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On the Identification of Linear Time-Invariant Systems from Input-Output Data

By B. GOPINATH

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This paper presents a new method for computing the parameters which determine the differential equations governing a linear time-invariant system with multiple inputs and outputs. Unlike earlier approaches the method presented does not involve computation of the impulse response. One of the main advantages of this method is its easy generalization to the case when the given data is contaminated with noise.

The identification of multiple input-output linear systems has been a problem of considerable interest because of its importance in circuit and control system theory. In circuit theory the problem is that of synthesizing a linear time invariant circuit to exhibit a prescribed input-output behavior. In control theory, however, the problem arises out of a need to model a given linear system with a suitable set of differential equations, given its input-output behavior. References 1, 2, and 3 deal with the problem of determining the parameters of the differential equation model from the impulse response. To the best of the author's knowledge, there is no published method which determines the impulse response from a finite segment of input-output data in the case of systems with more than one input and output.

I. THE "STATE INVARIANT" DESCRIPTION

In most applications of identification techniques, one is given only a record of input sequences and a record of output sequences, rather than the impulse response function. In these cases it seems best to get an internal description of the system directly from these data; that is, avoid the intermediate step of synthesizing the impulse response. In many applications the structure of systems that are being identified remains the same, while values of parameters change. Therefore, it is convenient to work in a certain coordinate frame which is fixed for the given system. Most important of all, a method of arriving at the values of parameters directly from input-output data is easier to analyze than the method in which impulse response is synthesized, since the sensitivity of intermediate computations required to obtain the impulse response matrix need not be analyzed.

The problem is therefore formulated as follows. Let Σ be a linear system in discrete time modeled by equations (1) and (2):

$$x(s+1) = Fx(s) + Gu(s) \tag{1}$$

$$y(s) = Hx(s). \tag{2}$$

$x(s) \in E^n$ (the "n" dimensional Euclidean space) is the state of Σ at time s; similarly $u(s)$ and $y(s)$ are the m-dimensional input and the p-dimensional output of Σ. F, G, H are real constant matrices of appropriate dimensions. Σ is assumed to be completely reachable and completely observable (for details about these terms see Ref. 4), namely

$$\text{rank of } [G, FG, \cdots, F^{n-1}G] = n \tag{3}$$

and

$$\text{rank of } [H', F'H', \cdots, F'^{n-1}H'] = n \tag{4}$$

where prime (′) denotes the transpose. Given a sequence of inputs $u(s)$ and outputs $y(s)$ for $s = 1, 2, \cdots, N$ (where N is sufficiently large), find a system $\hat{\Sigma}$ of the same dimension as Σ namely n such that $\hat{\Sigma}$ simulates the input-output behavior of Σ.

Remark 1: It is clear that there are some sequences $u(s)$ which will not be sufficient to uniquely specify $\hat{\Sigma}$. Theorems, presented in Section II, give sufficient conditions for $u(s)$ and N which uniquely determine $\hat{\Sigma}$.

Remark 2: When $\hat{\Sigma}$ is uniquely determined it will be shown that the state of $\hat{\Sigma}$ is uniquely related to the state of Σ. In fact the $\hat{F}$, $\hat{G}$, and

$\hat{H}$ of $\hat{\Sigma}$ will be related to the F, G, and H of Σ by a nonsingular transformation such that $HF^iG = \hat{H}\hat{F}^i\hat{G}$ which implies that the impulse responses of Σ and $\hat{\Sigma}$ are identical. Notice that for any nonsingular T

$$\hat{H} = HT^{-1} \qquad \hat{F} = TFT^{-1} \qquad \hat{G} = TG$$

implies that

$$HF^iG = \hat{H}\hat{F}^i\hat{G}.$$

The main difficulty in obtaining a direct algorithm is in getting at the state $x(s)$ from output sequences when the parameters of the system are not known. When, for example, H in the equation below is identity, or equivalently the output itself is the state, it is easy to find an internal description from sequences of inputs and outputs. From writing this equation as

$$x(s+1) = [F \quad G]\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}$$

$$y(s) = Hx(s) = x(s),$$

it follows that given enough observations one can solve for F and G from the above equation for most nontrivial input sequences (see Theorem 2). An easy way is to multiply both sides of this equation by $[x'(s) \ u'(s)]$ and sum from $s = 1$ to $s = N$ where N is the number of observations:

$$\sum_{s=1}^{N} \{x(s+1)[x'(s) \quad u'(s)]\} = [F \quad G] \sum_{s=1}^{N} \left\{ \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} [x'(s) \quad u'(s)] \right\}.$$

Whenever the matrix multiplying $[F \ G]$ in the above equation has an inverse, there exists a unique solution for F and G.

