Delta Modulation Granular Quantizing Noise

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We present a statistical analysis of a single integration delta modulation system in which slope overload effects are negligible. In defining the delta modulation signal ensemble, we identify a binary phase parameter and show that when this parameter is random, the signal statistics are stationary, provided the input is stationary. Thus the delta modulation correlation functions depend on a single time variable and have Fourier transforms that are the power spectra of the delta modulation signals.

After deriving the delta modulation correlation statistics and power density spectra, we use these functions to investigate the properties of the delta modulation granular quantizing noise. We demonstrate the ratio of input signal power to the quantizing noise power of three signals that approximate the system input. These signals are the integrated delta modulation signal, the signal at the output of the ideal low-pass interpolation filter usually considered in delta modulation studies, and the signal at the output of the optimum interpolation filter. We determine the properties of this filter by referring to the derived spectral density functions.

I. BACKGROUND

Delta modulation (ΔM) systems are subject to two types of quantizing distortion, generally referred to as granular quantizing noise and slope overload noise. The overload noise arises when the analog input to the delta modulator changes at a rate greater than the maximum average slope of the signal generated in the delta modulator feedback loop. The granular noise is analogous to pulse code modulation (PCM) quantizing noise; it arises because the ΔM signal is a discrete-time discrete-amplitude representation of a continuous-amplitude process.

After the discovery of ΔM in the early 1950's, two statistical analyses of distortion effects appeared.[1] Van de Weg considered a delta modulator, constrained so that slope overload effects are negligible, and analyzed the effects of granular quantizing noise in a manner that paralleled Bennett's analysis of quantizing noise effects in a pulse code modulation (PCM) system constrained to be free of overload.[2,3] Zetterberg, in 1955, published a study of both types of distortion as part of an extensive mathematical analysis of the ΔM process.[4] Zetterberg's expression for granular noise power is less precise than van de Weg's. His results pertaining to slope overload have recently been revised.[5]

Eleven years after the appearance of Zetterberg's paper an independent analysis of slope overload noise was published by O'Neal whose effort was supported by S. O. Rice.[6] O'Neal used van de Weg's formula to predict the granular noise power but obtained slope overload characteristics that differed from those derived by Zetterberg. The reason for the two solutions to the same problem is investigated in a recent paper by Protonotarios.[5] This paper gives new expressions for the slope overload noise that are more accurate than any previously obtained. Like O'Neal, Protonotarios uses van de Weg's characterization of the granular quantization effects.

Although van de Weg's formula for granular quantizing noise power has been experimentally verified over an important range of operating conditions, his statistical characterization is inadequate for certain analytical purposes. A principal difficulty in this characterization is the nonstationarity of the ΔM signal ensemble. Because the statistics are nonstationary it is not possible to calculate correlation coefficients by Fourier transformation of the power spectral density function, derived by van de Weg as a mean square amplitude spectrum.

To admit the techniques of stationary time series analysis to the study of ΔM signals, we generalize the signal ensemble by defining a binary phase parameter. We derive correlation statistics directly as average products and show that if the phase is random with both values equiprobable, the ensemble is stationary. Thus we are able to compute power density spectra as Fourier transforms of the correlation functions and to compare the new formula for granular quantizing noise with that given by van de Weg. We find that over the range of operating speeds considered by van de Weg and O'Neal that van de Weg's formula is a good approximation to the one presented here. For very low speeds van de Weg's approximations break down while the formulas we present in this paper are applicable to all ΔM sampling rates.

An additional advantage of this analysis is the presentation of cross-correlation statistics and the cross-power spectrum of the ΔM signal and the analog waveform it represents. We use the cross-power spectrum to derive the transfer function of the optimal interpolation filter for ΔM. We compare the output noise power of this filter with that of the ideal low-pass filter usually considered in ΔM studies. The correlation statistics presented here have also been used in the synthesis of optimal digital filters.[7]

II. THE ΔM SYSTEM

The delta modulator shown in Fig. 1 transforms the continuous signal $y(t)$ to the binary sequence

$$\cdots, b_{-1}, b_0, b_1, \cdots$$

in which b_n may have the value $+1$ or -1. The modulator generates binary symbols at τ second intervals according to the sign of $e(t)$, the error signal. This error is the difference of $y(t)$ and $x(t)$, the integrated ΔM signal generated in the modulator feedback loop. The term $x(t)$ is the integral of the binary impulses weighted by the "step size," δ. Thus $x(t)$ has a step of $+\delta$ or $-\delta$ at each sampling instant and is otherwise constant. At the ΔM receiver, this integrated ΔM signal is recovered by a replica of the modulator feedback loop and an analog signal, $\hat{y}(t)$, is generated by means of the interpolating filter with impulse response $h(\cdot)$. The signal $\hat{y}(t)$ is an approximation to the system input, and in this paper the fidelity of the ΔM system will be measured by the mean square error,

$$\eta = E\{[y(t) - \hat{y}(t)]^2\}, \tag{1}$$

in which $E\{\cdot\}$ is the expectation operator. We assume that the binary signal processed by the receiver is identical to the one generated at the modulator. The effects of transmission errors are not considered.

The two ΔM parameters are τ, the sampling interval, and δ, the step size. The quantizing distortion decreases monotonically with increasing

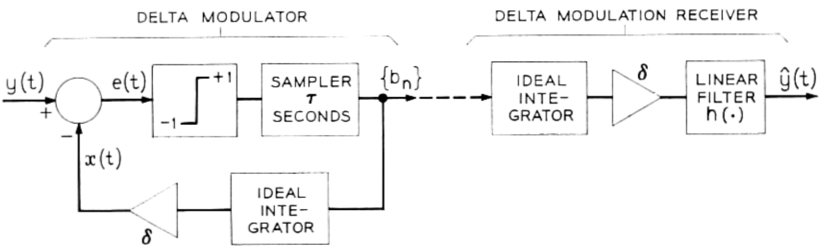

Fig. 1 — The delta modulation system.

sampling rate, $f_s = 1/\tau$, while for a fixed rate the value of the step size determines the mix of granular quantizing noise and slope overload noise in the quantizing noise signal, $y(t) - \hat{y}(t)$. In this paper we consider only the granular quantizing noise; thus we postulate a system in which δ is set such that δ/τ, the maximum average slope of $x(t)$, is exceeded by the slope of $y(t)$ with very low probability. To serve this aim we follow van de Weg and establish the condition that δ/τ is four times the root mean square slope of $y(t)$. This condition is analogous to the "4σ loading" assumed by Bennett in his analysis of a PCM system with negligible overload effects.[3] For gaussian signals, the probability that the slope of $y(t)$ is greater than δ/τ is less than 4×10^{-5}.

