Digital PM Spectra by Transform Techniques

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We derive an expression for the power spectrum of a class of nonstationary processes with periodicity in the two-dimensional autocorrelation function such that $R(t_1, t_2) = R(t_1 + T, t_2 + T)$. *Such a class includes many of the common digital signals. The method of derivation is based on the double Fourier transform which relates the spectrum of any signal to its autocorrelation function. This points to a very simple method of finding digital spectra.*

The results are applied to derive the general expression for the power spectrum of a wave phase modulated with a pulse stream $\sum_{-\infty}^{\infty} a_n g(t - nT)$. *The only restrictions on the pulse stream are that the* a_n*'s are independent and have identical probability distributions, and* $g(t)$ *is integrable and of finite length.*

I. INTRODUCTION

The power spectrum of a signal $x(t)$ can be defined in many ways. Every definition, however, has to yield some measure of the expected power at the output of a narrow bandpass filter, as a function of the center frequency of the filter.

If $x(t)$ is deterministic then the square of the magnitude of its Fourier transform represents energy density as a function of frequency. If the signal has finite length the energy is finite and we can define the power density to be the energy density divided by the length of the signal. If the signal has infinite length the energy may be infinite, but for realizable signals we can still define the power spectrum by operating on a finite time interval and finding the limit as the interval approaches infinity. This limit may include a set of δ-functions.

If $x(t)$ is a random signal the direct way of defining the power spectrum is to find the Fourier transform of a sample function $x_o(t)$ on a finite time interval T_o, take the magnitude square, divide by T_o, average over all possible $x_o(t)$, and finally take the limit as $T_o \rightarrow \infty$.[1]

If $x(t)$ is stationary, the power spectrum is proportional to the Fourier transform of the autocorrelation function. This is a very useful property which often simplifies the task of finding the power spectrum.

The use of transform techniques can be extended to nonstationary processes by means of a double Fourier transform.[2] There is a very simple relationship between the double Fourier transform of the autocorrelation function $R(t_1, t_2)$ of $x(t)$ and the expected energy (or power) as a function of frequency. Through the proper definitions this technique includes both stationary and deterministic processes as special cases.

One would not expect that the term power spectrum would have much meaning in the general case of nonstationary processes. For one thing it would require infinite observation time to measure the spectrum. For some special classes of nonstationary processes we can talk about power spectra. For instance if the autocorrelation function is a function of the time difference and only slowly varying with time we can talk about locally stationary processes.

Another class of signals where power spectrum has a meaning is where periodicity in $R(t_1, t_2)$ exists. We will study signals for which

$$R(t_1, t_2) = R(t_1 + T, t_2 + T). \tag{1}$$

We will show that for this class of signals the transform technique can be used to derive a simple expression, equation (26), for the power spectrum. It is believed that this method of arriving at the power spectrum is simpler than the direct way used by Anderson and Salz in their treatment of digital FM spectra.[3]

Equation (26) is given in such a form that it can easily be used for many of the common digital signals. It should be especially useful when the digital pulses are overlapping. We apply it to digital PM which was not included in Ref. 3.

II. GENERAL CONSIDERATIONS

A signal $x(t)$ has the autocorrelation†

$$R(t_1, t_2) = E[x(t_1) \cdot x^*(t_2)]. \tag{2}$$

† The symbol* denotes complex conjugation.

We define†

$$\Gamma(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(t_1, t_2) \exp(-j\omega_1 t_1 + j\omega_2 t_2)\, dt_1\, dt_2 . \quad (3)$$

Papoulis has derived a number of properties for the transform pair (see Chapter 12 of Ref. 2).

Define

$$\rho(\tau) = \int_{-\infty}^{\infty} R(t + \tau, t)\, dt \quad (4)$$

and

$$W(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\tau) e^{-j\omega\tau}\, d\tau. \quad (5)$$

$W(\omega)$ is the average energy spectrum and it can be shown that

$$W(\omega) = \frac{1}{2\pi} \Gamma(\omega, \omega). \quad (6)$$

If $x(t)$ has infinite length we can define the average power spectrum as

$$S(\omega) = \lim_{T_o \to \infty} \frac{W_o(\omega)}{T_o} \quad (7)$$

where $W_o(\omega)$ is the energy spectrum of $x(t)$ taken over an interval of length T_o.