In the case when $y(s)$ is not the state itself but only a linear function of the state, the problem is much more complex and one has to select certain appropriate components of the output sequence for an external description in terms of the observables, namely $y(i)$ and $u(i)$. The selection of the right components can be done by introducing an operator to be called the selector matrix as defined below.

In describing the theory of the direct identification method, considerable use is made of the input-output description to be detailed below.

Definition: $\mathcal{S}$ will denote the set of $k \times l$ matrices ($k \leqq l$) with the following properties:

$$(i)\ S = \{s_{ij}\} \text{ where } s_{ij} = 0 \text{ or } 1. \tag{5}$$

$$(ii)\ \forall_i,\ s_{ij} = 1 \text{ for one and only one } j, \text{ say } j_i. \tag{6}$$

$$(iii)\ j_1 < j_2 < \cdots < j_k, \quad j_i \leqq l, \quad i \leqq k. \tag{7}$$

Examples of matrices belonging to $\mathcal{S}$ are

$$[1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and so on.}$$

Any matrix $S \in \mathcal{S}$ will be referred to as a selector matrix, because S operating on a linear space E^l transforms it into a linear space E^k by mapping every vector $x \in E^l$ to a vector $y \in E^k$ by selecting the components $j_1, \cdots, j_k$ of $x \in E^l$.

The description presented is an "external" description in the sense that the dynamical equations are given in terms of quantities which can be observed from outside, that is, values of input and values of output.

Consider a completely reachable and completely observable discrete time system Σ represented as follows

$$x(s+1) = Fx(s) + Gu(s), \tag{8}$$

$$y(s) = Hx(s), \quad H: p \times n; \quad F: n \times n; \quad G: n \times m. \tag{9}$$

"Completely observable" implies*

$$\rho([F' : H']) = n. \tag{10}$$

$\rho(A)$ = rank of A. Therefore, $\exists$ an $S \in \mathcal{S}$, such that

$$S \begin{bmatrix} H \\ \vdots \\ HF^{n-1} \end{bmatrix} = T \text{ where } T \text{ is nonsingular;} \tag{11}$$

that is, T^{-1} exists. Without loss of generality it can be assumed from remark 2 that $T = I$ so far as the external description is concerned. Using equations (8) and (9) repeatedly, it follows that

$$\begin{aligned} y(s) &= Hx(s), \\ y(s+1) &= Hx(s+1) = HFx(s) + HGu(s) \\ y(s+n-1) &= HF^{n-1}x(s) + HF^{n-2}Gu(s) + \cdots + HGu(s+n-2). \end{aligned} \tag{12}$$

Let

* $[F' : H'] \triangleq [H', F'H', \cdots, F'^{(n-1)}H']$.

$$\bar{y}'(s) \triangleq [y'(s) \quad y'(s+1) \cdots y'(s+n-1)], \tag{13}$$

$$\bar{u}'(s) \triangleq [u'(s) \quad u'(s+1) \cdots u'(s+n-1)]. \tag{14}$$

Then, writing equation (12) in vector form, and also using equations (13) and (14), it follows that

$$\bar{y}(s) = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} x(s) + \begin{bmatrix} 0 & 0 & & 0 \\ HG & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ HF^{n-2}G & HF^{n-1}G & \cdots & 0 \end{bmatrix} \bar{u}(s). \tag{15}$$