If $y(t)$ is a sample function of a stationary stochastic process, the stated design condition may be expressed in terms of $S_{yy}(f)$, the power spectral density of the process. The important parameters of $S_{yy}(f)$ are its average,

$$\sigma^2 = \int_{-\infty}^{\infty} S_{yy}(f)\, df = E\{[y(t)]^2\}, \tag{2}$$

the mean square signal, and its effective bandwidth,[8]

$$f_e = \left[\frac{\int_{-\infty}^{\infty} f^2 S_{yy}(f)\, df}{\int_{-\infty}^{\infty} S_{yy}(f)\, df} \right]^{\frac{1}{2}}. \tag{3}$$

The rms slope of $y(t)$ is $2\pi\sigma f_e$. Thus the condition that the maximum average slope of $x(t)$ equal four times the rms slope of $y(t)$ may be expressed as

$$\delta/\tau = 8\pi\sigma f_e$$

or

$$\beta = \delta/\sigma = 8\pi f_e \tau = 8\pi/F \tag{4}$$

in which we have related the ΔM parameters to the important signal parameters. Thus, β is the step size as a multiple of the rms signal and $F = f_s/f_e$ is the sampling rate as a multiple of the effective bandwidth.

Equation (4) establishes β for each sampling rate; in the analysis of granular quantizing noise to be presented, it is the sampling rate that is considered to be the independent variable of the ΔM system. Studies of slope overload indicate that for minimal total quantizing noise, βF, instead of remaining constant as it does here, should increase with increasing F.[5,6] In the numerical examples given by O'Neal and by Protonotarios, the value of βF that results in minimal total quantizing noise approximates 8π for the highest sampling rate considered.

III. THE SCOPE OF THE ANALYSIS

The signals processed in the ΔM system have been analyzed as realizations of discrete-time (sampled-data) random processes. The transmitted binary sequence, $\{b_n\}$, the integrated ΔM signal, $x(t)$, and the analog output, $\hat{y}(t)$, are all determined by the values of the analog input at the sampling instants, $n\tau$ ($n = \cdots, -1, 0, 1, \cdots$). Thus the analysis reported here consists of derivations of the statistical properties of $\{x_n\} = \{x(n\tau)\}$, the integrated ΔM sequence and $\{e_n\} = \{e(n\tau)\}$, the error sequence, from the statistics of $\{y_n\} = \{y(n\tau)\}$, the input signal sequence.

If $y(t)$ is drawn from a stationary process with auto-covariance function $\sigma^2\rho(\cdot)$ [the Fourier transform of $S_{yy}(f)$], the covariance coefficients of the ΔM signals may be expressed as functions of the statistics, $\rho_n = \rho(n\tau)$. The derived covariance functions are $E\{x_ix_j\}$, the autocovariance of the integrated ΔM signal, and $E\{y_ix_j\}$, the cross-covariance of this signal and the analog input. A property of the definition (in Section 5.1) of the ensemble of sequences $\{x_n\}$ is its stationarity in the wide sense. (Van de Weg considers a somewhat different ensemble, one that has nonstationary statistics.) Thus the covariances are functions of the single time variable, $\mu = j - i$, and we denote them r_μ (the autocovariance) and ϕ_μ (the cross-covariance) respectively. Also of interest is Q_μ, the error covariance function given by

$$Q_\mu = E\{e_ne_{n+\mu}\} = \sigma^2\rho_\mu + r_\mu - \phi_\mu - \phi_{-\mu}.$$

It is shown in Section 5.4 that the covariance statistics, ϕ_μ, are proportional to $\sigma^2\rho_\mu$, the autocovariance of the continuous input. Thus $\phi_\mu = \phi_{-\mu}$ and the error covariance function is given by

$$Q_\mu = \sigma^2\rho_\mu + r_\mu - 2\phi_\mu. \tag{5}$$

Because the processes under consideration are stationary, their power density spectra are Fourier cosine series with coefficients given by the covariance statistics defined above. The spectra are periodic in frequency over intervals of $1/\tau$ Hz; they are denoted with asterisks in keeping with conventions of sampled data analysis. We apply the

Fourier series representation:

$$A^*(f) = a_0 + 2 \sum_{n=1}^{\infty} a_n \cos 2\pi n f \tau$$

so that

$$a_n = 2\tau \int_0^{f_s/2} A^*(f) \cos 2\pi n f \tau \, df. \tag{6}$$

In the sequel we will denote these Fourier transform relationships between $A^*(f)$ and a_n by $A^*(f) \longleftrightarrow a_n$.

The power density spectrum, $S^*_{yy}(f)$, of the samples of the analog input is related to $S_{yy}(f)$, the power spectrum of the continuous input signal, by

$$\sigma^2 \rho_\mu \leftrightarrow S^*_{yy}(f) = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} S_{yy}(f + nf_s). \tag{7}$$

It follows that if $y(t)$ is bandlimited to $W < f_s/2$ Hz, there is no aliasing distortion and

$$S^*_{yy}(f) = \frac{1}{\tau} S_{yy}(f), \quad \text{for} \quad | f | < f_s/2. \tag{8}$$

The other transform pairs of interest are $S^*_{xx}(f) \leftrightarrow r_\mu$, $S^*_{ee}(f) \leftrightarrow Q_\mu$, and $S^*_{xy}(f) \leftrightarrow \phi_\mu$. $S^*_{xx}(f)$ and $S^*_{ee}(f)$ are the power spectral density functions of the integrated ΔM signal and the error signal, respectively. $S^*_{xy}(f)$ is the cross-power spectrum of the integrated ΔM signal and the analog input. Equation (5) implies that the four power density spectra are related by

$$S^*_{ee}(f) = S^*_{xx}(f) + S^*_{yy}(f) - 2S^*_{xy}(f). \tag{9}$$

These spectral density functions and $H(f)$, the transfer function of the interpolating filter, determine the value of the output quantizing noise power defined in equation (1).† Thus,

$$\eta = 2\tau \int_0^{f_s/2} \{S^*_{yy}(f) - 2\text{Re}\,[H(f)S^*_{xy}(f)] + | H(f) |^2 S^*_{xx}(f)\} \, df \tag{10}$$

so that the transfer function of the optimal interpolation filter, that is, that which minimizes η, is the (nonrealizable) Wiener filter,[9,10]

† It is assumed here that $H(f)$ processes a sequence of ideal impulses. In Fig. 1 the filter input is a sequence of flat pulses of τ second duration so that when a filter described in this analysis is to be included in a real system, its transfer function should be weighted to compensate for the aperture effect.[3]

$$H_{\rm opt}(f) = \frac{S^*_{xy}(f)}{S^*_{xx}(f)}, \quad \text{for} \quad |f| \leqq f_s/2$$
$$= 0, \qquad \text{for} \quad |f| > f_s/2. \tag{11}$$

