An Extended Correlation Function of Two Random Variables Applied to Mobile Radio Transmission

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The definition, properties, and application of an extended correlation function of two random variables involving two common parameters are described and applied to mobile radio systems. The correlation functions of a predetection diversity combined signal (using a scheme of phase equalizing by multiple heterodyning) and of a directional antenna array signal are derived with the help of the extended correlation function.

These correlation functions can be used to determine parameter values giving minimum correlation between two signals desirable for diversity systems. One can also obtain the power spectra by taking the Fourier transform of these correlation functions. Thus extended correlation functions promise to be useful.

I. INTRODUCTION

If two random variables depend on only one common parameter, such as time or distance, conventional correlation formula can be applied to the two variables. However, if both of these variables involve not one but two common variable parameters, then the correlation formula found in the current literature is limited.* Since such cases occur in some of our mobile radio problems, as we discuss later, we need to define an extended correlation function and outline its properties and applications.

II. DERIVATION OF AN EXTENDED CORRELATION FUNCTION OF TWO RANDOM VARIABLES INVOLVING TWO COMMON PARAMETERS

A conventional normalized correlation function of two random variables r_1 and r_2, both of which are functions of one parameter

* Prior to acceptance of this paper for publication, the author was advised that a similar concept was discovered independently by A. Papoulis in his recently published book.[1]

d[that is, $r_1(d_1)$ and $r_2(d_2)$] can be expressed as[2]

$$\rho_{12}(d_1, d_2) = \frac{R_{12}(d_1, d_2) - m_1 m_2}{\sigma_1 \sigma_2}$$

$$= \frac{\langle r_1(d_1) r_2(d_2) \rangle_{av} - \langle r_1(d_1) \rangle_{av} \langle r_2(d_2) \rangle_{av}}{[\langle r_1^2(d_1) \rangle_{av} - \langle r_1(d_1) \rangle_{av}^2]^{\frac{1}{2}} \cdot [\langle r_2^2(d_2) \rangle_{av} - \langle r_2(d_2) \rangle_{av}^2]^{\frac{1}{2}}} \quad (1)$$

where m's are the mean values, σ^2's are the covariances, $\rho_{12}(d_1, d_2)$ is in the range $0 \leqq | \rho_{12}(d_1, d_2) | \leqq 1$, and $R_{12}(d_1, d_2) = \langle r_1(d_1) r_2(d_2) \rangle_{av}$ is the correlation function.†

Supposing a random variable $r_1(D_1; d_1)$ is a function of two parameters D_1 and d_1, and another variable $r_2(D_2; d_2)$ is a function of two parameters D_2 and d_2; the normalized correlation functions of these two variables can be deduced from equation (1):

$$\rho_{12}(D_1, D_2; d_1, d_2)$$
$$= \frac{R_{12}(D_1, D_2; d_1, d_2) - m_1 m_2}{\sigma_1 \sigma_2}$$
$$= \frac{\langle r_1(D_1, d_1) r_2(D_2, d_2) \rangle_{av} - \langle r_1(D_1, d_1) \rangle_{av} \langle r_2(D_2, d_2) \rangle_{av}}{[\langle r_1^2(D_1, d_1) \rangle_{av} - \langle r_1(D_1, d_1) \rangle_{av}^2]^{\frac{1}{2}} [\langle r_2^2(D_2, d_2) \rangle_{av} - \langle r_2(D_2, d_2) \rangle_{av}^2]^{\frac{1}{2}}}. \quad (2)$$

If the problem we are dealing with is a stationary random process for both of the parameters D and d, then

$$R_{12}(D_1, D_2; d_1, d_2) = R_{12}(D_1 - D_2; d_1 - d_2),$$
$$\langle r_k(D_k, d_k) \rangle_{av} = \langle r_k(0, 0) \rangle_{av} = m_k,$$
$$\langle r_k^2(D_k, d_k) \rangle_{av} - m_k^2 = \langle r_k^2(0, 0) \rangle_{av} - m_k^2 = \sigma_k^2,$$

where $k = 1, 2$. Now m_k and σ_k are constants and we may let $D = D_1 - D_2$ and $d = d_1 - d_2$. Then equation (2) becomes

$$\rho_{12}(D; d) = \frac{R_{12}(D; d) - m_1 m_2}{\sigma_1 \sigma_2}. \quad (3)$$

We call $\rho_{12}(D; d)$ a normalized extended correlation function of the first kind. Also we note that $\rho_{12}(D; d)$ in equation (3) is always smaller than $\rho_{12}(0; 0)$ which is equal to one:

$$\rho_{12}(D; d) \leqq \rho_{12}(0; 0) = 1.$$

† The terms "correlation function $R_{12}(d_1, d_2)$ and normalized correlation function $\rho_{12}(d_1, d_2)$" are adopted from Ref. 3, p. 59.