Let†

$$\begin{bmatrix} 0 & 0 & 0 & & 0 & 0 & 0 \\ HG & 0 & 0 & & 0 & 0 & 0 \\ HFG & HG & 0 & & 0 & 0 & 0 \\ \cdot & HFG & HG & & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ HF^{n-3}G & HF^{n-4}G & HF^{n-5}G & \cdots & HG & 0 & 0 \\ HF^{n-2}G & HF^{n-3}G & HF^{n-4}G & \cdots & HFG & HG & 0 \end{bmatrix} \triangleq R_1 ; \tag{16}$$

then multiplying both sides of equation (15) by S, using the comments given below equation (11), it follows that

$$S\bar{y}(s) = x(s) + SR_1\bar{u}(s). \tag{17}$$

Once again, using equation (9),

$$x(s+1) = Fx(s) + Gu(s),$$

which because of equation (17), with s replaced by $s + 1$, reduces to

$$x(s+1) = S\bar{y}(s+1) - SR_1\bar{u}(s+1); \tag{18}$$

substituting equation (9) for $x(s)$ in equation (17) gives

$$S\bar{y}(s+1) = F(S\bar{y}(s) - SR_1\bar{u}(s)) + S\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} Gu(s) + SR_1\bar{u}(s+1). \tag{19}$$

† The last column of zeroes in R_1 is added so that $\bar{y}$ and $\bar{u}$ may be consistently defined.

Since it has been shown that [see equation (11)]

$$S\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} = I, \tag{20}$$

it follows that

$$S\bar{y}(s+1) = FS\bar{y}(s) + R\bar{u}(s), \tag{21}$$

where‡

$$R \triangleq -FSR_1 + S\begin{bmatrix} HG & 0 & 0 & 0 & 0 \\ HFG & HG & 0 & 0 & 0 \\ HF^2G & HFG & 0 & 0 & 0 \\ \cdot & \cdot & 0 & \cdot & \cdot \\ HF^{n-1}G & HF^{n-2}G & \cdot & \cdot & HG \end{bmatrix}. \tag{22}$$

Equation (21) gives a relation between the input sequence $u(i)$ and the output sequence $y(i)$ which does not involve the state. It is an external description in the sense that the variables in equation (21), namely $u(i)$ and $y(i)$, can be measured externally. From equation (22) it follows that if R is partitioned as

$$[R_0 \quad R_1 \cdots R_{n-1}], \quad R_i : \quad n \times m \quad \forall i, \tag{23}$$

then

$$R_{n-1} = S\begin{bmatrix} 0 \\ \vdots \\ HG \end{bmatrix}. \tag{24}$$

It is obvious how one obtains the columns of the second product in equation (22). To obtain the contribution from $-FSR_1$, notice from equation (16) that S times the second column from the end of R_1 is, from equation (24), merely R_{n-1}. Therefore, the second column of FSR_1 from the end is simply FR_{n-1} and therefore

‡ In adding $SR_1\bar{u}(s+1)$ to the second term in equation (20), the last column of R_1 may be dropped because it is all zeroes.

$$R_{n-2} = -FR_{n-1} + S\begin{bmatrix} 0 \\ \vdots \\ HG \\ HFG \end{bmatrix} \tag{25}$$

Now notice that $R_{n-2} + FR_{n-1}$ is S times the third column from the end of R_1. Therefore the third column from the end of R is

$$R_{n-3} = -FR_{n-2} - F^2R_{n-1} + S\begin{bmatrix} 0 \\ \vdots \\ HG \\ HFG \\ HF^2G \end{bmatrix}.$$

Continuing in the same way,

$$R_{n-4} = -FR_{n-3} - F^2R_{n-2} - F^3R_{n-1} + S\begin{bmatrix} 0 \\ \cdot \\ 0 \\ HG \\ HFG \\ HF^2G \\ HF^3G \end{bmatrix}$$

and finally

$$S\begin{bmatrix} HG \\ HFG \\ \vdots \\ HF^{n-1}G \end{bmatrix} = R_0 + FR_1 + \cdots + F^{n-1}R_{n-1} \, . \tag{26}$$

Now, since it was possible to choose a basis such that

$$S\begin{bmatrix} HG \\ \vdots \\ HF^{n-1}G \end{bmatrix} = IG,$$

one has

$$G = R_0 + FR_1 + \cdots F^{n-1}R_{n-1} . \tag{27}$$

Equation (21) may be written in the form

$$S\bar{y}(s + 1) = [F \quad R]\begin{bmatrix} S\bar{y}(s) \\ \bar{u}(s) \end{bmatrix}$$

and may in principle be solved for F and R. Thus from equations (27) it is clear that if the values for $u(i)$ and $y(i)$ were given and S were known, one could also solve for one set of values for F; and since in most cases H is full rank, H can be assumed to be $[I \ 0]$.