The associated minimal quantizing noise power is

$$\eta_{\min} = 2\tau \int_0^{f_s/2} \left\{ S^*_{yy}(f) - \frac{[S^*_{xy}(f)]^2}{S^*_{xx}(f)} \right\} df. \tag{12}$$

In previous ΔM studies it was assumed that $y(t)$ is bandlimited to W Hz and that the interpolation is performed by a perfect low pass filter with transfer function

$$H_{lpf}(f) = 1, \quad \text{for} \quad |f| \leqq W$$
$$= 0, \quad \text{for} \quad |f| > W.$$

Equation (10) indicates that the quantizing noise power associated with this filter is

$$\eta_{lpf} = 2\tau \int_0^W [S^*_{yy}(f) + S^*_{xx}(f) - 2S^*_{xy}(f)]\, df$$
$$= 2\tau \int_0^W S^*_{ee}(f)\, df. \tag{13}$$

Thus the quantizing noise power associated with the low-pass filter is the portion of the power of the error signal that lies within the band of the analog input. By substituting the Fourier series with coefficients Q_μ into equation (13) we arrive at the formula for the low pass filter quantizing noise in terms of the error covariance coefficients:

$$\eta_{lpf} = \frac{1}{R}\left[Q_0 + 2 \sum_{n=1}^{\infty} Q_n \frac{\sin\left(\frac{\pi n}{R}\right)}{\left(\frac{\pi n}{R}\right)} \right] \tag{14}$$

in which $R = f_s/2W$ is the bandwidth expansion ratio of the ΔM system. It is the ratio of the ΔM sampling rate to the Nyquist sampling rate of the input signal. The ratio, F/R, of the two normalized sampling rates is $2W/f_e$, twice the ratio of the highest frequency spectral component of $y(t)$ to the effective bandwidth.

IV. PRINCIPAL RESULTS

4.1 *Covariance Coefficients*

By means of the formulas of the preceding sections, the characteristics of granular quantizing noise may be expressed in terms of the correlation statistics ρ_μ, r_μ, and ϕ_μ. These quantities depend on the nature of the analog input and on the normalized sampling rate, F. Details of the derivations of r_μ and ϕ_μ, when the input is drawn from a stationary gaussian process, are given in the subsequent sections of this paper. Here we present the covariance formulas and use them to investigate the quantizing noise properties.

As multiples of the mean square input, the autocovariance coefficients of the integrated ΔM signal are

$$\frac{r_0}{\sigma^2} = 1 + 4 \sum_{k=1}^{\infty} \exp\left[-\frac{F^2k^2}{32}\right] + \frac{64\pi^2}{F^2}\left\{\frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2} \exp\left[-\frac{F^2k^2}{32}\right]\right\}$$

$$\frac{r_\mu}{\sigma^2} = \rho_\mu\left\{1 + 4 \sum_{k=1}^{\infty} \exp\left[-\frac{F^2k^2}{32}\right]\right\} + \frac{64}{F^2} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{mk} [1 + (-1)^{m+k}]$$
$$\cdot\left\{\exp\left[-\frac{F^2(k^2 + m^2 - 2mk\rho_\mu)}{128}\right] - \exp\left[-\frac{F^2(k^2 + m^2 + 2mk\rho_\mu)}{128}\right]\right\}$$
$$\text{for } \mu \text{ even,}$$

$$\frac{r_\mu}{\sigma^2} = \rho_\mu\left\{1 + 4 \sum_{k=1}^{\infty} \exp\left[-\frac{F^2k^2}{32}\right]\right\} + \frac{128}{F^2} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{mk} (-1)^m$$
$$\cdot\left\{\exp\left[-\frac{F^2(k^2 + m^2 - 2mk\rho_\mu)}{128}\right] - \exp\left[-\frac{F^2(k^2 + m^2 + 2mk\rho_\mu)}{128}\right]\right\}$$
$$\text{for } \mu \text{ odd.} \qquad (15)$$

The cross-covariance function of $\{x_n\}$ and $\{y_n\}$ is proportional to $\sigma^2\rho_\mu$, the autocovariance function of $\{y_n\}$. Thus

$$\frac{\phi_\mu}{\sigma^2} = c\rho_\mu \qquad (16)$$

where

$$c = 1 + 2 \sum_{k=1}^{\infty} \exp\left[-\frac{F^2k^2}{32}\right]. \qquad (17)$$

Q_μ, the autocovariance of the error signal, is related to ρ_μ, r_μ, and ϕ_μ through equation (5). Therefore

$$\frac{Q_0}{\sigma^2} = \frac{64\pi^2}{F^2}\left\{\frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2} \exp\left[-\frac{F^2k^2}{32}\right]\right\}, \qquad (18)$$

$$\frac{Q_\mu}{\sigma^2} = \frac{64}{F^2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mk} [1 + (-1)^{m+k}]\left\{\exp\left[-\frac{F^2(k^2 + m^2 - 2mk\rho_\mu)}{128}\right]\right.$$

$$- \exp\left[-\frac{F^2(k^2 + m^2 + 2mk\rho_\mu)}{128}\right]\Bigg\}, \quad \text{for} \quad \mu \text{ even},$$

$$\frac{Q_\mu}{\sigma^2} = \frac{128}{F^2} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{mk} (-1)^m \Bigg\{ \exp\left[-\frac{F^2(k^2 + m^2 - 2mk\rho_\mu)}{128}\right]$$

$$- \exp\left[-\frac{F^2(k^2 + m^2 + 2mk\rho_\mu)}{128}\right]\Bigg\}, \quad \text{for} \quad \mu \text{ odd}. \tag{19}$$

4.2 *The Minimal Output Quantizing Noise Power*

The proportionality of the autocovariance of the input signal and the cross-covariance of the input and the integrated ΔM signal implies that the related spectra are also proportional: $S_{xy}^*(f) = cS_{yy}^*(f)$. When this relationship is substituted into equation (12), the formula for the quantizing noise power at the output of an optimal interpolation filter, the result is

$$\eta_{\min} = 2\tau \int_0^{f_s/2} S_{yy}^*(f)\left[1 - \frac{c^2 S_{yy}^*(f)}{(2c - 1)S_{yy}^*(f) + S_{ee}^*(f)}\right] df \tag{20}$$

in which equation (9) has been used to substitute for $S_{xx}^*(f)$. By algebraic manipulation equation (20) may be shown to be identical to

$$\eta_{\min} = \frac{c^2}{(2c - 1)^2}\left\{2\tau \int S_{ee}^*(f)\, df - 2\tau \int \frac{[S_{ee}^*(f)]^2}{(2c - 1)S_{yy}^*(f) + S_{ee}^*(f)}\, df\right\}$$