Combining equations (2), (3), (6), and (7) we have

$$S(\omega) = \lim_{T_o \to \infty} \frac{\int_{-T_o/2}^{T_o/2} \int_{-T_o/2}^{T_o/2} E[x(t_1) \cdot x^*(t_2)] \exp[-j\omega(t_1 - t_2)]\, dt_1\, dt_2}{2\pi T_o}. \quad (8)$$

If we change the order of integration and expectation and observe that the double integral is separable and that one integral is the conjugate of the other we get

$$S(\omega) = \lim_{T_o \to \infty} \frac{E\left[\left| \int_{-T_o/2}^{T_o/2} x(t) e^{-j\omega t}\, dt \right|^2\right]}{2\pi T_o} \quad (9)$$

which is the definition of power spectrum given by Rice.[1]

A large class of digital signals has the property

$$R(t_1, t_2) = R(t_1 + T, t_2 + T). \quad (10)$$

Papoulis has shown that in this case $\Gamma(\omega_1, \omega_2)$ consists of line masses

† Notice the signs in the exponential.

in the (ω_1, ω_2)-plane. The line mass on $\omega_1 = \omega_2$ gives the average power spectrum $S(\omega)$ such that

$$\Gamma(\omega_1, \omega_2) = \Gamma_r(\omega_1, \omega_2) + (2\pi)^2 S(\omega_1)\, \delta(\omega_1 - \omega_2) \tag{11}$$

where $\Gamma_r(\omega_1, \omega_2)$ has no line masses on $\omega_1 = \omega_2$. From the definition of $\Gamma(\omega_1, \omega_2)$, equation (3), we get

$$\Gamma(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(t_1, t_2) \exp(-j\omega_1 t_1 + j\omega_2 t_2)\, dt_1\, dt_2 . \tag{12}$$

Let

$$t_1 = t + \tau \tag{13}$$

$$t_2 = t \tag{14}$$

then

$$\Gamma(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(t + \tau, t) \cdot \exp[-j\omega_1 \tau - j(\omega_1 - \omega_2)t]\, d\tau\, dt. \tag{15}$$

We divide the t-axis in intervals of length T and get

$$\Gamma(\omega_1, \omega_2) = \sum_{q=-\infty}^{\infty} \int_{t=qT}^{(q+1)T} \int_{\tau=-\infty}^{\infty} R(t + \tau, t) \cdot \exp[-j\omega_1 \tau - j(\omega_1 - \omega_2)t]\, d\tau\, dt. \tag{16}$$

From equation (10) it follows that we have periodicity in t; in each interval $t \epsilon \{qT, (q + 1)T\}$ we let

$$t \to t + qT$$

and equation (16) becomes

$$\Gamma(\omega_1, \omega_2) = \sum_{q=-\infty}^{\infty} \int_{t=0}^{T} \int_{\tau=-\infty}^{\infty} R(t + \tau, t) \exp(-j\omega_1 \tau) \cdot \exp[-j(\omega_1 - \omega_2)(t + qT)]\, d\tau\, dt \tag{17}$$

or

$$\Gamma(\omega_1, \omega_2) = \int_{t=0}^{T} \int_{\tau=-\infty}^{\infty} R(t + \tau, t) \exp[-j\omega_1 \tau - j(\omega_1 - \omega_2)t]\, d\tau\, dt \cdot \frac{2\pi}{T} \sum_{-\infty}^{\infty} \delta\left(\omega_1 - \omega_2 - \frac{2\pi n}{T}\right). \tag{18}$$

The set of δ-functions gives us the line masses mentioned earlier. $S(\omega)$ is the mass on $\omega_1 = \omega_2$, and from equations (18) and (11)

$$S(\omega) = \frac{1}{2\pi T} \int_{t=0}^{T} \int_{\tau=-\infty}^{\infty} R(t + \tau, t) e^{-j\omega\tau} \, d\tau \, dt. \tag{19}$$

