THE BELL SYSTEM TECHNICAL JOURNAL

Volume 48 | February 1969 | Number 2

The Application of Delta Modulation to Analog-to-PCM Encoding

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(Manuscript received September 24, 1968)

This paper describes a pulse code modulation (PCM) encoder that employs a delta modulator as the analog-to-digital converter. A tapped binary shift register and an up-down counter cause delta modulation signals to be converted to a uniformly quantized PCM format. When tap weights are optimized with respect to a minimum mean square error criterion, the number of shift register stages necessary to obtain a fixed level of output quantizing noise varies inversely with delta modulation sampling rate. Because the tap weights may be rounded to a modest number of binary places, the arithmetic operations are simple to implement. A significant portion of the delta modulation-to-PCM converter may be time shared among several signals.

I. INTRODUCTION

Although delta modulation (ΔM) has been the subject of many theoretical and experimental studies, instances of its practical application are, to date, rare. In commercial systems, the most prominent digital representation of continuous signals is pulse code modulation (PCM). Relative to PCM, ΔM has the advantage of admitting simpler means of analog-to-digital and digital-to-analog conversion. However, the ΔM sampling rate is higher than that of PCM and in many cases the transmission rate is also higher.

Since the discovery of delta modulation in the early 1950's,[1, 2] investigators have proposed various modifications of the elementary, single-integration ΔM system (for example, double integration ΔM[2], high information ΔM[3], continuous ΔM[4]) for the purpose of decreasing the required sampling rate. Generally, lower rates are achieved by means of operations on the continuous signals presented to and appearing in the delta modulator and thus at the expense of equipment complexity.

In this paper we propose a PCM encoder that incorporates the simple single-integration delta modulator and a transversal digital filter that converts the ΔM sequence to a uniformly quantized PCM sequence. For this encoder, we demonstrate an inverse relationship between the ΔM sampling rate and the number of digital filter stages required to achieve a specified level of PCM quantizing noise power. With this encoder the advantages of ΔM, in particular the simple means of analog-to-digital conversion, may be combined with those of PCM: a linear representation of the continuous signal and in many cases a lower transmission rate than that required by simple ΔM. This union of ΔM and PCM is achieved with a digital filter which may be readily implemented with integrated circuit devices.

In the following sections of this paper, we demonstrate the validity of a transversal filter as a ΔM-PCM converter by reference to a ΔM-analog-PCM signal processing sequence. We then criticize the digital filter design method that is based on simulation of the analog system and we proceed to demonstrate the relevance of a mean square error design criterion. The associated synthesis method results in a class of encoders for each output signal-to-noise ratio. Within each class the required sampling rate is inversely related to the number of digital filter stages.

II. THE ΔM SYSTEM

Fig. 1 shows a delta modulator that transforms a continuous input signal $y(t)$ to the binary sequence

$$\{b_n\} = \cdots, b_{-1}, b_0, b_1, \cdots \tag{1}$$

in which b_i may have the value $+1$ or -1. These binary symbols are generated at τ second intervals; we assume that $y(t)$ is a member of a class of signals band-limited to W Hz. Thus we can represent $y(t)$ by a sequence of PAM samples generated at the rate of $2W$ per second. In order to simplify the ΔM-PCM conversion, we constrain the delta modulator such that its output rate, $1/\tau$, is an integral multiple of the PAM rate. The ratio of its input (ΔM) rate to its output (PAM) rate is a basic parameter of the ΔM-PCM converter. This ratio is the integer

$$R = \frac{1}{2W\tau}. \tag{2}$$

The other basic parameter is the step size, δ, the gain of the modulator feedback loop. Assume that the linear filter in this loop is an ideal integrator with impulse response

$$\begin{aligned} f(t) &= 1 \quad \text{for} \quad t \geqq 0 \\ &= 0 \quad \text{for} \quad t < 0 \end{aligned} \tag{3}$$

so that the integrated ΔM signal, $x(t)$, contains a step of height $\pm\delta$ at each ΔM sampling instant and is constant between sampling instants. R and δ determine the inherent quantizing noise of the delta modulator, and δ, in a practical system, is selected to provide a proper balance between granular[9,11] quantizing noise (that predominates for high values of δ) and slope-overload noise (the predominant form for low values of δ).[5,6,7] In the analysis to follow, assume that no slope overload occurs; to make this assumption valid, we specify the parameter δ such that δ/τ, the maximum slope of the analog signal that may be reconstructed from $\{b_n\}$, is four times the rms slope of $y(t)$. If $y(t)$ is a sample function of a stationary time series with power density spectrum $Y(f)$, the spectrum of the time derivative of $y(t)$ is $(2\pi f)^2 Y(f)$, and the parameter β, the ΔM step size as a multiple of rms signal amplitude, is

$$\beta = \frac{\delta}{\sigma} = 8\pi\tau f_e \tag{4}$$

where we define

$$f_e = \left[\frac{\int_0^W f^2 Y(f)\, df}{\int_0^W Y(f)\, df} \right]^{\frac{1}{2}} \tag{5}$$

as the "effective frequency width" of the analog input. If f_e is expressed as a constant times W, the cutoff frequency of $Y(f)$, it is clear that equation (4) is proportional to τW, so that for a given spectral density function, βR is a constant.

The ΔM design condition of equation (4) is identical to the one adopted by van de Weg[9] in his analysis of the granular quantizing noise of ΔM and it is analogous to the "4σ loading" which is the basis

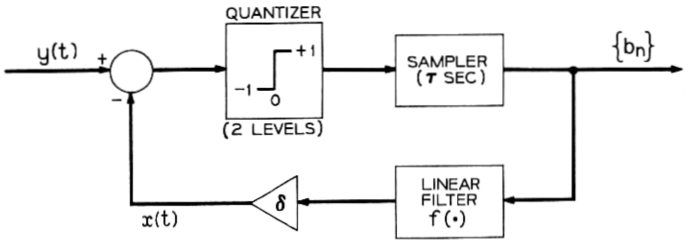

Fig. 1 — Delta modulator.

of Bennett's analysis of PCM quantizing noise.[10] If slope overload noise as well as granular noise is taken into account, analytic and simulation studies[6, 7] indicate that for minimal overall quantizing noise, the parameter βR [a constant in equation (4)] should increase with increasing R. For $R = 32$, equation (4) approximates the optimal step size, but for lower values of R the total quantizing noise inherent in $\{b_n\}$ may be reduced if a lower δ is accepted. A decrease in δ reduces the quantity of granular noise in the ΔM system while increasing the slope overload content.