II. THE MINIMAL REPRESENTATION AND THE DIRECT ALGORITHM.

It was shown in Section I that, corresponding to every internal description of Σ which is completely controllable and completely observable, there is a description in the form of equation (21). In this section we show that from the knowledge of the values of $u(i)$, $i = 1, \cdots, N$, and $y(i)$, $i = 1, \cdots, N$, one can get the internal description of Σ under very general conditions on $u(i)$. Central to the discussion are a few results which are presented in the form of theorems for the sake of clarity and precision.

Given $u(i)$, $i = 1, \cdots, N$, the inputs to a system Σ of dimension n which is completely observable and completely reachable, and the corresponding outputs $y(i)$, $i = 1, \cdots, N$, the following propositions hold true:

Note 1: It will be assumed in the following that the column dimension k of the selector matrix is always a multiple of p; further if $k = rp$, then

$$(r - 1)p \leqq j_l \leqq rp.$$

It is obvious that there is no loss of generality involved in this assumption. (l is the row dimension of S.)

Note 2: In the definition of $\bar{y}(s)$ and $\bar{u}(s)$ in equations (13) and (14), the n should be replaced by r defined in Note 1 above.

Theorem 1: Let S *be* l × k (= rp) ; *then*

$$\rho\left\{S\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{r-1} \end{bmatrix}\right\} < 1 \tag{28}$$

implies that

$$\sum_{s=1}^{N-r+1} \begin{bmatrix} S\bar{y}(s) \\ \bar{u}(s) \end{bmatrix} [\bar{y}'(s)S' \quad \bar{u}'(s)] \text{ is a singular matrix} \tag{29}$$

for every sequence u(i), i = *1, 2,* $\cdots$, N.

Proof: Multiplying equation (10) on the left by S and replacing n by r we have,

$$S\bar{y}(s) = S\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{r-1} \end{bmatrix} x(s) + SR_1\bar{u}(s). \tag{30}$$

Because of equation (28) $\exists$ a vector, $z \neq 0$, and in E^l such that

$$z'S\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{r-1} \end{bmatrix} = 0. \tag{31}$$

Therefore, multiplying equation (30) on the left by z' gives

$$z'S\bar{y}(s) = z'SR_1\bar{u}(s). \tag{32}$$

Therefore,

$$[z' \quad -z'SR_1]\begin{bmatrix} S\bar{y}(s) \\ \bar{u}(s) \end{bmatrix} = 0 \qquad \forall s \leqq N - r + 1 \tag{33}$$

which implies that

$$\sum_{s=1}^{N-r+1} \begin{bmatrix} S\bar{y}(s) \\ \bar{u}(s) \end{bmatrix} [\bar{y}'(s)S' \quad \bar{u}'(s)] \text{ is singular.} \qquad \text{QED}$$

Theorem 2: If Σ *is completely observable and completely reachable, the matrices* F, G, *and* H *are* n $\times$ n, n $\times$ m, *and* p $\times$ n *respectively; then* $\exists$ *an* S: n $\times$ np *such that*

$$T \triangleq \sum_{s=1}^{n+nm} \begin{bmatrix} S\bar{y}(s) \\ \bar{u}(s) \end{bmatrix} [\bar{y}'(s)S' \quad \bar{u}(s)] > 0 \text{ almost surely} \tag{34}$$

where u(i) *are random variables having a joint nonlattice distribution.**

Proof: The first step in the proof consists of establishing Lemma 1.
Lemma 1: T > 0 *if and only if*

$$\sum_{s=1}^{n+nm} \begin{bmatrix} x(s) \\ \bar{u}(s) \end{bmatrix} [x'(s) \quad \bar{u}(s)] > 0. \tag{35}$$