$$- \frac{(c - 1)^2}{2c - 1}\, 2\tau \int S_{yy}^*(f)\, df \tag{21}$$

in which the integrals are taken over the set of f in $0 \leqq f \leqq f_s/2$ for which $S_{yy}^*(f) \neq 0$. The third integral in equation (21) is $\sigma^2/2\tau$; if the input is bandlimited to W Hz [with $S_{yy}^*(f) \neq 0$ for $|f| < W$], the first integral is that given in (13), $\eta_{lpf}/2\tau$. It follows that equation (21) may be rewritten as

$$\eta_{\min} = \frac{c^2}{(2c - 1)^2}\left[\eta_{lpf} - 2\tau \int_0^W \frac{[S_{ee}^*(f)]^2}{(2c - 1)S_{yy}^*(f) + S_{ee}^*(f)}\, df\right]$$

$$- \frac{(c - 1)^2}{2c - 1}\, \sigma^2. \tag{22}$$

Thus for a bandlimited signal, equation (22) relates the quantizing noise power at the output of a low pass interpolation filter to the noise at the output of an optimal filter. As the sampling rate increases, $c \to 1$ and the integral in equation (22), of a quadratic form of the coefficients, Q_μ, becomes negligible relative to η_{lpf} which is the integral of a linear form.

Thus for high sampling rates, $\eta_{\min} \approx \eta_{lpf}$, indicating that the transfer function of the optimal filter is nearly flat over frequencies at which $S_{yy}^*(f) \neq 0$ and is zero where $S_{yy}^*(f) = 0$.

4.3 *Approximations*

The infinite series in the formulas for the covariance coefficients converge rapidly, and in many cases of practical interest, entire series contribute negligibly to the values of the coefficients. For example, if the input possesses a flat power spectrum, cutoff at W Hz, the effective bandwidth is $W/(3)^{\frac{1}{2}}$ and the normalized sampling rate is related to the bandwidth expansion ratio by $F = 2(3)^{\frac{1}{2}} R$. Thus for $R \geq 12$ ΔM samples per Nyquist interval, the single summations in equations (15), (17), (18) and (19) consist of powers of a e^{-54} or less. These summations are added to 0.25 or to $\pi^2/3$ and thus have negligible effect on the values of the covariance coefficients. In the double summations, only the terms obtained with the two indices equal contribute significantly to the total when F is high. These double summations may, therefore, be replaced by single sums and we have the following approximations:

$$\frac{r_0}{\sigma^2} \approx 1 + \frac{64\pi^2}{3F^2}$$

$$\frac{r_\mu}{\sigma^2} \approx \rho_\mu + \frac{256}{F^2} \sum_{k=1}^{\infty} \frac{(-1)^{\mu k}}{k^2} \exp\left(-\frac{F^2k^2}{64}\right) \sinh\left(\frac{F^2k^2\rho_\mu}{64}\right) \tag{23}$$

$$c \approx 1, \qquad \frac{\phi_\mu}{\sigma^2} \approx \rho_\mu \tag{24}$$

$$\frac{Q_0}{\sigma^2} \approx \frac{64\pi^2}{3F^2} = \beta^2/3$$

$$\frac{Q_\mu}{\sigma^2} \approx \frac{256}{F^2} \sum_{k=1}^{\infty} \frac{(-1)^{\mu k}}{k^2} \exp\left(-\frac{F^2k^2}{64}\right) \sinh\left(\frac{F^2k^2\rho_\mu}{64}\right). \tag{25}$$

If in equation (25) we approximate $\sinh x$ by $e^x/2$† and substitute the result in equation (14) for η_{lpf} , we obtain van de Weg's formula for the granular noise power. Van de Weg claims its validity for $R \geqq 2$ samples per Nyquist interval. Our precise formula for Q_μ , equation (19), leads to noise power characteristics that are valid for all sampling rates.

† This leads to a small but nonzero value of Q_μ as $\mu \rightarrow \infty$ and $\rho_\mu \rightarrow 0$. Retention of the e^{-x} term in the approximate formula for Q_μ results in $Q_\infty = 0$ and thus avoids an anomaly and a source of numerical error in van de Weg's noise power formula.

4.4 *Signal-to-Noise Ratio Characteristics*

In this section we demonstrate the nature of the derived quantizing noise characteristics by illustrating the effect of the ΔM sampling rate on the quantizing noise powers, η_{lpf} and $\eta_{\min}$, and on Q_0, the mean square error at the input to the interpolation filter. In particular, Fig. 2 shows on a dB scale, $S_{opt} = \sigma^2/\eta_{\min}$, the output signal-to-noise ratio of an optimal interpolation filter; $S_{lpf} = \sigma^2/\eta_{lpf}$, the signal-to-noise ratio of a low pass filter; and $S_0 = \sigma^2/Q_0$ the signal-to-noise ratio prior to interpolation. The data in Fig. 2 pertain to the case of a zero-mean stationary gaussian input with a flat bandlimited power spectrum. The signal-to-noise ratios are shown as functions of R, the number of ΔM samples per Nyquist interval.

For high sampling rates, equation (25) indicates that Q_0 is approximately $\delta^2/3$, the mean square value of a random variable distributed uniformly over an interval of length 2δ. Thus with increasing R, S_0 rises at the rate of 20 dB per decade. At high sampling rates S_{lpf} and S_{opt} are nearly identical. Their slope is 30 dB per decade as indicated by equation (14) which is a linear combination of the error covariance coefficients (proportional to R^{-2}), weighted by $1/R$.

At low sampling rates, S_0 and S_{lpf} become very low (−15 dB at the Nyquist rate) while S_{opt} tends toward unity, corresponding to a filter that generates zero output (the mean input), and thus has a mean square error of σ^2.

V. DERIVATION OF COVARIANCE STATISTICS

Although the ΔM system considered in this paper is identical to the one studied by van de Weg and the values obtained for granular noise power are virtually the same as his over a wide range of transmission speeds, the method of analysis used in obtaining the present results differs considerably from van de Weg's. Van de Weg formulated the ensemble of integrated ΔM signals as a nonstationary process; he was thus unable to compute spectral characteristics from derived covariance statistics. Instead of considering correlation properties, van de Weg began with the amplitude spectrum of a sample function of the integrated ΔM signal ensemble. He then calculated the power density spectrum as the mean square amplitude spectrum.