We return to the original variables t_1 and t_2 which are absolute times. Using equations (13) and (14) we get

$$S(\omega) = \frac{1}{2\pi T} \int_{t_2=0}^{T} \int_{t_1=-\infty}^{\infty} R(t_1, t_2) \exp[-j\omega(t_1 - t_2)] \, dt_1 \, dt_2 . \tag{20}$$

We segment the t_1-axis in intervals of length T

$$S(\omega) = \frac{1}{2\pi T} \sum_{q=-\infty}^{\infty} \int_{t_1=qT}^{(q+1)T} \int_{t_2=0}^{T} R(t_1, t_2) \cdot \exp[-j\omega(t_1 - t_2)] \, dt_2 \, dt_1 . \tag{21}$$

The kth term of the sum is

$$S_k(\omega) = \frac{1}{2\pi T} \int_{t_1=kT}^{(k+1)T} \int_{t_2=0}^{T} R(t_1, t_2) \exp[-j\omega(t_1 - t_2)] \, dt_1 \, dt_2 . \tag{22}$$

From the definition of autocorrelation of equation (2):

$$R(t_1, t_2) = R^*(t_2, t_1). \tag{23}$$

Substitute equation (23) into equation (22) and change the variables such that

$$t_1 \to t_2 - kT \quad \text{and} \quad t_2 \to t_1 - kT.$$

Because $R(t_1, t_2)$ is periodic as shown by equation (10) it follows that

$$S_k(\omega) = \frac{1}{2\pi T} \int_{t_1=-kT}^{(-k+1)T} \int_{t_2=0}^{T} R^*(t_1, t_2) \exp[j\omega(t_1 - t_2)] \, dt_2 \, dt_1 \tag{24}$$

or

$$S_k(\omega) = S^*_{(-k)}(\omega). \tag{25}$$

We can then combine terms in equation (21) where $q > 0$ in pairs and equation (21) becomes

$$S(\omega) = \frac{1}{2\pi T} \left\{ \int_0^T \int_0^T R(t_1, t_2) \exp[-j\omega(t_1 - t_2)] \, dt_1 \, dt_2 + 2 \operatorname{Re} \left[\sum_{q=1}^{\infty} \int_{t_1=qT}^{(q+1)T} \int_0^T R(t_1, t_2) \exp[-j\omega(t_1 - t_2)] \, dt_1 \, dt_2 \right] \right\}. \tag{26}$$

III. DIGITAL PHASE MODULATION

Represent the signal by

$$x(t) = V_c \cos (\omega_c t + \phi(t)) \tag{27}$$

where

$$\phi(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT). \tag{28}$$

We assume that the a_n's are independent with identical probability distribution and $g(t)$ is integrable and of finite length such that $g(t) \equiv 0$ outside the time interval $\{0, pT\}$. Otherwise $g(t)$ is arbitrary.

The Appendix shows by means of transforms that the one sided spectrum of $x(t)$ is

$$S(\omega_c + \omega) = \frac{V_c^2}{2} \Gamma(\omega, \omega) \tag{29}$$

where ω is the difference between the actual frequency and the carrier frequency. $\Gamma(\omega_1, \omega_2)$ is defined by equation (3) with

$$R(t_1, t_2) = E\{\exp [j\phi(t_1) - j\phi(t_2)]\}. \tag{30}$$

If we substitute equation (28) into equation (30) and use the fact that the a_n's are independent we get

$$R(t_1, t_2) = \prod_{n=-\infty}^{\infty} E[\exp \{ja_n[g(t_1 - nT) - g(t_2 - nT)]\}]. \tag{31}$$

To show that $R(t_1 + T, t_2 + T) = R(t_1, t_2)$ let $t_1 \to t_1 + T$ and $t_2 \to t_2 + T$ and reindex such that $n \to n + 1$.

$$R(t_1 + T, t_2 + T) = \prod_{n=-\infty}^{\infty} E[\exp \{ja_{n+1}[g(t_1 - nT) - g(t_2 - nT)]\}]. \tag{32}$$

Since we assumed identical probability distributions the periodicity follows.