The analytic work reported in this paper follows Bennett and van de Weg by constraining the ΔM parameters to conform to equation (4) and by assuming that no slope overload occurs. [If $y(t)$ is drawn from a gaussian random process, the probability that its derivative exceeds δ/τ is less than 4×10^{-5}.]

III. THE ΔM-PCM CONVERTER

Fig. 2 shows a ΔM-PCM converter that consists of an analog-to-PCM encoder operating on the analog signal produced by a ΔM receiver. This receiver contains a replica of the delta modulator feedback loop and a linear filter that in many analytic studies is considered to possess an ideal low-pass transfer function. This filter

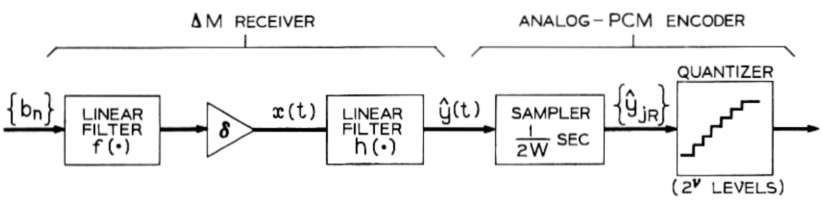

Fig. 2 — ΔM-analog-PCM converter.

processes the integrated ΔM signal, $x(t)$, and rejects the portion of its quantizing noise that lies outside the W Hz bandwidth of the original analog signal. In Fig. 2, ΔM signals are accepted by the converter at τ second intervals and PAM samples are presented to the PCM quantizer every $1/2W = R\tau$ seconds. The PAM sample that approximates $y_{jR} = y(jR\tau)$ is $\hat{y}_{jR} = \hat{y}[(jR + M)\tau]$ in which the delay of the filter $h(\cdot)$ is assumed to be $M\tau$ seconds. This output sample is related to $\{b_n\}$, the ΔM input of Fig. 2, by

$$\hat{y}_{jR} = \delta \sum_{n=0}^{\infty} g_n b_{jR+M-n} \tag{6}$$

where $g_n = g(n\tau)$ and $\delta g(\cdot)$ is the impulse response of the ΔM receiver. Thus, $g(\cdot)$ is the convolution of $f(\cdot)$ and $h(\cdot)$.

In the case of single integration ΔM, $f(\cdot)$ is the unit step function and $g(\cdot)$ is the unit step response of the filter $h(\cdot)$, that is,

$$g(t) = \int_0^t h(u)\, du. \tag{7}$$

Thus $h(\cdot)$ may be scaled such that

$$\lim_{n\to\infty} g_n = 1$$

which implies that a number N exists such that equation (6) may be approximated with arbitrary accuracy by

$$\hat{y}_{jR} \cong \delta \sum_{n=0}^{N-1} g_n b_{jR+M-n} + \delta \sum_{n=N}^{\infty} b_{jR+M-n}. \tag{8}$$

The second term in equation (8) may be realized by an up-down counter operating on the ΔM input delayed by $N\tau$ seconds. The first term is the weighted sum of the outputs of a tapped binary shift register. The structure implied by equation (8) is shown in Fig. 3.

Figure 3 represents a transversal filter with analog coefficients generating a discrete-time analog output, $\{\hat{y}_{jR}\}$. The required quantization of $\{\hat{y}_{jR}\}$ may be realized by means of the quantization of the filter coefficients.* The purpose of the converter implies the digitization of Fig. 3 and its implementation (with finite-precision arithmetic) as the entire ΔM-PCM converter.

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* In this paper, a PCM encoder is assumed to consist of a sampler and quantizer only. In a transmission system this encoder would be followed by a "channel encoder" that represents the quantized signal in an appropriate (for example, binary or multilevel) format.

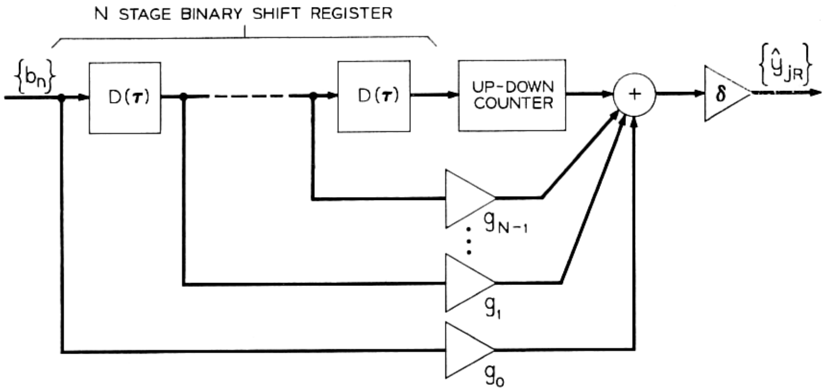

Fig. 3 — Digital ΔM-PCM converter [$D(\tau)$ denotes a τ second delay].