In the work reported in this paper, the ensemble of integrated ΔM signals is stationary in the wide sense, so that the power spectra are Fourier transforms of the covariance functions whose derivations are described in the remainder of this paper. The difference between van

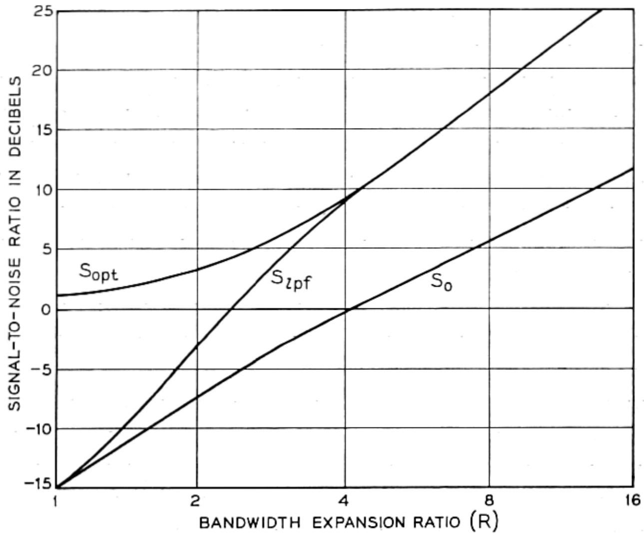

Fig. 2 — Quantizing noise characteristics.

de Weg's signal ensemble and ours lies in the role of the binary phase parameter defined in the Section 5.1.

5.1 *The Integrated ΔM Signal Ensemble*

The integrated ΔM signal, $\{x_n\}$, is a discrete-time discrete-amplitude function. The signal ranges over values $k\delta$ ($k = 0, \pm 1, \pm 2, \cdots$), and the absence of slope overload implies that x_n takes on the value of the allowed quantization level nearest to y_n. (In overload conditions, x_n and y_n may differ considerably.) At any sampling instant, the set of allowed quantization levels of a given signal is either the odd-parity subset of quantization levels,

$$\pm\delta, \pm 3\delta, \pm 5\delta, \cdots \tag{26}$$

or the even-parity subset

$$0, \pm 2\delta, \pm 4\delta, \cdots . \tag{27}$$

This restriction to a subset of the $k\delta$ follows from the ΔM mechanism which constrains each sample of $\{x_n\}$ to differ by $\pm\delta$ from its predecessor. Thus if $x_0 = 2k\delta$, $x_1 = (2k \pm 1)\delta$ and any sample that may be written x_{2m} ($m = 0, \pm 1, \pm 2, \ldots$) is constrained to an even-parity value. Similarly the subsequence $\{x_{2m+1}\}$ ranges over the odd-parity set of quantization levels.

Thus in the absence of slope overload, x_n is the result of processing y_n with a uniform PCM quantizer with quantization intervals of length 2δ. Either x_n is the output of the even-parity quantizer, with levels given by equation (27) or the output of the odd-parity quantizer with levels given in equation (26). The input-output characteristics of the two quantizers are shown in Fig. 3.

In defining the ΔM signal ensemble, van de Weg assumed that the "initial condition," $x_0 = 2k\delta$, applies to all sequences $\{x_n\}$. In van de Weg's analysis, therefore, all samples in $\{x_{2m}\}$ are generated by the even-parity quantizer and all samples in $\{x_{2m+1}\}$ are generated by the odd parity quantizer. Thus the probability functions of x_{2m} and x_{2m+1} differ and the ensemble of sequences $\{x_n\}$ is nonstationary.

We now generalize van de Weg's formulation of the integrated ΔM signal ensemble by observing that the ΔM system may also generate signals with the initial condition, $x_0 = (2k-1)\delta$. In this event $\{x_{2m}\}$ is the output of the odd-parity quantizer of Fig. 3 and $\{x_{2m+1}\}$ is the output of the even-parity quantizer. We shall refer to the initial condition that applies to a given $\{x_n\}$ as the "phase" of the signal. Thus we define the two phase states:

A_1: $\{x_{2m}\}$ generated by the even-parity quantizer
A_2: $\{x_{2m}\}$ generated by the odd-parity quantizer.

A delay of a signal by τ seconds results in a phase reversal from A_1 to A_2 or from A_2 to A_1.

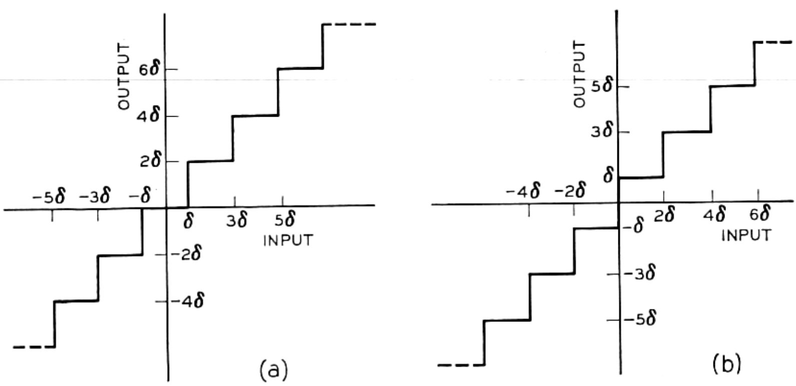

Fig. 3 — Two uniform quantizers: (a) even-parity quantizer, (b) odd-parity quantizer.

If we admit signals with both phases to the ΔM ensemble, the statistics of the ensemble of $\{x_n\}$ depend on the relative frequency of occurrence, that is, on the prior probability of the two phases. Van de Weg's ensemble is a "coherent" one for which the prior probability function is

$$\Pr\{A_1\} = 1, \qquad \Pr\{A_2\} = 0. \tag{28}$$

In this paper we study the statistics of the noncoherent ensemble in which

$$\Pr\{A_1\} = \Pr\{A_2\} = \tfrac{1}{2}. \tag{29}$$

The correlation analysis begins with the derivation of probability functions conditioned on each of the two phases. Marginal probabilities may be calculated on the basis of a prior probability function as

$$\Pr\{x_n = k\delta\} = \Pr\{A_1\}\Pr\{x_n = k\delta \mid A_1\} + \Pr\{A_2\}\Pr\{x_n = k\delta \mid A_2\}. \tag{30}$$

When equation (29) is used in the computation of equation (30), the result is independent of n. Similarly the joint marginal probability of x_n and $x_{n+\mu}$ is independent of n when equation (29) is accepted. When equation (28) is accepted, as it is in van de Weg's analysis, both the single and joint probability functions depend on the parity of n and the covariance statistics are functions of two time variables.

In principle, either equation (28) or (29) may be applicable to the operation of a particular ΔM system. In practice, numerical results based on the two phase conditions are usually quite similar. In analytic work, there is a considerable advantage offered by equation (28), the noncoherence assumption. It admits the techniques of stationary time series analysis to the investigation of questions of interest.

5.2 *The Probability Distribution of* x_n

Here we derive the probability function of a sample, x_n, of the integrated ΔM signal. The probabilities conditioned on A_1 and A_2 depend on whether n is even or odd, but the marginal probability function is independent of n when A_1 and A_2 are equiprobable.