In order to use equation (26) we now have to evaluate $R(t_1, t_2)$ for

$$0 \leqq t_1 \leqq \infty \quad \text{and} \quad 0 \leqq t_2 \leqq T.$$

Since $t_2 \in \{0, T\}$, $g(t_2 - nT)$ will contribute in the factors when

$$-(p - 1) \leqq n \leqq 0.$$

Now $t_1 \in \{qT, (q+1)T\}$ and $g(t_1 - nT)$ will contribute when

$$-(p-1) + q \leqq n \leqq q.$$

In order to determine for which n's only one will contribute and for which n's both will contribute to the factors in equation (31) let us look at Fig. 1. We see that if $q < p$ then there will be factors where both contribute. Equation (21) becomes

$$R(t_1, t_2) = \prod_{-p+1}^{-p+q} E\{\exp[-jag(t_2 - nT]\} \cdot \prod_{-p+q+1}^{0} E(\exp\{ja[g(t_1 - nT) - g(t_2 - nT)]\}) \cdot \prod_{1}^{q} E\{\exp[jag(t_1 - nT)]\}. \tag{33}$$

In the first product we let $n \to n - p$ and in the second $n \to n - p + q$. We also let $t_1 \to t_1 + qT$ and this part of equation (26) becomes

$$\frac{1}{\pi T} \operatorname{Re}\Bigg\{\sum_{q=1}^{p-1} e^{-jq\omega T} \int_0^T \int_0^T \prod_{n=1}^{q} [E(\exp\{jag[t_1 + (q-n)T]\}) \cdot E(\exp\{-jag[t_2 + (p-n)T]\})] \cdot \prod_{n=1}^{p-q} E[\exp(ja\{g[t_1 + (p-n)T] - g[t_2 + (p-q-n)T]\})] \cdot \exp[-j\omega(t_1 - t_2)]\, dt_1\, dt_2\Bigg\}. \tag{34}$$

The first double integral in equation (26) we get from equation (33) with $q = 0$. Let $n \to -n$,

$$\frac{1}{2\pi T} \int_0^T \int_0^T \prod_{n=0}^{p-1} E(\exp\{ja[g(t_1 + nT) - g(t_2 + nT)]\}) \cdot \exp[-j\omega(t_1 - t_2)]\, dt_1\, dt_2 . \tag{35}$$

We have yet to find the terms where $p \leqq q \leqq \infty$. If we look at Fig. 1, we see that

$$R(t_1, t_2) = \prod_{-p+1}^{0} E\{\exp[-jag(t_2 - nT)]\} \cdot \prod_{-p+q+1}^{q} E\{\exp[jag(t_1 - nT)]\}. \tag{36}$$

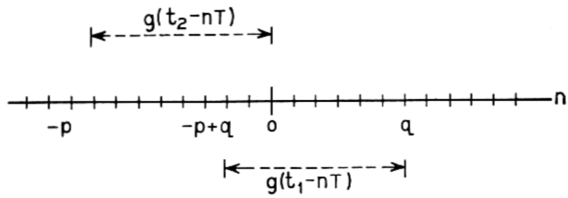

Fig. 1 — Contributions to $R(t_1,\ t_2)$.

In the first product let $n \to -n$, and in the second let $n \to -n + q$, and finally $t_1 \to t_1 + qT$. Substituting the result into terms in equation (26) where $q \geqq p$ we get

$$\frac{1}{\pi T}\left| \int_0^T \prod_{n=0}^{p-1} E\{\exp\,[jag(t+nT)]\}\cdot e^{-i\omega t}\,dt \right|^2 \cdot \mathrm{Re}\cdot\left[\sum_{q=p}^{\infty} e^{-iq\omega T}\right]$$

$$= \frac{1}{\pi T}\left| \int_0^T \prod_{n=0}^{p-1} E\{\exp\,[jag(t+nT)]\}e^{-i\omega t}\,dt \right|^2$$

$$\cdot\left\{\mathrm{Re}\left[\sum_{q=1}^{\infty} e^{-jq\omega T} - \sum_{q=1}^{p-1} e^{-jq\omega T}\right]\right\} \qquad (37)$$

which becomes

$$\frac{1}{\pi T}\left| \int_0^T \prod_{n=0}^{p-1} E\{\exp\,[jag(t+nT)]\}e^{-i\omega t}\,dt \right|^2$$

$$\cdot\left\{\frac{\pi}{T}\sum_{m=-\infty}^{\infty}\delta\left[\omega-\frac{2\pi m}{T}\right]+\frac{1}{2}-\cos\left[\frac{(p-1)\omega T}{2}\right]\frac{\sin\left(\frac{p\omega T}{2}\right)}{\sin\frac{\omega T}{2}}\right\} \qquad (38)$$