3.1 *Simulation of the Analog System*

If this digital converter is designed to simulate the analog-interface structure of Fig. 2, the filter coefficients, g_n, may be derived as samples, separated by τ seconds, of the unit step response of the filter $h(\cdot)$. In order to find N, the required number of coefficients, a number ϵ is chosen to represent the maximum tolerable truncation error. This error is bounded in the following expression

$$\epsilon \leqq \delta \sum_{n=N}^{\infty} | 1 - g(n\tau) | \cong \frac{\delta}{\tau} \int_{N\tau}^{\infty} | 1 - g(t) | \, dt. \tag{9}$$

Because δ/τ is constant with changing sampling rates the truncation error requirement implies a constant value of $N\tau$ or N/R. Thus the number of filter stages is proportional to the ΔM sampling rate when the design procedure is based in this manner on the simulation of the analog interface converter. A similar conclusion was reached by O'Neal who employed a different simulation procedure but found the number of required filter stages to be $N = 10R$.†

3.2 *Critique of the Analog Simulation Method*

The principal goal of the present study is to demonstrate the relationship between the ΔM speed (proportional to R) and the digital filter complexity (indicated by N), required to obtain PCM output signals of a given quality. As R increases, the difference between

† O'Neal, J. B., unpublished memorandum relating to the computer simulation, the results of which are given in Ref. 6.

the integrated ΔM signal [$x(t)$ in Fig. 2] and the analog input is reduced; this implies that in order to achieve a fixed output fidelity, the noise rejection requirements of the digital filter may be relaxed. It would be desirable to represent the reduced performance demands on the converter as a decrease in its complexity, N. Since the simulation design technique achieves the opposite result we use a minimum mean square error design which admits the calculation of optimum coefficient values for any N and the derivation of the desired trade-off between ΔM speed and converter complexity.

3.3 *Minimal Mean Square Error Design*

The converter in Fig. 3 derives an estimate of y_{jR} that is a linear combination of a finite set of sample values of $x(t)$. Thus we are able to use the covariance statistics of the sampled data sequences $\{y_{jR}\} = \{y(jR\tau)\}$ and $\{x_n\} = \{x(n\tau)\}$ to calculate optimal (with respect to mean square error) values of the filter coefficients.

We consider the desired PAM sample, $\hat{y}_{jR}$, to be an estimate of y_{jR} that is based on the statistical evidence of the ΔM sequence, $\cdots$, $b_{-1}, b_0, b_1, \cdots, b_{jR+M}$, beginning in the indefinite past relative to $t = jR\tau$ and terminating with the $(jR + M)$th binary symbol. If M is negative, the estimation process involves prediction of y_{jR} on the basis of information available (at the delta modulator) prior to t = $jR\tau$. In the case considered here, M will range over nonnegative numbers and the estimator will have a lag of $M\tau$ seconds.

The linear estimation design problem involves the specification of the infinite set of coefficients α_n in the estimation formula

$$\hat{y}_{jR} = \sum_{n=-\infty}^{jR+M} \alpha_{jR-n} b_n \,. \tag{10}$$

This problem will be approached by a consideration of the sample values of $x(t)$, the integrated ΔM signal:

$$x(t) = \delta \sum_{n=-\infty}^{k} b_n \quad \text{for} \quad k\tau \leqq t < (k+1)\tau.$$

Each sample value, $x_n = x(n\tau)$, depends on the entire history of the ΔM sequence. The character of $x(t)$ and its relation to $y(t)$, the original analog signal, are illustrated in Fig. 4. Except when the system is in a slope overload condition, x_n differs from the analog signal by no more than two times the step size. Thus,

$$|x_n - y_n| \leqq 2\,\delta. \tag{11}$$

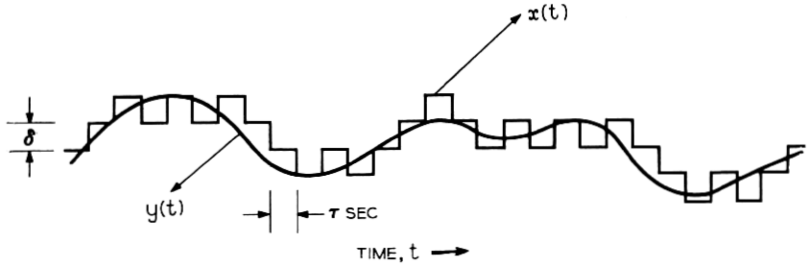

Fig. 4 — Analog input, $y(t)$, and integrated ΔM signal, $x(t)$.

Equation (11) and Fig. 4 suggest that x_n is closely correlated with y_{jR} for values of n near jR. Such correlation implies the usefulness of the $N = 2M + 1$ samples, $x_{jR-M}, x_{jR}, \ldots, x_{jR+M}$ in the estimation of y_{jR}. These N sample values comprise the statistical evidence of the estimate,

$$\hat{y}_{jR} = \sum_{k=-M}^{M} a_k x_{jR-k} \; . \tag{12}$$

Adoption of equation (12) requires the derivation of only a finite set of coefficients, a significant improvement over the situation suggested by equation (10). The two estimation procedures are identical, however, because the infinite set of α_n in equation (10) may be calculated directly from the N values of a_k as

$$\begin{aligned} \alpha_n &= \delta \sum_{k=-M}^{n} a_k \qquad -M \leqq n \leqq M \\ \alpha_n &= \delta \sum_{k=-M}^{M} a_k = \alpha_M \qquad n > M. \end{aligned} \tag{13}$$

Thus equation (12) may be rewritten,

$$\hat{y}_{jR} = \sum_{n=-M}^{M} \alpha_n b_{jR-n} + \alpha_M \sum_{n=M+1}^{\infty} b_{jR-n} \; , \tag{14}$$

which is equivalent to equation (8) with $N = 2M + 1$ and $\delta g_{n+M} = \alpha_n$. Thus the estimation procedure of equation (12) may be realized by the structure shown in Fig. 3.