Under the condition A_1, the samples $\{x_{2m}\}$ are outputs of the even-parity quantizer so that $x_{2m} = 2k\delta$ when

$$(2k - 1)\delta \leqq y_{2m} < (2k + 1)\delta.$$

If $\{y_n\}$ is a sample function of a stationary zero-mean gaussian process with variance σ^2, we have

$$\Pr\{x_{2m} = 2k\delta \mid A_1\} = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{(2k-1)\delta}^{(2k+1)\delta} \exp\left(-\frac{u^2}{2\sigma^2}\right) du$$
$$\Pr\{x_{2m} = (2k-1)\delta \mid A_1\} = 0. \tag{31}$$

The samples $\{x_{2m+1}\}$ are generated by the odd parity quantizer so that

$$\Pr\{x_{2m+1} = 2k\delta \mid A_1\} = 0$$
$$\Pr\{x_{2m+1} = (2k-1)\delta \mid A_1\} = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \int_{(2k-2)\delta}^{2k\delta} \exp\left(-\frac{u^2}{2\sigma^2}\right) du. \tag{32}$$

Under the condition A_2, the complementary probability function applies:

$$\Pr\{x_{2m} = 2k\delta \mid A_2\} = \Pr\{x_{2m+1} = (2k-1)\delta \mid A_2\} = 0,$$
$$\Pr\{x_{2m} = (2k-1)\delta \mid A_2\} = \Pr\{x_{2m+1} = (2k-1)\delta \mid A_1\}, \tag{33}$$
$$\Pr\{x_{2m+1} = 2k\delta \mid A_2\} = \Pr\{x_{2m} = 2k\delta \mid A_1\}.$$

By combining equations (30) to (33), one may demonstrate that $\Pr\{x_n = k\delta\}$ depends on n (in particular on whether n is even or odd) for all prior probabilities of A_1 and A_2 except the equiprobable pair given in equation (29). Thus equation (29) is a necessary condition for stationarity. When this condition is imposed and $\beta = \delta/\sigma$ incorporated, the formula for the marginal probability of x_n becomes

$$\Pr\{x_n = k\delta\} = \frac{1}{2(2\pi)^{\frac{1}{2}}} \int_{(k-1)\beta}^{(k+1)\beta} \exp\left(-\frac{u^2}{2}\right) du. \tag{34}$$

From equation (34), the moments of x_n may be calculated. We have

$$E\{x_n\} = \frac{\delta}{2(2\pi)^{\frac{1}{2}}} \sum_{k=-\infty}^{\infty} k \int_{(k-1)\beta}^{(k+1)\beta} \exp\left(-\frac{u^2}{2}\right) du = 0 \tag{35}$$

and

$$E\{x_n^2\} = r_0 = \frac{\delta^2\beta}{(2\pi)^{\frac{1}{2}}} \sum_{k=-\infty}^{\infty} k^2 \int_0^1 \exp\left[-\frac{\beta^2}{2}(v+k)^2\right] dv, \tag{36}$$

which is equivalent to the form of r_0 given in equation (15). The derivation of equation (15) from (36) is demonstrated in Section A.2 of the appendix.

5.3 *The Joint Probability of* x_n *and* $x_{n+\mu}$

For each phase condition, the expression of the joint conditional probability of x_n and $x_{n+\mu}$ depends on the parity of n and the parity of μ. For phase A_1, x_n and $x_{n+\mu}$ are both generated by the even-parity quantizer when n and μ are even numbers. Thus the conditional probability that $x_n = 2k\delta$ and $x_{n+\mu} = 2l\delta$ is the probability that

$$(2k - 1)\delta \leqq y_n < (2k + 1)\delta \quad \text{and} \quad (2l - 1)\delta \leqq y_{n+\mu} < (2l + 1)\delta.$$

Thus for n and μ both even,

$$\Pr\{x_n = 2k\delta, x_{n+\mu} = 2l\delta \mid A_1\}$$
$$= \frac{1}{2\pi\sigma^2(1 - \rho_\mu^2)^{\frac{1}{2}}} \int_{(2k-1)\delta}^{(2k+1)\delta} \int_{(2l-1)\delta}^{(2l+1)\delta} \exp\left[-\frac{u^2 + v^2 - 2\rho_\mu uv}{2\sigma^2(1 - \rho_\mu^2)}\right] du\, dv$$

$$\Pr\{x_n = (2k - 1)\delta, x_{n+\mu} = l\delta \mid A_1\}$$
$$= \Pr\{x_n = k\delta, x_{n+\mu} = (2l - 1)\delta \mid A_1\} = 0. \tag{37}$$

Similarly we derive conditional probability expressions for the eight cases listed under step 1 in Table I. The four marginal probabilities indicated under step 2 are calculated as

$$\Pr\{x_n = k\delta, x_{n+\mu} = l\delta\} = \tfrac{1}{2}\Pr\{x_n = k\delta, x_{n+\mu} = l\delta \mid A_1\}$$
$$+ \tfrac{1}{2}\Pr\{x_n = k\delta, x_{n+\mu} = l\delta \mid A_2\}. \tag{38}$$

Among the four cases there are only two different formulas. One is applicable to even values of μ and the other to odd values of μ. When μ is even, x_n and $x_{n+\mu}$ are generated by the same quantizer and when μ is odd they are generated by different quantizers. The marginal joint probability function is independent of n. It may be expressed in terms of the double integral expression

$$p(k, l, \mu) = \frac{1}{4\pi(1 - \rho_\mu^2)^{\frac{1}{2}}} \int_{(k-1)\beta}^{(k+1)\beta} \exp\left(-\frac{v^2}{2}\right)$$
$$\cdot \int_{(l-1)\beta}^{(l+1)\beta} \exp\left[-\frac{(u - v\rho_\mu)^2}{2(1 - \rho_\mu^2)}\right] du\, dv \tag{39}$$

as

$$\begin{aligned} \Pr\{x_n = k\delta, x_{n+\mu} = l\delta\} &= p(k, l, \mu) \quad \text{for} \quad k + l + \mu \text{ even} \\ &= 0 \qquad\qquad\quad \text{for} \quad k + l + \mu \text{ odd}. \end{aligned} \tag{40}$$

TABLE I—STEPS IN DERIVING Pr $\{x_n = k\delta, x_{n+\mu} = l\delta\}$

Step 1 Conditional probabilities obtained for cases	Step 2 Marginal probabilities	Step 3 Identical expressions except for cases
n even, μ even A_1 n even, μ even A_2	n even, μ even	μ even
n odd, μ even A_1 n odd, μ even A_2	n odd, μ even	
n even, μ odd A_1 n even, μ odd A_2	n even, μ odd	μ odd
n odd, μ odd A_1 n odd, μ odd A_2	n odd, μ odd	

The autocovariance coefficient, r_μ, is the expected product of x_n and $x_{n+\mu}$:

$$r_\mu = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (k\delta)(l\delta) \Pr\{x_n = k\delta, x_{n+\mu} = l\delta\}. \tag{41}$$

Substitution of equation (40) into (41) results in

$$\begin{aligned} r_\mu &= \delta^2 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (2k)(2l)p(2k, 2l, \mu) \\ &\quad + \delta^2 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (2k-1)(2l-1)p(2k-1, 2l-1, \mu) \quad \text{for } \mu \text{ even} \\ r_\mu &= 2\delta^2 \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (2k)(2l-1)p(2k, 2l-1, \mu) \quad \text{for } \mu \text{ odd.} \end{aligned} \tag{42}$$

Section A.3 of the appendix outlines the derivation of equation (15) from (42) and (39).