Proof of Lemma 1: If $T \ngtr 0$, $\exists$ z' such that $z' = [z_1' \quad z_2'] \neq 0$, and

$$z_1' S\bar{y}(s) + z_2'\bar{u}(s) = 0 \quad \forall s. \tag{36}$$

Since Σ is completely observable, multiplying equation (17)

$$S\bar{y}(s) = x(s) + SR_1\bar{u}(s) \tag{37}$$

on the left by z_1' one obtains

$$z_1'S\bar{y}(s) = z_1'x(s) + z_1'SR_1\bar{u}(s). \tag{38}$$

Combining equations (36) and (38),

$$z_1'x(s) + (z_1'SR_1 - z_2')\bar{u}(s) = 0 \qquad \forall s, \tag{39}$$

and

$$[z_1' \;, z_1'SR_1 - z_2'] \neq 0;$$

for if $[z_1' \;, z_1'SR_1 - z_2'] = 0$, then $[z_1' \quad z_2'] = 0$, which contradicts $z \neq 0$. Therefore,

$$\sum_{s=1}^{n+nm} \begin{bmatrix} x(s) \\ \bar{u}(s) \end{bmatrix} [x'(s) \quad \bar{u}'(s)] \ngtr 0. \tag{40}$$

Now suppose $T > 0$. Let

$$\sum_{s=1}^{n+nm} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} [x'(s) \quad \bar{u}(s)] \ngtr 0.$$

Then $\exists$ a $z' = [z_1' \quad z_2'] \neq 0$ such that

$$z_1'x(s) + z_2'\bar{u}(s) = 0 \quad \forall s. \tag{41}$$

Again multiplying equation (37) by z_1' and using equation (41), it follows that

$$z_1'S\bar{y}(s) = -z_2'\bar{u}(s) + z_1'SR_1\bar{u}(s) \tag{42}$$

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* A nonlattice distribution is one in which no nonzero probability mass is concentrated on a surface less than the dimension of the random variable.

$$z_1' S\bar{y}(s) + (z_2' - z_1' SR_1) \quad \bar{u}(s) = 0 \qquad \forall s. \tag{43}$$

Once again $[z_1', (z_2' - z_1' SR_1)] \neq 0$ since $z \neq 0$, which contradicts $T > 0$. The proof of Lemma 2 will now complete the proof of Theorem 2.

Lemma 2: If (i) Σ *is completely controllable,* (ii) u(i) *are random variables with a joint nonlattice distribution, then the* $(n + nm) \times (n + nm)$ *matrix*

$$\begin{bmatrix} x(1) & \cdots & x(n + nm) \\ \bar{u}(1) & \cdots & \bar{u}(n + nm) \end{bmatrix} \tag{44}$$

is almost surely nonsingular.

Proof: From Lemma A.2 in Appendix A of Ref. 5 it follows that if

$$z(s + 1) = F_1 z(s) + G_1 u(s), \quad \text{with} \quad F_1 (n + nm) \times (n + nm), \tag{45}$$

then $[z(1), \cdots, z(n + nm)]$ is nonsingular with probability one, if F_1, G_1 is completely controllable. Further, from equations (8) and the definition of u, it is clear that

$$\begin{bmatrix} x(s+1) \\ u(s+1) \\ \cdot \\ \cdot \\ u(s+n) \end{bmatrix} = \begin{bmatrix} F & G & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & I & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & I \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \\ \cdot \\ \cdot \\ u(s+n-1) \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ I \end{bmatrix} . \; u\,(s + n) \tag{46}$$

Therefore, identifying F_1 and G_1 as

$$\begin{bmatrix} F & G & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & I & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ \cdot \\ \vdots \\ I \end{bmatrix}$$

respectively, equation (44) follows since it can easily be shown that $[F \quad G]$ controllable implies that $[F_1 \quad G_1]$ is completely controllable. Lemma 2 implies that the matrix in equation (40) is positive definite since in general A nonsingular $\rightarrow A^T A > 0$, which implies equation (34) by (i).

III. THE COMPUTATIONAL METHOD

The main part of the algorithm, as would be expected from the discussion in Section II, is to determine the right selector matrix. Once this has been done it is easy to solve for the parameters. In order to utilize certain properties of the matrix $[F' \; \vdots \; H]$, a class of matrices $\bar{\mathcal{S}} \subset \mathcal{S}$ is detailed below since $\bar{\mathcal{S}} \subset \mathcal{S}$, the number of different selector matrices one has to try, is smaller than $\mathcal{S}$.