We now consider the mean square estimation error of equation (12)

$$\eta = E\{(y_{jR} - \hat{y}_{jR})^2\} \tag{15}$$

or

$$\eta = \sigma^2 - 2 \sum_{k=-M}^{M} a_k E\{y_{jR} x_{jR-k}\} + \sum_{k=-M}^{M} \sum_{l=-M}^{M} a_k a_l E\{x_{jR-k} x_{jR-l}\} \tag{16}$$

where $E\{\cdot\}$ is the expectation operator. The expected values on the right side of equation (16) are, respectively, the cross-covariance function of $\{y_{jR}\}$ and $\{x_n\}$ and the autocovariance function of $\{x_n\}$. It has been shown that if $y(t)$ is a member of a stationary ensemble, the sequence of samples $\{x_n\}$ is also stationary.[11] Thus we may adopt the notation

$$\Phi_k = E\{y_{jR}x_{jR-k}\} \tag{17}$$

for the cross-covariance which depends only on $|k|$ and

$$r_\mu = E\{x_{jR-k}x_{jR-l}\} \tag{18}$$

for the autocovariance which is a function only of $|\mu| = |l - k|$. Equation (16) will now be expressed in matrix notation in terms of Φ and A defined as column vectors ($N \times 1$ matrices) with components Φ_k and $a_k(-M \leqq k \leqq M)$ respectively, and in terms of ψ, defined as the $N \times N$ autocovariance matrix with components $\psi_{k,l} = r_{l-k}(-M \leqq k, l \leqq M)$. Thus we have

$$\eta = \sigma^2 - 2A^T\Phi + A^T\psi A. \tag{19}$$

If the mean value of $y(t)$ is zero the coefficients for which η is minimized are given by[12]

$$A^* = \psi^{-1}\Phi \tag{20}$$

and the minimal mean square error is

$$\eta_{\min} = \sigma^2 - \Phi^T\psi^{-1}\Phi \tag{21}$$

which is the result of substituting A^* in equation (20) for A in equation (19).

The Appendix shows that when $y(t)$ is a sample function of a stationary gaussian process the estimation error may be expressed in terms of the quantizing noise correlation vector Q with components $Q_k(-M \leqq k \leqq M)$, the correlation coefficients of the error samples $\{y_n - x_n\}$. Equation (21) may be approximated by

$$\eta_{\min} \cong Q_0 - Q^T\psi^{-1}Q. \tag{22}$$

IV. ENCODER CHARACTERISTICS

The Appendix gives formulas for the correlation coefficients, r_μ and Φ_k associated with the encoding of an analog input signal that is a member of a stationary gaussian ensemble. These formulas have been applied to the calculation of filter coefficients [equation (20)] and the quantizing noise power [equation (21)] of the PCM samples generated by optimal linear processing of the ΔM sequence. In particular, the relationships among the following three parameters have been investigated: (*i*) R, the bandwidth expansion ratio, (*ii*) N, the number of coefficients, and (*iii*) $S = \sigma^2/\eta_{\min}$ the signal-to-quantizing-noise ratio.

This signal-to-noise ratio is a function of the spectrum of the processed signal and of N and R which have been treated as independent variables in the calculations. In a practical design procedure, S would be the independent variable, specified according to the system fidelity criterion. With S fixed, N and R vary inversely; in practice their values would be selected as a compromise between the objectives of achieving low ΔM speed (low R) and a simple converter structure (low N).

Figure 5 pertains to a system whose gaussian input process has a flat spectrum band-limited to W Hz. The solid curves show S as a function of R for various values of N and the broken curve indicates the result of optimal analog processing of the integrated ΔM signal.

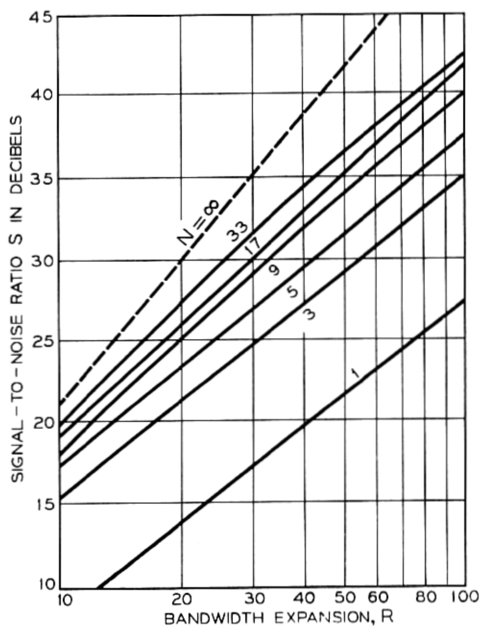

Fig. 5 — Performance curves, flat spectrum.

This curve, the ratio of σ^2 to η_∞ [see Appendix, equation (42)], is a bound on the solid curves. It corresponds to the signal-to-noise ratio of a transversal filter with an unlimited number of stages. The slope of the broken curve is 30 dB/decade.

The lowest curve in Fig. 5 indicates the signal-to-noise ratio of the encoder with a digital filter with one coefficient. For $R \geqq 16$, this coefficient may be set equal to unity with little (less than 0.5 dB) loss in estimation accuracy. With a unit coefficient, the converter reduces to an up-down counter alone, and the encoder consists only of a delta modulator and a counter as shown in Fig. 6. In this case each PCM output is a sample of the integrated ΔM signal, $x(t)$. Sampling this signal (whose bandwidth is RW Hz) at the Nyquist rate of the analog input results in aliasing of the high frequency noise components of $x(t)$ into the W Hz signal band of the coded signal. The noise power in the output of Fig. 6 is the entire mean square error of $\{x_n\}$,

$$Q_0 = E\{(y_n - x_n)^2\} \cong \delta^2/3, \tag{23}$$

so that the signal-to-noise ratio increases with ΔM speed at the rate of 20 dB/decade.

The effect of setting $N = 3$, which involves the addition of a 2-stage tapped binary shift register to Fig. 6 is (for $R \geqq 16$) a signal-to-noise ratio improvement of more than 7 dB or a ΔM speed reduction by a factor of 2.4 for a fixed value of S. Two additional stages ($N = 5$) further increase S by 2 dB or reduce the ΔM speed required for fixed S to less than one-third of that required when only an up-down counter is employed. Figure 5 demonstrates the manner in which further (through diminishing) improvements are obtained with the incorporation of additional coefficients and stages of delay.

