac{1}{(X_2 - X_1)} \sum_{r_m=0}^{2^l-1} \sum_{s_m=0}^{2^l-1} \Pr_m \{r_m \mid s_m\} \cdot \int_{X_1 + s_m (X_2 - X_1)/2^l}^{X_1 + (s_m+1)(X_2-X_1)/2^l} \left| x_m - X_1 - (r_m + \tfrac{1}{2})\left(\frac{X_2 - X_1}{2^l}\right) \right| dx_m .$$

However,

$$\int_{X_1 + s_m (X_2 - X_1)/2^l}^{X_1 + (s_m+1)(X_2-X_1)/2^l} \left| x_m - X_1 - (r_m + \tfrac{1}{2})\left(\frac{X_2 - X_1}{2^l}\right) \right| dx_m = \left(\frac{X_2 - X_1}{2^l}\right)^2 (| r_m - s_m | + \tfrac{1}{4}\, \delta_{r_m s_m})$$

where

$$\begin{aligned} \delta_{r_m s_m} &= 1 \quad \text{for} \quad r_m = s_m \\ &= 0 \quad \text{for} \quad r_m \neq s_m . \end{aligned}$$

Thus,

$$\mathrm{ASE}_m = \frac{X_2 - X_1}{2^{2l}} \sum_{r_m=0}^{2^l-1} \sum_{s_m=0}^{2^l-1} | r_m - s_m | \Pr_m \{r_m \mid s_m\} + \frac{X_2 - X_1}{4 \cdot 2^{2l}} \sum_{r_m=0}^{2^l-1} \sum_{s_m=0}^{2^l-1} \Pr_m \{r_m \mid s_m\}\, \delta_{r_m s_m} .$$

The average numerical error for the mth sample was defined in equation (6) as

$$\text{ANE}_m = \frac{1}{2^l} \sum_{r_m=0}^{2^l-1} \sum_{s_m=0}^{2^l-1} | r_m - s_m | \Pr_m \{r_m \mid s_m\}.$$

In addition, it can be shown that for a channel in which the errors are independent of the symbols actually transmitted,

$$\sum_{r_m=0}^{2^l-1} \sum_{s_m=0}^{2^l-1} \Pr_m \{r_m \mid s_m\} \; \delta_{r_m s_m} = 2^l \Pr_m \{0 \mid 0\}.$$

Therefore,

$$\text{ASE}_m = \left(\frac{X_2 - X_1}{2^l}\right)(\text{ANE}_m + \tfrac{1}{4} \Pr_m \{0 \mid 0\}).$$

APPENDIX C

Theorem 2 of Reference 2

A significant-bit code is a code in which the $(k-k_0)$ most significant positions are protected by what is referred to as a base code and the remaining k_0 positions are transmitted unprotected. For the base code when used alone, let $\Pr_B\{0 \mid 0\}$ denote the probability that the output of the decoder is the zero message when the input to the encoder is the zero message. Also, let ANE_B denote the average numerical error of the base code. The average numerical error for the significant-bit code is given by Theorem 2 of Ref. 2:

Theorem 2: Let the base code be defined as above. For a binary symmetric channel with independent errors and when all messages are equally likely to be transmitted,

$$\text{ANE}_{\text{SB}} = \Pr_B \{0 \mid 0\} p \sum_{i=1}^{k_0} 2^{i-1} q^{k_0 - i} + 2^{k_0} \text{ANE}_\text{B} \,.$$

Uncoded transmission is the special case where $k = k_0$. Thus, the average numerical error for uncoded transmission can be obtained by letting $\text{ANE}_\text{B} = 0$ and $\Pr_B\{0 \mid 0\} = 1$ when $k = k_0$.

APPENDIX D

Proof that the Average System Error for Uncoded Transmission Increases Monotonically with p

In Section V, equation (9) gives the average system error for uncoded transmission as

$$\text{ASE}_{\text{UC}} = \frac{1}{2^l}\left(p \sum_{i=1}^{l} 2^{i-1}q^{l-i} + \frac{q^l}{4}\right).$$

After differentiating with respect to q and grouping terms,

$$\frac{d\text{ASE}_{\text{UC}}}{dq} = \frac{1}{2^l}\left[-\left(l - \frac{l}{4}\right)q^{l-1} - \sum_{i=1}^{l-1} (l - i)(2^i - 2^{i-1})q^{l-i-1}\right].$$

For $\frac{1}{2} < q < 1$,

$$\frac{d\text{ASE}_{\text{UC}}}{dq} < 0.$$

Thus, ASE_{UC} decreases monotonically as q goes from ½ to 1 or, alternatively, ASE_{UC} increases monotonically as p runs from 0 to ½.