5.4 *The Joint Distribution of* y_n *and* $x_{n+\mu}$

Here we consider the joint probability function of a discrete random variable, $x_{n+\mu}$ and a continuous random variable y_n. Once again the marginal distributions are independent of n when the two phases are equiprobable. For $\mu = 0$, the marginal probability function is

$$\Pr\{y_n = u,\, x_n = k\delta\} = \frac{1}{2\sigma(2\pi)^{\frac{1}{2}}}\exp\left(-\frac{u^2}{2\sigma^2}\right)du \quad \text{for} \quad (k-1)\delta \leqq u < (k+1)\delta$$
$$= 0 \quad \text{for other values of } u. \tag{43}$$

The expected value of $x_n y_n$ may be computed as

$$E\{x_n y_n\} = \phi_0 = \frac{\delta\sigma}{2(2\pi)^{\frac{1}{2}}}\sum_{k=-\infty}^{\infty} k \int_{(k-1)\beta}^{(k+1)\beta} u \exp\left(-\frac{u^2}{2}\right)du \tag{44}$$

which is shown in Section A.4 of appendix to be $c\sigma^2$, with c given by equation (17).

For other values of μ, the conditional probability function of u and $k\delta$ is the probability that $y_n = u$ and $(k-1)\delta \leqq y_{n+\mu} < (k+1)\delta$, provided $k\delta$ is an output level of the quantizer that processes $y_{n+\mu}$. The marginal probability function may be written as

$$\Pr\{y_n = u,\, x_{n+\mu} = k\delta\}$$
$$= \frac{1}{4\pi\sigma^2(1-\rho_\mu^2)^{\frac{1}{2}}}\int_{(k-1)\delta}^{(k+1)\delta} \exp\left[-\frac{u^2+v^2-2uv\rho_\mu}{2\sigma^2(1-\rho_\mu^2)}\right]dv\,du \tag{45}$$

from which cross-covariance coefficient ϕ_μ may be calculated as

$$\phi_\mu = \sum_{k=-\infty}^{\infty} k\delta \int_{-\infty}^{\infty} u \Pr\{y_n = u,\, x_{n+\mu} = k\delta\}\,. \tag{46}$$

If equation (45) is substituted into (46) and the integration with respect to u is performed first, the result is

$$\phi_\mu = \frac{\rho_\mu\delta\sigma}{2(2\pi)^{\frac{1}{2}}}\sum_{k=-\infty}^{\infty} k \int_{(k-1)\beta}^{(k+1)\beta} v \exp\left(-\frac{v^2}{2}\right)dv \tag{47}$$

which is equation (44) multiplied by ρ_μ.

APPENDIX

Applications of the Poisson Sum Formula to the Derivation of Covariance Coefficients

A.1 *Basic Formula*[11]

$$f(x,\, t) = \sum_{n=-\infty}^{\infty} \exp\left[-t(x+n)^2\right]. \tag{48}$$

$$f(x,\, t) = \left(\frac{\pi}{t}\right)^{\frac{1}{2}}\left[1 + 2\sum_{k=1}^{\infty}\exp\left(-\frac{\pi^2k^2}{t}\right)\cos 2\pi kx\right]. \tag{49}$$

A.1.1 *Even Terms*

$$g(x, t) = \sum_{n=-\infty}^{\infty} \exp\left[-t(x + 2n)^2\right] = f\left(\frac{x}{2}, 4t\right) \tag{50}$$

$$g(x, t) = \frac{1}{2}\left(\frac{\pi}{t}\right)^{\frac{1}{2}}\left[1 + 2\sum_{k=1}^{\infty} \exp\left(-\frac{\pi^2 k^2}{4t}\right)\cos \pi k x\right]. \tag{51}$$

A.1.2 *Odd Terms*

$$h(x, t) = \sum_{n=-\infty}^{\infty} \exp\left[-t(x + 2n - 1)^2\right] = f(x, t) - g(x, t) \tag{52}$$

$$h(x, t) = \frac{1}{2}\left(\frac{\pi}{t}\right)^{\frac{1}{2}}\left[1 + 2\sum_{k=1}^{\infty} (-1)^k \exp\left(-\frac{\pi^2 k^2}{4t}\right)\cos \pi k x\right]. \tag{53}$$

A.2 *Mean Square Value of* x_n

Equation (36) may be developed in terms of the partial derivatives of equation (48):

$$f_t(x, t) = -\sum_{n=-\infty}^{\infty} (x + n)^2 \exp\left[-t(x + n)^2\right] \tag{54}$$

and

$$f_x(x, t) = -2t\sum_{n=-\infty}^{\infty} (x + n) \exp\left[-t(x + n)^2\right]. \tag{55}$$

Equations (54) and (55) may be combined to form

$$\sum_{n=-\infty}^{\infty} n^2 \exp\left[-t(x + n)^2\right] = x^2 f(x, t) + \frac{x}{t} f_x(x, t) - f_t(x, t). \tag{56}$$

If the order of summation and integration in equation (36) is reversed, the resulting integrand is identical in form to the left side of equation (56). Thus equation (49) may be substituted into the right side of equation (56) and the three terms integrated over $0 \leqq x \leqq 1$. The result is

$$\begin{aligned}\int_0^1 \sum_{n=-\infty}^{\infty} n^2 \exp\left[-t(x + n)^2\right] dx \\ = \frac{1}{2t}\left(\frac{\pi}{t}\right)^{\frac{1}{2}}\left[1 + 4\sum_{k=1}^{\infty} \exp\left(-\frac{\pi^2 k^2}{t}\right)\right] \\ + \left(\frac{\pi}{t}\right)^{\frac{1}{2}}\left[\frac{1}{3} + \sum_{k=1}^{\infty}\left(\frac{1}{\pi k}\right)^2 \exp\left(-\frac{\pi^2 k^2}{t}\right)\right].\end{aligned} \tag{57}$$

The variable, t, in equation (57) is related to equation (36) by $t =$

$\beta^2/2 = 32\pi^2/F^2$; when this latter form is substituted for t, the form of r_0 given in equation (15) results.