Notice some of the functions that the transversal filter may be said to perform. In particular its noise reduction effects may be viewed as the result of filtering in the frequency domain, of statistical estimation, or of digital interpolation. Viewed as a substitute for the analog filter in Fig. 2, the transversal filter may be said to reject the out-of-band components of the error signal, $y(t) - x(t)$. As the number of stages increases, the filter cutoff becomes sharper and the output noise power is reduced.[13] Considered as an estimator of a random variable, the filter bases its estimates on an increasing number of correlated data as N increases. The quadratic form, $Q^T\psi^{-1}Q$, consisting of correlation coefficients, increases with increasing N and therefore the output noise power given by equation (22) decreases. Finally, the filter may be viewed as an interpolator. As the number

Fig. 6 — Encoder structure for $N = 1$.

of filter stages increases, the ΔM speed may be decreased and a proportionally greater step size may be tolerated in the delta modulator. The resolution of the ΔM signal is thus reduced while the accuracy of the PCM output is maintained due to the interpolation performed by the digital filter between increasingly separated ΔM quantization levels.

4.1 *Digitization of Coefficient Values*

The data of Fig. 5 apply strictly to an analog estimate obtained with analog coefficients. ("Analog" numbers in this context mean numbers quantized to the precision of the numerical methods used in calculating the coefficient values.) In a practical application of the ΔM–PCM conversion method, the coefficients would exist in digital form; in order to investigate this situation, we will consider the effect of rounding the values of the derived coefficients to a limited number of binary places.

In the discussion that follows, it will be convenient to divide equation (14) by δ and to consider the estimator of y_{jR}/δ, given by

$$\frac{\hat{y}_{jR}}{\delta} = \sum_{n=-M}^{M-1} \gamma_n b_{jR-n} + \gamma_M \sum_{n=M}^{\infty} b_{jR-n} \tag{24}$$

in which $\gamma_i = \alpha_i/\delta$. The coefficients of equation (24) are related to the components of A, the coefficient vector, by the sums

$$\gamma_n = \sum_{k=-M}^{n} a_k \qquad -M \leqq n \leqq M. \tag{25}$$

In the digitization of the ΔM–PCM converter, the set of analog coefficients, $\gamma_{-M}, \ldots, \gamma_0, \ldots, \gamma_M$, is rounded to L binary places by means of the computation

$$\gamma_i(L) = 2^{-(L-1)} \times [\text{the greatest integer} \leqq 2^{(L-1)}\gamma_i + 0.5]. \tag{26}$$

Corresponding to the limited-precision coefficients of equation (26) is a quantized coefficient vector,

$$A(L) = [a_{-M}(L), \cdots, a_0(L), \cdots, a_M(L)]^T,$$

whose components are calculated according to

$$a_{-M}(L) = \gamma_{-M}(L)$$

$$a_i(L) = \gamma_i(L) - \gamma_{i-1}(L) \qquad -M + 1 \leqq i \leqq M. \tag{27}$$

The mean square error of the digitized converter may be calculated from equation (19),

$$\eta(L) = \sigma^2 - 2[A(L)]^T \Phi A(L) + [A(L)]^T \psi A(L). \tag{28}$$

For each estimator considered in the derivation of Fig. 5, the signal-to-noise ratio, $\sigma^2/\eta(L)$, has been computed for $L = 1, 2, \ldots, 15$ binary places in each coefficient.

Under the assumption that the quantization of coefficients is not allowed to degrade the signal-to-noise ratio by more than 0.5 dB, Table I has been obtained. L^* in this table is the minimum value of L for which the inequality

$$10 \log [\eta(L)/\eta_{\min}] < 0.5$$

is valid over all $R \geqq 10$.

When the coefficients are rounded to L^* places according to Table I, we find in every case that $\gamma_M(L) = 1$ so that equation (24) may be specified as

$$\frac{\hat{y}_{jR}}{\delta} = \sum_{n=-M}^{M-1} \gamma_n(L^*) b_{jR-n} + \sum_{n=M}^{\infty} b_{jR-n}. \tag{29}$$

The second summation in equation (29) is the output of the up-down counter (which requires no weighting); the first summation may be implemented by means of a tapped binary shift register with $2M = N - 1$ stages.

It is also the case that for $n < M$, $\gamma_n(L^*) < 1$, so that each of the $N - 1$ tap weights is a proper binary fraction that may be represented by $L^* - 1$ bits. The output of the counter ranges over the set of integers and the tapped shift register serves as an interpolator so that the resultant PCM output is more finely quantized than the

TABLE I — EFFECT OF COEFFICIENT ROUND-OFF

Required coefficient accuracy (L^*)	1	4	4	5	7	8
Number of coefficients (N)	1	3	5	9	17	33

sample values of the integrated ΔM signal, of which it is a weighted sum.

4.2 *Other Signal Spectra*

The data of Fig. 5 relate to the processing of analog input signals that possess a flat band-limited spectrum. Converter performance characteristics have also been obtained for analog input signals with spectral density functions of the general form

$$Y(f) = \frac{2\pi}{f_c \tan^{-1}\left(\frac{2\pi W}{f_c}\right)} \times \frac{f_c^2}{(2\pi f)^2 + f_c^2} \qquad |f| \leqq W$$
$$= 0 \qquad |f| > W. \tag{30}$$

Such spectra result when band limited white noise is processed by a low-pass RC filter with corner frequency $f_c = 1/RC$.

Corresponding to the examples analyzed by O'Neal, two spectra conforming to equation (30) have been considered.[6] In one case the ratio f_c/W, of corner frequency to cutoff, is 0.25 and in the other case this ratio is 0.068. These spectra relate to broadcast television and *Picturephone*® visual telephone signals, respectively; the performance curves are given in Figs. 7 and 8. The shapes of these two families of curves are similar to one another; both resemble the curves in Fig. 5. The principal difference betwen any two corresponding curves is a vertical translation whose magnitude is the squared ratio, expressed in dB, of the two relevant step sizes.

Thus, for any of the three spectra considered and a given N and R, the signal-to-noise ratio in dB is approximately the value of S given in Fig. 5 plus the correction

$$-10 \log [3 f_e^2] \text{ dB}. \tag{31}$$

(For signals with a flat spectral density function $f_e^2 = \frac{1}{3}$.)