Substituting equations (35), (34), and (38) into equations (29) and (26)

$$S(\omega_c+\omega) = \frac{V_c^2}{2}\left\{\frac{1}{2\pi T}\left[\int_0^T\int_0^T \prod_{n=0}^{p-1} E(\exp\,\{ja[g(t_1+nT)-g(t_2+nT)]\})\right.\right.$$

$$\cdot\exp\,[-j\omega(t_1-t_2)]\,dt_1\,dt_2 + \left|\int_0^T \prod_{n=0}^{p-1} E\{\exp\,[jag(t+nT)]\}e^{-i\omega t}\,dt\right|^2$$

$$\left\{1-2\cdot\cos\left[\frac{(p-1)\omega T}{2}\right]\frac{\sin\left(\frac{p\omega T}{2}\right)}{\sin\left(\frac{\omega T}{2}\right)}\right\} + 2R_e\left\{\sum_{q=1}^{p-1} e^{-iq\omega T}\right.$$

$$
\cdot \int_0^T \int_0^T \prod_{n=1}^{q} [E(\exp \{jag[t_1 + (q - n)T]\})
$$

$$
\cdot E(\exp \{-jag[t_2 + (p - n)T]\})] \prod_{n=1}^{p-q} E[\exp (ja\{g[t_1 + (p - n)T]
$$

$$
- g[t_2 + (p - q - n)T]\})] \exp [-j\omega(t_1 - t_2)] \, dt_1 \, dt_2 \Big\} \Bigg]
$$

$$
+ \frac{1}{T^2} \sum_{m=-\infty}^{\infty} \left| \int_0^T \prod_{n=0}^{p-1} E\{\exp [jag(t + nT)]\}
$$

$$
\cdot \exp \left(-j \frac{2\pi mt}{T}\right) dt \right|^2 \delta\left(\omega - \frac{2\pi m}{T}\right) \Bigg\} \tag{39}
$$

IV. EXAMPLE OF PM SPECTRA

A complete evaluation of equation (39) is not attempted here. We compute $S(\omega_c + \omega)$ for the case when $g(t)$ is a rectangular pulse of duration $\tau < T$, and height 1. We assume that a has r levels which are equidistant and equally probable such that

$$
a^{(k)} = \alpha_o + \frac{k - 1}{r} \alpha \tag{40}
$$

$$
P\{a^{(k)}\} = \frac{1}{r}. \tag{41}
$$

First we rewrite equation (39) with p = 1:

$$
S(\omega_c + \omega) = \frac{V_c^2}{2} \left[\frac{1}{2\pi T} \int_0^T \int_0^T E(\exp \{ja[g(t_1) - g(t_2)]\}) \right.
$$

$$
\cdot \exp [-j\omega(t_1 - t_2)] \, dt_1 \, dt_2
$$

$$
+ \frac{1}{T^2} \left| \int_0^T E[e^{jag(t)}] \exp [-jwt] \, dt \right|^2
$$

$$
\left. \cdot \left(\sum_{m=-\infty}^{\infty} \delta\left\{\omega - \frac{2\pi m}{T}\right\} - \frac{T}{2\pi} \right) \right]. \tag{42}
$$

The expected values in equation (42) will be

$$
E(\exp \{ja[g(t_1) - g(t_2)]\})
$$

$$
= \frac{1}{r} \sum_{k=1}^{r} \exp \left\{ j \left[\alpha_0 + \frac{k - 1}{r} \alpha \right] [g(t_1) - g(t_2)] \right\} \tag{43}
$$

and

$$E[e^{jag(t)}] = \frac{1}{r} \sum_{k=1}^{r} \exp \left[j\left(\alpha_o + \frac{k-1}{r} \alpha\right) g(t) \right]. \tag{44}$$