APPENDIX E

Parity Check Matrices for Codes Considered in Section VI

$\text{ASE}_{(3,1)}$: A (3, 1) perfect single error-correcting code to protect the most significant position.

$\xi = 7$:

$$H = \begin{bmatrix} 1\ 0\ 0\ 0\ 0 & I_2 \\ 1\ 0\ 0\ 0\ 0 & \end{bmatrix} \qquad \alpha = 1 \qquad l = 5$$

$\xi = 15$:

$$H = \begin{bmatrix} 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & I_2 \\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & \end{bmatrix} \qquad \alpha = 1 \qquad l = 13$$

$\text{ASE}_{(3,1),(3,1)}$: Independent (3, 1) perfect single error-correcting codes to protect the two most significant positions.

$\xi = 7$:

$$H = \begin{bmatrix} 1\ 0\ 0 & \\ 1\ 0\ 0 & I_4 \\ 0\ 1\ 0 & \\ 0\ 1\ 0 & \end{bmatrix} \qquad \alpha = 1 \qquad l = 3$$

$\xi = 15$:

$$H = \begin{bmatrix} 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & \\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & I_4 \\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & \\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & \end{bmatrix} \qquad \alpha = 1 \qquad l = 11$$

ASE$_{(7,4)}$: A (7, 4) perfect single error-correcting code to protect the four most significant positions.

$\xi = 7$:

$$H = \begin{bmatrix} 1\ 1\ 1\ 0 & \\ 1\ 1\ 0\ 1 & I_3 \\ 1\ 0\ 1\ 1 & \end{bmatrix} \qquad \alpha = 1 \qquad l = 4$$

$\xi = 15$:

$$H = \begin{bmatrix} 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & \\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & I_3 \\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 & \end{bmatrix} \qquad \alpha = 1 \qquad l = 12$$

ASE$_{(15,11)}$: A (15, 11) perfect single error-correcting code.
$\xi = 15$:

$$H = \begin{bmatrix} 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 & \\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0 & I_4 \\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1 & \\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1 & \end{bmatrix} \qquad \alpha = 1 \qquad l = 11$$

ASE$'_{(15,11)}$: *A* (15, 11) perfect single error-correcting code in a significant-bit packed code.

$\xi = 7$:

$$H = \begin{bmatrix} 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 0 & & 0 & & 0 & & 0 & & \\ 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 0 & 0_6 & 0 & 0_6 & 0 & 0_6 & 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 0 & 0_6 & I_4 \\ 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & \\ 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & & \end{bmatrix}$$

$$\alpha = 11 \qquad l = 7 \qquad \rho = 0.36 \qquad R = 0.950$$

$\xi = 15$:

$$H = \left[\begin{array}{ccccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 0 & & 0 & & 0 & & 0 & & \\
1 & 0_{14} & 1 & 0_{14} & 1 & 0_{14} & 1 & 0_{14} & 0 & 0_{14} & 0 & 0_{14} & 0 & 0_{14} & 1 & 0_{14} & 1 & 0_{14} & 1 & 0_{14} & 0 & 0_{14} & I_4 \\
1 & & 1 & & 0 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & \\
1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & &
\end{array}\right]$$

$$\alpha = 11 \qquad l = 15 \qquad \rho = 0.36 \qquad R = 0.976$$

$\mathrm{ASE}'_{(31,26)}$: A (31, 26) perfect single error-correcting code in a significant-bit packed code.