For spectra in the form of equation (30), this correction is[6]

$$11.19 - 10 \log \left\{ \frac{2\pi(f_c/W)}{\tan^{-1}[2\pi(W/f_c)]} - \left(\frac{f_c}{W}\right)^2 \right\}$$
$$= 11.35 \text{ dB} \quad \text{for} \quad f_c/W = 0.25$$
$$= 16.89 \text{ dB} \quad \text{for} \quad f_c/W = 0.068. \tag{32}$$

For speech signal processing, realistic analytic results are more difficult to obtain. Signal spectra vary from speaker to speaker; the

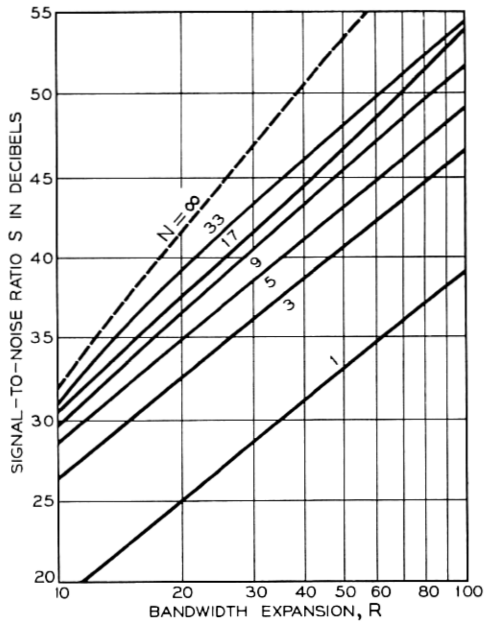

Fig. 7 — Performance curves, RC spectrum, $f_c/W = 0.25$.

portion of the speech spectrum to be conveyed depends on the application. A particularly severe problem arises when a range of signal levels must be accommodated by the encoder. To treat this problem, it may be possible to introduce companding to the delta modulator; but the solution currently favored is the acceptance of a higher ΔM sampling rate than would be required if the signal level were fixed and the introduction of a digital compandor[14] to operate on the uniformly quantized signal produced by the ΔM–PCM converter.

4.3 *A Design Example*

This section demonstrates the form of a converter designed to process the digital representation of analog signals with a band-limited RC spectrum in which $f_c/W = 0.068$. We assume that a PCM signal-to-noise ratio of at least 41 dB is required, and refer to Fig. 8 to determine values of N and R for which this requirement may be met. For each value of N illustrated in Fig. 8, the required R is given in Table II. Thus the converter in which $N = 5$ and $R = 24$ meets the stated objective. Table III gives the optimal coefficients a_i, α_i, and the limited precision set,

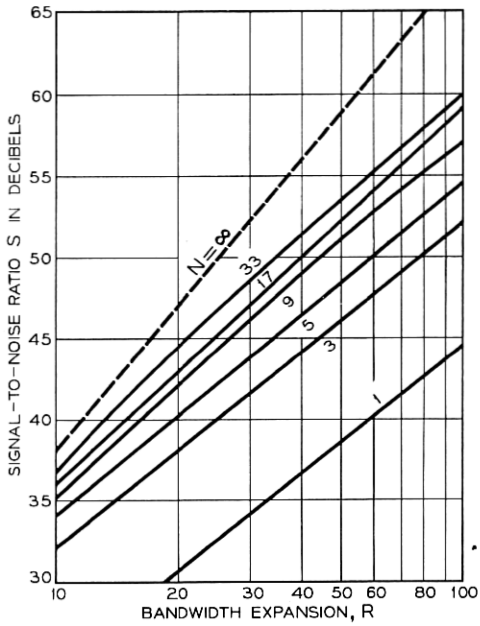

Fig. 8 — Performance curves, RC spectrum, $f_c/W = 0.068$.

$\gamma_i(4)$, for $-2 \leqq i \leqq 2$. The signal-to-noise ratio associated with the analog coefficients is 41.9 dB and that associated with the coefficients rounded to four binary places is 41.8 dB. (When the coefficients are limited in precision to three binary places, the value of S is reduced to 39.9 dB.)

Figure 9 demonstrates, in the schematic form of Fig. 3, one implementation of the converter. The indicated arithmetic operations need be performed only once for every 24 ΔM inputs accepted by the converter. These operations consist of modification of the sign of each coefficient (multiplication by ±1), addition of four 3-bit numbers, and addition of their sum to the counter output. The sign modification and coefficient addition operations (shown in the broken box) may be performed by a combination of Boolean logic elements with four binary

TABLE II — VALUES OF N AND R FOR $S \geqq 41$ dB IN FIG. 8

R	67	29	21	18	17	15	13
N	1	3	5	9	17	33	∞

TABLE III—CONVERTER COEFFICIENTS, $N = 5$

i	ϵ_i	α_i	$\gamma_i(4)$ (binary form)
−2	0.15996	0.15996	0.001
−1	0.22575	0.38571	0.011
0	0.22866	0.61437	0.101
1	0.22575	0.84013	0.111
2	0.15996	1.00009	1.000

inputs and five outputs. Because the logical processor is required to operate during only one of every 24 basic time intervals of the system, the possibility exists for it to be shared among other digital signals.

Notice that, in general, for converters in which $N = 5$, it has been found that coefficient values may be truncated to four binary places without unacceptable degradation of converter performance. Furthermore, the coefficients, $\gamma_i(4)$, have, in general, the values given in Table III. Thus, regardless of the spectrum of the analog input and the sampling rate and step size of the delta modulator, the same converter structure may be employed to produce a near-maximum (within 0.5 dB) PCM signal-to-noise ratio. This signal-to-noise ratio is a function primarily of the step size of the delta modulator. Except in the manner that it determines the step size [through equation (4)], the precise shape of the spectrum of the input signal has only a secondary influence on system performance. With regard to the ΔM sampling rate, variations in R affect the converter only by varying the number of ΔM inputs accepted between the generation of

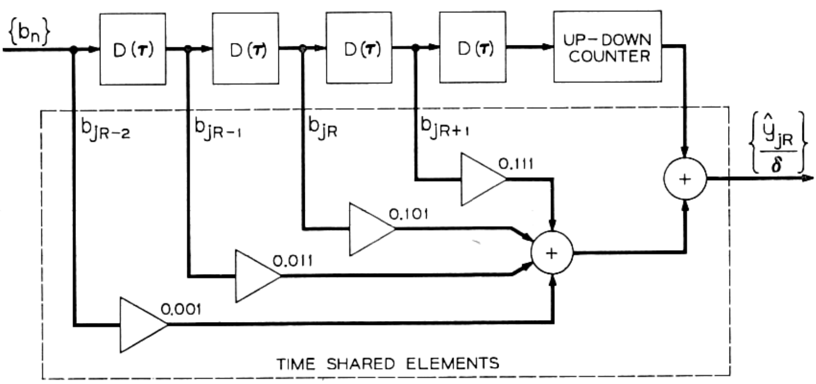

Fig. 9 — Converter structure for $N = 5$.