From equations (43) and (44) it follows that

$$\int_0^T \int_0^T E(\exp \{ja[g(t_1) - g(t_2)]\}) \exp [-j\omega(t_1 - t_2)] \, dt_1 \, dt_2$$

$$= \frac{1}{r} \sum_{k=1}^{r} \left| \int_0^T \exp \left[j\left(\alpha_o + \frac{k-1}{r} \alpha\right) \right] g(t) - j\omega t \, dt \right|^2 \tag{45}$$

since we can separate the integrals and because the integral over t_2 is the conjugate of the integral over t_1. Also

$$\left| \int_0^T E[e^{jag(t)}] e^{-j\omega t} \, dt \right|^2$$

$$= \frac{1}{r^2} \left| \sum_{k=1}^{r} \int_0^T \exp \left[j\left(\alpha_o + \frac{k-1}{r} \alpha\right) \right] g(t) - j\omega t \, dt \right|^2. \tag{46}$$

Let us set

$$\int_0^T \exp \left[j\left(\alpha_o + \frac{k-1}{r} \alpha\right) g(t) - j\omega t \right] dt = F_k(\omega) \tag{47}$$

Then equation (42) becomes

$$S(\omega_c + \omega) = \frac{V_c^2}{2} \left\{ \frac{1}{2\pi T} \left[\sum_{k=1}^{r} \frac{1}{r} \mid F_k(\omega) \mid^2 - \left| \sum_{k=1}^{r} \frac{1}{r} F_k(\omega) \right|^2 \right] \right.$$

$$\left. + \frac{1}{T^2} \sum_{m=-\infty}^{\infty} \left| \sum_{k=1}^{r} \frac{1}{r} F_k\left(\frac{2\pi m}{T}\right) \right|^2 \delta\left(\omega - \frac{2\pi m}{T}\right) \right\}. \tag{48}$$

With $g(t) = 1$ for $t \in \{0, \tau\}$ and 0 for $t \in \{\tau, T\}$ we get (after some trigonometric manipulations)

$$S(\omega_c + \omega) = \frac{V_c^2}{2} \left\{ \frac{\tau^2}{2\pi T} \left[\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} \right]^2 \left\{ 1 - \frac{1}{r^2} \left[\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2r}\right)} \right]^2 \right\} \right.$$

$$+ \sum_{m=-\infty}^{\infty} \left\{ \frac{\tau^2}{r^2 T^2} \left[\frac{\sin\left(\frac{m\pi\tau}{T}\right)}{\frac{m\pi\tau}{T}} \right]^2 \left[\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2r}\right)} \right]^2 + \left(1 - \frac{\tau}{T}\right)^2$$

$$\cdot\left[\frac{\sin m\pi\left(1-\frac{\tau}{T}\right)}{m\pi\left(1-\frac{\tau}{T}\right)}\right]^2 + \frac{2}{r}\frac{\tau}{T}\left(1-\frac{\tau}{T}\right)\frac{\sin\left(\frac{m\pi\tau}{T}\right)}{\frac{m\pi\tau}{T}}\cdot\frac{\sin\left[m\pi\left(1-\frac{\tau}{T}\right)\right]}{m\pi\left(1-\frac{\tau}{T}\right)}$$

$$\cdot\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2r}\right)}\cdot\cos\left[m\pi+\alpha_o+\frac{\alpha(r-1)}{2r}\right]\delta\left[\omega-\frac{2\pi m}{T}\right]\Bigg]\Bigg\}. \tag{49}$$

The continuous part of the spectrum is independent of α_o. Only the random deviations from α_o, and the pulse length determine the continuous part of the spectrum. The term α_o is related to the periodicity of the signal and consequently the spectrum spikes will be functions of α_o.

There are some interesting special cases of equation (49). If $(\alpha/r) = n\cdot 2\pi$ then the continuous part of the spectrum disappears. We would expect this because from equation (40) it follows that during each pulse a phase excursion of α_o plus an integer number of 2π is made. This is the same as modulating with a periodic signal where in one period we have $\alpha_o g(t)$.