Definition: $\bar{\mathcal{S}}$ is the set of matrices $S \in \mathcal{S}$ such that S is $l \times k$; then

(*i*) k is an integral multiple of p and further, if $k = rp$, then

$$(r - 1)p < j_l \leqq rp$$

where j_l is as defined in equation (5).

(*ii*)
$$j_i - j_{i-1} \leqq 2p.$$

Observe that by (i) there always exists an $S \in \bar{\mathcal{S}}$ such that equation (11) holds, since as can easily be proved, if

$$\rho([H', F'H', \cdots, F'^{s}H']) = \rho([H', F'H', \cdots, F'^{(s+1)}H']) = q$$

then

$$\rho([H', F' H', F'^{(s+j)}H']) = q \qquad j = 0, 1, 2, \cdots ;$$

so that in spite of condition (ii) in the above definition, there exist an $S \in \bar{\mathcal{S}}$ such that equation (11) holds.

(*iii*) The formulas (21) and (27) are still valid for any $S \in \bar{\mathcal{S}}$ satisfying equation (11), with n replaced everywhere by r defined in condition (i) in the definition of $\bar{\mathcal{S}}$ above.

Now from Theorems 1 and 2 and the above discussion, the direct algorithm can be summarized as follows.

It can be assumed without loss of generality that: (*i*) $N \geqq n + (m + 1)n$; that is, there is a sufficient number of observations to determine the internal description uniquely. n is the minimal dimension of the system to be identified. (*ii*) H has full rank. (*iii*) $n \geqq p$.

Step 1: Since $n \leqq \bar{N}/(m + 2)$, let $\bar{N}$ be the largest integer $\leqq N/(m + 2)$. Then $n \leqq N$. In order to arrive at the right S, one starts with an $S \in \mathcal{S}$ of row dimension $\bar{N}$ and tests the nonsingularity of

$$T \triangleq \sum_{s=1}^{(m+1)\bar{N}} \begin{bmatrix} S\bar{y}(s) \\ \bar{u}(s) \end{bmatrix} [\bar{y}'(s)S' \quad \bar{u}'(s)]$$

for all $S \in \mathcal{S}$ and having row dimension $\bar{N}$. If T is nonsingular, $\bar{N} = n$. If T is singular, then reduce the row dimension of S by 1 and repeat the test. Repeat the procedure until T becomes nonsingular. The row dimension of S will then be n; let r be as defined in condition (i) in the definition of $\bar{S}$; that is, S is $n \times rp$.

Step 2: Solve for F, R as follows.

$$[F \quad R] = \left\{ \sum_{s=1}^{(m+1)\bar{N}} Sy(s+1)[\bar{y}(s)S' \quad \bar{u}'(s)] \right\} T^{-1}$$

Step 3: Solve for G from the following formula.

$$G = R_0 + FR_1 + \cdots + F^{r-1}R_{r-1},$$

where $S{:}n \times rp$ and R_i are the partitions of R such that

$$R = [R_0 \quad R_1 \cdots R_{r-1}]$$

and $R_i = n \times m \; i = 0, \cdots, r-1$. H can be assumed to be $[I \; 0]$ where the identity has dimension p.

In the case when Σ is a continuous-time system, the algorithm presented above applies with appropriate modifications. In the definitions of $\bar{y}(s)$ and $\bar{u}(s)$, s now assumes values in $\mathcal{R}$ and $y(s+i)$ should be replaced by $y^{(i)}(s)$ evaluated at s. The summation signs should be replaced by integration over an interval. The formulas for the parameters become

$$[F \quad R] = \int_t^{t+\epsilon} S\bar{y}^{(r)}(s)[\bar{y}'(s)S' \quad \bar{u}'(s)]\, ds\, T^{-1}$$

$$T = \int_t^{t+\epsilon} \begin{bmatrix} \bar{S}y(s) \\ \bar{u}(s) \end{bmatrix} [\bar{y}'(s)S' \quad \bar{u}'(s)]\, ds.$$

G can be obtained from R exactly as in the above algorithm for the discrete time case.

In the case when observations are contaminated with noise, this method can be generalized to yield consistent estimates for the parameters (see Ref. 5).

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