PCM outputs. The configuration of the logical processor (arithmetic unit) is not altered by changes in ΔM sampling rate.

V. CONCLUSIONS

The PCM encoder described combines the principal advantage of delta modulation, a simple means of analog-to-digital conversion, with the advantages of PCM, a linear representation of the coded signal and often fewer bits per second than are required with ΔM. The result of adopting the minimum mean square error criterion, in the synthesis of the ΔM–PCM converter, is a family of encoders for each possible level of output quantizing noise and thus considerable design flexibility.

The transversal filter in the converter has a particularly simple configuration. The sequence of signals to be processed is represented by one bit per sample so that the only required arithmetic operations are addition and subtraction of prespecified coefficient values. These coefficients may be rounded to a modest number of binary places and the digital filter may be considered to be a combination of elementary Boolean logic elements rather than an arithmetic unit. With the exception of the up-down counter, the elements of the ΔM–PCM converter may be time-shared among several signals.

In addition to being applicable as a general-purpose PCM encoder, the device described in this paper may be adopted to serve in a digital communication system which performs local office switching of signals coded in a ΔM format and trunk transmission of PCM signals. In this application there would be a ΔM modem for each analog station and a limited number of ΔM–PCM converters for processing trunk calls. Elements of such a system are shown schematically in Fig. 10.

APPENDIX

ΔM Signal Statistics, Gaussian Inputs[11]

Expressions have been derived for the matrices required in the application of equations (20) and (21) to the determination of filter coefficients and values of noise power under the assumption that $y(t)$, the input to the delta modulator, is a member of a gaussian ensemble. The correlation coefficients Φ_k and r_μ are presented in this section and the following section expresses the ΔM and PCM quantizing noise characteristics in terms of these correlation characteristics.

The covariance statistics are expressed as functions of the ΔM step

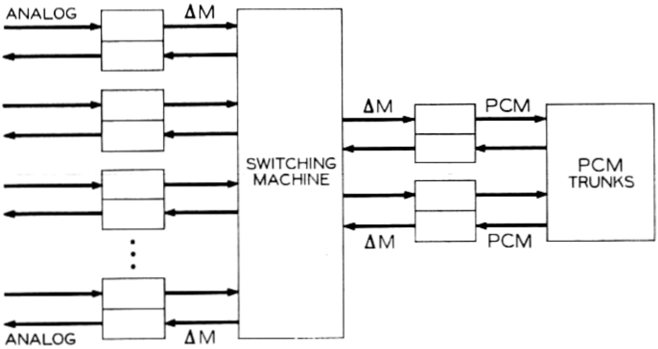

Fig. 10 — An application in which the delta modulator and the ΔM-PCM converter are separated.

size, δ, its normalized value, $\beta = \delta/\sigma$, and the correlation coefficients of the analog input,

$$\rho_k = \frac{1}{\sigma^2} E\{y(n\tau)y[(n+k)\tau]\}. \tag{33}$$

These coefficients are samples, taken at τ second intervals, of the autocorrelation function, $\rho(\cdot)$, of $y(t)$. The power density spectrum, $Y(\cdot)$, is the Fourier transform of $\sigma^2\rho(\cdot)$.

The cross-covariance of the integrated ΔM sequence, $\{x_n\}$, and the samples of the analog input, $\{y_{jR}\}$, is proportional to the autocovariance of $\{y_j\}$. Thus,

$$\Phi_0 = \sigma^2\left\{1 + 2\sum_{k=1}^{\infty} \exp\left[-\frac{2\pi^2k^2}{\beta^2}\right]\right\}$$

and in general

$$\Phi_k = \rho_k \Phi_0 . \tag{34}$$

For the autocovariance of $\{x_n\}$, we have the more complicated formulas:

$$r_0 = \sigma^2\left\{1 + 4\sum_{k=1}^{\infty} \exp\left[-\frac{2\pi^2k^2}{\beta^2}\right]\right\} + \delta^2\left\{\frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{\pi^2k^2} \exp\left[-\frac{2\pi^2k^2}{\beta^2}\right]\right\}$$

$$r_\mu = \rho_\mu\sigma^2\left\{1 + 4\sum_{k=1}^{\infty} \exp\left[-\frac{2\pi^2k^2}{\beta^2}\right]\right\} + \frac{\delta^2}{\pi^2}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{mk}[1 + (-1)^{m+k}]$$

$$\cdot\left\{\exp\left[-\frac{\pi^2(k^2 + m^2 - 2mk\rho_\mu)}{2\beta^2}\right] - \exp\left[-\frac{\pi^2(k^2 + m^2 + 2mk\rho_\mu)}{2\beta^2}\right]\right\}$$

for μ even

$$r_\mu = \rho_\mu \sigma^2 \left\{ 1 + 4 \sum_{k=1}^{\infty} \exp\left[-\frac{2\pi^2 k^2}{\beta^2} \right] \right\} + \frac{2\delta^2}{\pi^2} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{mk} (-1)^m$$

$$\cdot \left\{ \exp\left[-\frac{\pi^2(k^2 + m^2 - 2mk\rho_\mu)}{2\beta^2} \right] - \exp\left[-\frac{\pi^2(k^2 + m^2 + 2mk\rho_\mu)}{2\beta^2} \right] \right\}$$

$$\text{for} \quad \mu \text{ odd.} \qquad (35)$$

In most cases of practical interest, the ΔM step size is a fraction of the rms input signal and the formulas in equations (34) and (35) may be simplified considerably. All of the single summations in these formulas involve powers of

$$\exp\left[-\frac{2\pi^2}{\beta^2} \right] = \exp\left[-\frac{19.7}{\beta^2} \right],$$

which for $\beta < 0.5$ is less than 10^{-34}. In this event we have the very accurate approximations

$$\Phi_0 \cong \sigma^2 \quad \text{and} \quad r_0 \cong \sigma^2 + \delta^2/3. \qquad (36)$$