A.3 *Autocovariance Coefficients*

In order to illustrate the derivation of equation (15) for r_μ from equation (42), we consider odd values of μ. By substituting into equation (42) the form of $p(k, l, \mu)$ given in equation (39) we write

$$r_\mu = \frac{2\delta^2}{4\pi(1 - \rho_\mu^2)^{\frac{1}{2}}} \sum_{k=-\infty}^{\infty} 2k \int_{(2k-1)\beta}^{(2k+1)\beta} \exp\left(-\frac{v^2}{2}\right) G(v)\, dv, \tag{58}$$

in which we have defined

$$G(v) = \beta \int_{-1-v\rho_\mu/\beta}^{1-v\rho_\mu/\beta} \sum_{l=-\infty}^{\infty} (2l - 1) \exp\left[-\frac{\beta^2(x + 2l - 1)^2}{2(1 - \rho_\mu^2)}\right] dx. \tag{59}$$

The integrand in equation (59) is related to the infinite series in equation (52) and its partial derivative with respect to x by

$$\sum_{n=-\infty}^{\infty} (2n - 1) \exp\left[-t(x + 2n - 1)^2\right] = -xh(x, t) - \frac{1}{2t} h_x(x, t), \tag{60}$$

in which the variable $t = \beta^2/2(1 - \rho_\mu^2)$. Into equation (60) we substitute the form of $h(x, t)$ given in equation (53) and perform the integration required in equation (59). The integral of the second term is zero so that $G(v)$ is β times the integral of the first term of equation (60). Thus equation (58) may be written in the form

$$\begin{aligned} G(v) = {} & [2\pi(1 - \rho_\mu^2)]^{\frac{1}{2}} \\ & \cdot\left\{\frac{\rho_\mu v}{\beta} + 2 \sum_{m=1}^{\infty} \frac{1}{\pi m} \exp\left[-\frac{\pi^2 m^2(1 - \rho_\mu^2)}{2\beta^2}\right] \sin \frac{\pi m \rho_\mu v}{\beta}\right\}, \end{aligned} \tag{61}$$

which must be weighted by $\exp(-v^2/2)$ and integrated according to equation (58).

Equations (58) and (61) thus show r_μ to be the sum of two terms. The first term consists of a constant, $\rho_\mu \sigma\delta/(2\pi)^{\frac{1}{2}}$, multiplying the sum

$$\sum_{k=-\infty}^{\infty} 2k \int_{(2k-1)\beta}^{(2k+1)\beta} v \exp\left(-\frac{v^2}{2}\right) dv = 2 \sum_{k=-\infty}^{\infty} \exp\left[-\frac{\beta^2}{2}(2k - 1)^2\right]. \tag{62}$$

This latter summation is in the form of equation (52) with $x = 0$, $t = \beta^2/2$ so that with the application of equation (53), (62) becomes

$$h\left(0, \frac{\beta^2}{2}\right) = \frac{(2\pi)^{\frac{1}{2}}}{\beta}\left[1 + \sum_{k=1}^{\infty} (-1)^k \exp\left(-\frac{\pi^2 k^2}{2\beta^2}\right)\right]. \tag{63}$$

The second term in the expression for r_μ may be written in the form,

$$\delta^2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{m=1}^{\infty} \frac{1}{\pi m} \exp\left[-\frac{\pi^2 m^2(1-\rho_\mu^2)}{2\beta^2}\right] \cdot \sum_{k=-\infty}^{\infty} 2k \int_{(2k-1)\beta}^{(2k+1)\beta} \exp\left(-\frac{v^2}{2}\right) \sin\frac{\pi m \rho_\mu v}{\beta}\, dv. \tag{64}$$

If the sine in this expression is developed in exponential form, the summation, ranging over k, in the above expression, has a form similar to the integral and sum in equation (59). If it is analyzed in the manner that $G(v)$ was reduced the following identity may be demonstrated:

$$\sum_{k=-\infty}^{\infty} k \int_{(2k-1)\beta}^{(2k+1)\beta} \exp\left(-\frac{v^2}{2}\right) \sin\frac{\pi m \rho_\mu v}{\beta}\, dv = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \exp\left(-\frac{\pi^2 m^2 \rho_\mu}{2\beta^2}\right)\left[\frac{\pi m \rho_\mu}{\beta^2} + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{\pi k} \times \exp\left(-\frac{\pi^2 k^2}{2\beta^2}\right) \sinh\left(\frac{\pi^2 k m \rho_\mu}{\beta^2}\right)\right]. \tag{65}$$

Thus equation (64) becomes

$$2\rho_\mu\sigma^2 \sum_{m=1}^{\infty} \exp\left(-\frac{\pi^2 m^2}{2\beta^2}\right) + 2\delta^2 \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)}{\pi^2 k m} \cdot \exp\left[-\frac{\pi^2(k^2+m^2)}{2\beta^2}\right] \sinh\left(\frac{\pi^2 k m \rho_\mu}{\beta^2}\right) \tag{66}$$

so that r_μ for μ odd, the sum of (66) and $\rho_\mu \sigma \delta/(2\pi)^{1/2}$ times (63), may be expressed as

$$r_\mu = \rho_\mu \sigma^2\left[1 + 4\sum_{k=1}^{\infty} \exp\left(-\frac{2\pi^2 k^2}{\beta^2}\right)\right] + 2\delta^2 \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^m}{\pi^2 k m} \cdot \exp\left[-\frac{\pi^2(k^2+m^2)}{2\beta^2}\right] \sinh\left(\frac{\pi^2 k m \rho_\mu}{\beta^2}\right). \tag{67}$$

If $8\pi/F = \beta^2$ is substituted in equation (67) the result is equation (15).

Similarly the formula given in equation (42) for r_μ when μ is even may be developed to demonstrate its identity to the formula in equation (15).

A.4 *Cross-Covariance*

Performing the integration indicated in equation (44) we have

$$\phi_0 = \frac{\delta\sigma}{2(2\pi)^{\frac{1}{2}}} \sum_{k=-\infty}^{\infty} k\left\{\exp\left[-\frac{\beta^2}{2}(k-1)^2\right] - \exp\left[-\frac{\beta^2}{2}(k+1)^2\right]\right\}$$

$$= \frac{\delta\sigma}{(2\pi)^{\frac{1}{2}}} \sum_{k=-\infty}^{\infty} \exp\left[-\frac{\beta^2}{2}k^2\right], \qquad (68)$$

which is equivalent to equation (48) with $x = 0$, $t = \beta^2/2 = [32\,\pi^2/F^2]$. Thus equation (49) may be substituted with the result given in equation (16):

$$\phi_0 = \sigma^2\left[1 + 2\sum_{k=1}^{\infty} \exp\left(-\frac{2\pi^2k^2}{\beta^2}\right)\right].$$

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