If $\tau \to T$ all spikes except the carrier spike go to zero; this gives

$$S(\omega_c+\omega) = \frac{V_c^2}{2}\left\{\frac{T}{2\pi}\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\right]^2 \left\{1-\frac{1}{r^2}\left[\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2r}\right)}\right]^2\right\}\right.$$

$$\left. + \frac{1}{r^2}\left[\frac{\sin\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2r}\right)}\right]^2 \delta(\omega)\right\} \tag{50}$$

which is a familiar result. This is independent of α_o because in this case α_o is just a constant phase angle added to the carrier at all times.

It is interesting to note that if $\alpha = n\cdot 2\pi$, the carrier spike will also disappear.

V. CONCLUSION

There is a large class of nonstationary processes which yield signals with periodicity in the two dimensional autocorrelation function such that $R(t_1+T, t_2+T) = R(t_1, t_2)$. Such a class includes many of the common digital signals.

A simple expression has been derived for the power spectrum of such a signal. It was done by means of the double Fourier transform which relates the spectrum of any signal to its autocorrelation function. For this class of signals the method is very powerful.

The results were applied to get the general expression for the power spectrum of a wave phase modulated with a pulse stream $\sum_{-\infty}^{\infty} a_n g(t - nT)$. The only restrictions on the pulse stream are that the a_n's are independent and have identical probability distributions and, $g(t)$ is integrable and of finite length. As an example a rectangular pulse of length $\tau < T$ was considered.

The same method can be used for digital FM to arrive at the expression given by Anderson and Salz.[3] It can also be shown that the expression given here for PM goes into the one for FM given in Ref. 3 with the accrued phase per pulse equal to 0. This corresponds to $\alpha_r = 0$ in Ref. 3, a case which was not treated there. We get this FM case by substituting

$$\omega_d \int_0^t g(t')\, dt'$$

for $g(t)$ in our equation (39) where ω_d is a frequency deviation parameter defined in Ref. 3.

APPENDIX

Carrier Translation

An angle modulated signal is represented by

$$x(t) = V_c \cos [\omega_c t + \phi(t)]. \tag{51}$$

The autocorrelation function of $x(t)$ is

$$R_x(t_1, t_2) = E[x(t_1) \cdot x^*(t_2)] \tag{52}$$

which becomes

$$\begin{aligned} R_x(t_1, t_2) = \frac{V_c^2}{4} (&\exp [j\omega_c(t_1 + t_2)] E\{\exp [j\phi(t_1) + j\phi(t_2)]\} \\ &+ \exp [-j\omega_c(t_1 + t_2)] E\{\exp [-j\phi(t_1) - j\phi(t_2)]\} \\ &+ \exp [j\omega_c(t_1 - t_2)] E\{\exp [j\phi(t_1) - j\phi(t_2)]\} \\ &+ \exp [-j\omega_c(t_1 - t_2)] E\{\exp [-j\phi(t_1) + j\phi(t_2)]\}). \end{aligned} \tag{53}$$

The average energy in the two first terms will for most practical cases of modulation go to zero. To show this we take the double Fourier transform of the first term

$$\Gamma_1(\omega_1, \omega_2) = \frac{V_c^2}{4} \mathfrak{F}_2\{\exp [j\omega_c(t_1 + t_2)]\} * \mathfrak{F}_2(E\{\exp [j\phi(t_1) + j\phi(t_2)]\}) \tag{54}$$

or

$$\Gamma_1(\omega_1, \omega_2) = \frac{V_c^2}{4} (2\pi)^2 \cdot \delta(\omega_1 - \omega_c) \cdot \delta(\omega_2 + \omega_c) * \mathfrak{F}_2(E\{\exp [j\phi(t_1) + j\phi(t_2)]\}) \tag{55}$$

where $\mathfrak{F}_2$ is a double Fourier transform operator and * means convolution in both t_1 and t_2. The plus sign in the second δ-function comes from the plus sign in front of ω_2 in the definition of the double Fourier transform of equation (3). Equation (55) shows that we can find $\Gamma_1(\omega_1, \omega_2)$ by first finding