In the formulas for r_μ, there are double sums of the difference of two exponential terms. When equation (36) applies, the second term is negligible and the first term has significant values only when the two indices of summation are equal. We thus have the approximation,

$$r_\mu \cong \rho_\mu \sigma^2 + \frac{2\delta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{\mu k}}{k^2} \exp\left[-\frac{\pi^2 k^2 (1 - \rho_\mu)}{\beta^2} \right], \qquad (37)$$

which is valid in most cases of practical interest.

A.1 *Quantizing Noise*

A consideration of the statistical properties of the quantizing noise signals that appear in the analog-ΔM–PCM encoder provides insight into the signal processing operations described in this paper. In the delta modulator, we have the error signal $y(t) - x(t)$, that is sampled and quantized to produce the ΔM sequence, $\{b_n\}$. This is a high frequency signal with effective bandwidth RW; the filter, $h(\cdot)$, in the ΔM-analog receiver of Fig. 2 has the role of rejecting its out-of-band components. Similarly, the digital ΔM–PCM converter rejects an increasing proportion of the power of the error signal as N, the number of filter stages, increases. Equation (6) indicates that the performance of a digital ΔM–PCM converter with an unlimited number of stages ($N = \infty$) is equivalent to that of the analog-interface converter of Fig. 2.

The autocovariance coefficients of $\{y_n - x_n\}$, the sequence of error samples, are

$$Q_k = E\{(y_n - x_n)(y_{n+k} - x_{n+k})\} = \sigma^2\rho_k + r_k - 2\Phi_k . \tag{38}$$

Equations (34) and (38) may be combined in the expression

$$\Phi_k = \frac{\Phi_0}{2\Phi_0 - \sigma^2}(r_k - Q_k) = \frac{C}{2C - 1}(r_k - Q_k) \tag{39}$$

in which we have set

$$C = \frac{\Phi_0}{\sigma^2} = 1 + 2\sum_{k=1}^{\infty} \exp\left[-\frac{2\pi^2k^2}{\beta^2}\right].$$

Equation (39) may now be substituted into equation (21) for minimum mean square error, $\eta_{\min}$. Appearing in the resulting expression for $\eta_{\min}$ are the terms $r^T\psi^{-1}$ and $\psi^{-1}r$ in which r denotes the column matrix with components $r_k(-M \leqq k \leqq M)$. The two matrix products are transposes of one another; both are identical to quite simple matrices. Thus,

$$r^T\psi^{-1} = [0, 0, \cdots, 1, \cdots, 0, 0] = [\psi^{-1}r]^T. \tag{40}$$

When equation (40) is applied, equation (21) may be expressed in terms of the quantizing noise statistics as

$$\eta_{\min} = \left(\frac{C}{2C - 1}\right)^2 (Q_0 - Q^T\psi^{-1}Q) - \frac{(C - 1)^2}{2C - 1}\sigma^2, \tag{41}$$

in which Q is the error covariance vector and Q_0 is the total power in the error signal, $\{y_n - x_n\}$. Thus $Q^T\psi^{-1}Q$ represents the amount of noise rejected by a digital filter whose coefficients are given by equation (20). Because ψ is positive definite, equation (41) indicates that such a filter cannot enhance the quantizing noise power of the system.

As N increases without limit, the mean square error approaches the quantizing noise power associated with the optimum analog interpolation filter of the ΔM signals. This quantity may be expressed in terms of the signal spectra as[11]

$$\eta_\infty = \lim_{N\to\infty} \eta_{\min} = \left(\frac{C}{2C - 1}\right)^2 \frac{1}{R}\left[Q_0 + 2\sum_{k=1}^{\infty} Q_k \frac{\sin\left(\frac{\pi k}{R}\right)}{\left(\frac{\pi k}{R}\right)} - \int_0^1 \frac{[E^*(fW)]^2}{(2C - 1)Y^*(fW) + E^*(fW)}\,df\right] - \frac{(C - 1)^2}{2C - 1}\sigma^2 \tag{42}$$

in which $Y^*(\cdot)$ and $E^*(\cdot)$ are the power spectral density functions of $\{y_n\}$, the sampled analog signal, and $\{y_n - x_n\}$, the sampled error signal, respectively. Thus

$$Y^*(fW) = \sigma^2\left[1 + 2 \sum_{k=1}^{\infty} \rho_k \cos \frac{\pi f}{R}\right] = \frac{1}{\tau} \sum_{n=-\infty}^{\infty} Y[W(f + 2nR)] \quad (43)$$

and

$$E^*(fW) = Q_0 + 2 \sum_{k=1}^{\infty} Q_k \cos \frac{\pi f}{R}. \quad (44)$$

The sum in equation (42) of Q_0 and the infinite series is the quantizing noise at the output of an ideal low pass filter with bandwidth W Hz. The integral represents the additional noise power reduction that results from an optimal, rather than a flat, filter transfer function.

When the approximations given in equations (36) and (37) are valid, the correlation coefficients that appear in equations (41) and (44) may be approximated by

$$\begin{aligned} Q_0 &\cong \frac{\delta^2}{3} \\ Q_\mu &\cong \frac{2\delta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{\mu k}}{k^2} \exp\left[-\frac{\pi^2 k^2(1 - \rho_\mu)}{\beta^2}\right]. \end{aligned} \quad (45)$$

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