When the difference D is equal to zero, then

$$\rho_{12}(0; d) = \rho_{12}(d).$$

Now we should illustrate and extend equation (3). We are going to find an extended correlation function of the first kind from a function $\epsilon(D; d_1, d_2, d_3, \cdots, d_m)$ where all the d's are function of D and another parameter α [that is, $d_i(D, \alpha)$ for $i = 1, m$], then

$$R_\epsilon(D; d_1, d_2, d_3, \cdots, d_m) = \langle \epsilon[0; d_1(0, 0), d_2(0, 0), d_3(0, 0), \cdots) \cdot \epsilon(D; d_1(D, \alpha), d_2(D, \alpha), \cdots, d_m(D, \alpha)] \rangle_{\text{av}},$$

and the normalized correlation function can be derived from equation (2) as

$$\rho_\epsilon(D; d_1, d_2, \cdots, d_m) = \frac{R_\epsilon(D; d_1, \cdots, d_m) - m_\epsilon^2}{\sigma_\epsilon^2}$$

where

$$m_\epsilon = \langle \epsilon[0; d_1(0, 0), d_2(0, 0), d_3(0, 0), \cdots, d_m(0, 0)] \rangle_{\text{av}}$$

$$\sigma_\epsilon^2 = \langle \epsilon^2[0; d_1(0, 0), d_2(0, 0), d_3(0, 0), \cdots, d_m(0, 0)] \rangle_{\text{av}} - m_\epsilon^2 .$$

If we consider the case $d_i(D, \alpha)$ is a constant for all D and α itself is a constant, then we may assign a new symbol $R_\epsilon(D \mid d_1, d_2, \cdots, d_m)$ which can be expressed as

$$R_\epsilon(D \mid d_1, d_2, \cdots, d_m) = \langle \epsilon(0; d_1, d_2, \cdots, d_m) \epsilon(D; d_1, d_2, d_3, \cdots, d_m) \rangle_{\text{av}} .$$

$R_\epsilon(D \mid d_1, d_2, d_3, \cdots, d_m)$ is a correlation function under a condition that all $d_1, d_2, d_3, \cdots, d_m$ are constants. The normalized correlation is

$$\rho_\epsilon(D \mid d_1, d_2, \cdots) = \frac{R_\epsilon(D \mid d_1, d_2, \cdots, d_m) - m_\epsilon^2}{\sigma_\epsilon^2}, \tag{4}$$

where m_ϵ and σ_ϵ have been defined previously. We call equation (4) the normalized correlation function of the second kind. As we will show in the Section III, the extended correlation function of first kind $\rho_{12}(D; d)$ and the extended correlation function of second kind $\rho_{12}(D \mid d)$ can be used to obtain the correlation of signals from two diversity scheme receivers easily.

In order to give physical meaning to these functions, let us consider the following two cases. Suppose that two base-station multibranch diversity receiver arrays are separated by a distance D. The antenna spacing between branch-antenna elements for the first array is d_1 and for the second array is d_2. Both receivers simultaneously receive the signal from a distant mobile radio unit. We would like to determine the values of d_1, d_2, and D to obtain the least cross-correlation desirable for the best diversity reception of these two received signals. The extended correlation of first kind $\rho_{12}(D;\, d_1,\, d_2)$ may be used in this case.

The second case assumes that a mobile radio multi-branch diversity receiver array, with given uniform antenna element spacing d, moves along the street with a constant speed V. The autocorrelation of a signal ϵ, received by the mobile receiver, can be obtained from the extended correlation function of the second kind $\rho_\epsilon(D \mid d)$. Alternatively, we can also consider a multielement directive antenna instead of the diversity scheme. In this paper, we only treat the latter case. The former case can be solved following the same technique.