$$\mathfrak{F}_2(E\{\exp [j\phi(t_1) + j\phi(t_2)]\})$$

and then move it in the (ω_1, ω_2)-plane so that the point (0, 0) falls at $(\omega_c, -\omega_c)$. (See Fig. 2.) To find the average energy we set $\omega_1 = \omega_2$. We see then, if

$$\mathfrak{F}_2(E\{\exp [j\phi(t_1) + j\phi(t_2)]\})$$

does not have any significant mass density for frequencies of the order of ω_c there will be no contribution from $\Gamma_1(\omega_1, \omega_2)$ falling on the line $\omega_1 = \omega_2$. This will be the assumption here, that is, the modulating functions do not produce sidebands as far as ω_c away from the carrier. Then it also follows that the second term in equation (53) will not contribute.

Now let us set

$$E\{\exp [j\phi(t_1) - j\phi(t_2)]\} = R(t_1, t_2). \tag{56}$$

Thus

$$E\{\exp [-j\phi(t_1) + j\phi(t_2)]\} = R^*(t_1, t_2). \tag{57}$$

If

$$\Gamma(\omega_1, \omega_2) = \mathfrak{F}_2[R(t_1, t_2)] \tag{58}$$

then from equation (3) it follows

$$\mathfrak{F}_2[R^*(t_1, t_2)] = \Gamma(-\omega_2, -\omega_1). \tag{59}$$

Taking the double Fourier transform of equation (53) then yields

$$\Gamma_x(\omega_1, \omega_2) = \frac{V_c^2}{4}(2\pi)^2 \cdot [\delta(\omega_1 - \omega_c) \cdot \delta(\omega_2 - \omega_c) * \Gamma(\omega_1, \omega_2) + \delta(\omega_1 + \omega_c) \cdot \delta(\omega_2 + \omega_c) * \Gamma(-\omega_2, -\omega_1)]. \quad (60)$$

If we look at Fig. 2 we see that the convolution just means sliding $\Gamma(\omega_1, \omega_2)$ and $\Gamma(-\omega_2, -\omega_1)$ along the line $\omega_1 = \omega_2$. We want the portion of $\Gamma_x(\omega_1, \omega_2)$ that is located on the line. The sliding process takes the portions of $\Gamma(\omega_1, \omega_2)$ and $\Gamma(-\omega_2, -\omega_1)$ that are already on the line and moves them to the points (ω_c, ω_c) and $(-\omega_c, -\omega_c)$, respectively. Setting $\omega_1 = \omega_2 = \omega_c + \omega$ in equation (60) and performing the convolution we get

$$\Gamma_x(\omega_c + \omega, \omega_c + \omega) = \frac{V_c^2}{4}[\Gamma(\omega, \omega) + \Gamma(-\omega, -\omega)]. \quad (61)$$

The second part in equation (61) is just the mirror image of the first and the one-sided spectrum becomes

$$S(\omega_c + \omega) = \frac{V_c^2}{2}\Gamma(\omega, \omega) \quad (62)$$

where

$$\Gamma(\omega_1, \omega_2) = \mathfrak{F}_2(E\{\exp[j\phi(t_1) - j\phi(t_2)]\}).$$

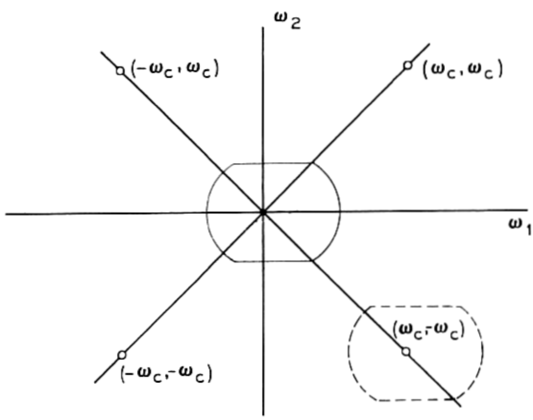

Fig. 2 — Carrier translation.

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