$\xi = 7$:

$$H = \left[\begin{array}{cccccccccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 0 & 0_6 & 0 & 0_6 & 0 & 0_6 & 0 & 0_6 & 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 0 & 0_6 & 0 & 0_6 \\
1 & & 1 & & 0 & & 0 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & \\
1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 &
\end{array}\right.$$

$$\left.\begin{array}{ccccccccccccccccccccccccc}
1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
0 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 0 & & 0 & & 0 & & 0 & & \\
0 & 0_6 & 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 0 & 0_6 & 0 & 0_6 & 0 & 0_6 & 1 & 0_6 & 1 & 0_6 & 1 & 0_6 & 0 & 0_6 & I_5 \\
0 & & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & \\
1 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 1 & & 0 & & 1 & & 1 & &
\end{array}\right]$$

$$\alpha = 26 \qquad l = 7 \qquad \rho = 0.19 \qquad R = 0.973$$

$\xi = 15$:

$$H = \left[\begin{array}{cccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
1 & 0_{14} & 1 & 0_{14} & 1 & 0_{14} & 1 & 0_{14} & 0 & 0_{14} & 0 & 0_{14} & 0 & 0_{14} \\
1 & & 1 & & 0 & & 0 & & 1 & & 1 & & 0 & \\
1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 &
\end{array}\right.$$

$$
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0\;0_{14} & 1\;0_{14} & 1\;0_{14} & 1\;0_{14} & 1\;0_{14} & 0\;0_{14} & 0\;0_{14} & 0\;0_{14} & 1\;0_{14} & 1\;0_{14} \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
$$

$$
\left.
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \\
1\;0_{14} & 1\;0_{14} & 0\;0_{14} & 0\;0_{14} & 0\;0_{14} & 1\;0_{14} & 1\;0_{14} & 1\;0_{14} & 0\;0_{14} & I_5 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 &
\end{array}
\right]
$$

$$\alpha = 26 \qquad l = 15 \qquad \rho = 0.19 \qquad R = 0.987$$

$\text{ASE}''_{(31,26)}$: A (31, 26) perfect single error-correcting code in a significant-bit packed code.

$\xi = 7$:

$$
H = \left[
\begin{array}{cccccc}
11 & 11 & 11 & 11 & 11 & 11 \\
11 & 11 & 11 & 11 & 00 & 00 \\
11\;0_5 & 11\;0_5 & 00\;0_5 & 00\;0_5 & 11\;0_5 & 11\;0_5 \\
11 & 00 & 11 & 00 & 11 & 00 \\
10 & 10 & 10 & 10 & 10 & 10
\end{array}
\right.
$$

$$
\left.
\begin{array}{cccccccc}
11 & 10 & 00 & 00 & 00 & 00 & 00 & \\
00 & 01 & 11 & 11 & 11 & 00 & 00 & \\
00\;0_5 & 01\;0_5 & 11\;0_5 & 10\;0_5 & 00\;0_5 & 11\;0_5 & 10\;0_5 & I_5 \\
11 & 01 & 10 & 01 & 10 & 11 & 01 & \\
10 & 11 & 01 & 01 & 01 & 10 & 11 &
\end{array}
\right]
$$

$$\alpha = 13 \qquad l = 7 \qquad \rho = 0.38 \qquad R = 0.948$$

$\xi = 15$:

$$H = \left[\begin{array}{cccccccccccc} 11 & & 11 & & 11 & & 11 & & 11 & & 11 & \\ 11 & & 11 & & 11 & & 11 & & 00 & & 00 & \\ 11 & 0_{13} & 11 & 0_{13} & 00 & 0_{13} & 00 & 0_{13} & 11 & 0_{13} & 11 & 0_{13} \\ 11 & & 00 & & 11 & & 00 & & 11 & & 00 & \\ 10 & & 10 & & 10 & & 10 & & 10 & & 10 & \end{array}\right.$$

$$\left.\begin{array}{ccccccccccccccc} 11 & & 10 & & 00 & & 00 & & 00 & & 00 & & 00 & & \\ 00 & & 01 & & 11 & & 11 & & 11 & & 00 & & 00 & & \\ 00 & 0_{13} & 01 & 0_{13} & 11 & 0_{13} & 10 & 0_{13} & 00 & 0_{13} & 11 & 0_{13} & 10 & 0_{13} & I_5 \\ 11 & & 01 & & 10 & & 01 & & 10 & & 11 & & 01 & & \\ 10 & & 11 & & 01 & & 01 & & 01 & & 10 & & 11 & & \end{array}\right]$$

$$\alpha = 13 \qquad l = 15 \qquad \rho = 0.38 \qquad R = 0.975$$

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