III. APPLICATION TO MOBILE RADIO PROBLEMS

3.1 *Derivation of the Correlation Function of a Signal Received from a Predetection Diversity Combining Receiver*

A multichannel predetection diversity combining system is a scheme for bringing a number of RF carriers to a common phase by means of multiple heterodyning. Then a linear combiner at the IF frequency is used to sum the individual channels.[4,5] A signal received from this system is called a predetection diversity combined signal.

Suppose that a signal consisting of multipath vertically polarized waves is received by an M-branch predetection combining mobile receiver with a M-antenna space diversity array. The M-antennas are spaced by d_1, d_2, $\cdots$ d_M respectively from an arbitrary common point. After the array has moved a distance D, the received signal ϵ, which is the sum of the M individual signal amplitudes received from M individual antennas, can be expressed as[6,7]

$$\begin{aligned}\epsilon(D;\, d_1,\, d_2,\, d_3,\, d_4,\, \cdots,\, d_M) &= r_1(D;\, d_1) + r_2(D;\, d_2) + r_3(D;\, d_3) \\ &\quad + \cdots + r_M(D;\, d_M) \\ &= \sum_1^M r_m(D;\, d_m), \end{aligned} \tag{5}$$

where all r_m are functions of distance D and antenna spacing d_m (see Appendix A). For a mobile radio signal,[7,8] or a long range fading signal,[9] the r_m are usually Rayleigh distributed. Suppose that all d's are constants, then the autocorrelation function of the signal given in equation (5) as a function of the separation distance D is an extended autocorrelation function of second kind which can be expressed as

$$\begin{aligned} R_\epsilon(D \mid d_1, d_2, d_3, d_4, \cdots) &= \langle \epsilon(0; d_1, d_2, d_3, d_4, \cdots)\epsilon(D; d_1, d_2, d_3, d_4)\rangle_{av} \\ &= \left\langle \left[\sum_1^M r_m(0; d_m)\right]\left[\sum_1^M r_m(D; d_m)\right]\right\rangle_{av} \\ &= \left\langle \sum_{m=1}^M \sum_{n=1}^M r_m(0; d_m) r_n(D; d_n)\right\rangle_{av} \\ &= \sum_{m=1}^M \sum_{n=1}^M R_{mn}(D; d_m - d_n). \end{aligned} \tag{6}$$

Using equation (3), this can also be written

$$R_\epsilon(D \mid d_1 \cdots d_M) = \rho_\epsilon(D \mid d_1, \cdots, d_M)(\sigma_\epsilon^2) + m_\epsilon^2, \tag{7}$$

where

$$\sigma_\epsilon^2 = \langle \epsilon^2(0; d_1, \cdots, d_M)\rangle - m_\epsilon^2, \tag{8}$$

$$\begin{aligned} \langle \epsilon^2(0; d_1, \cdots, d_M)\rangle_{av} &= \left\langle \left(\sum_1^M r_m(0; d_m)\right)^2 \right\rangle_{av} \\ &= \sum_1^M \sum_1^M \langle r_m(0; d_m) r_n(0; d_n)\rangle_{av} \\ &= \sum_1^M \sum_1^M R_{mn}(0; d_m - d_n). \end{aligned} \tag{9}$$

Substituting equation (8) into equation (7), and combining equations (6) and (7), we obtain

$$\rho_\epsilon(D \mid d_1, \cdots, d_M) = \frac{\sum_1^M \sum_1^M R_{mn}(D; d_m - d_n) - m_\epsilon^2}{\sum_1^M \sum_1^M R_{mn}(0; d_m - d_n) - m_\epsilon^2}. \tag{10}$$

The terms $R_{mn}(D; d_m - d_n)$ can be found from equation (3);

$$R_{mn}(D; d_m - d_n) = \rho_{mn}(D; d_m - d_n)\sigma_m\sigma_n + m_m m_n \tag{11}$$

and

$$m_\epsilon^2 = \langle \epsilon(0; d_1 \cdots d_m) \rangle_{\text{av}}^2 = \left\langle \sum_1^M r_m(0; d_m) \right\rangle_{\text{av}}^2$$

$$= \left[\sum_1^M \langle r_m(0; d_m) \rangle_{\text{av}} \right]^2$$

$$= \sum_1^M \sum_1^M m_m m_n \, . \tag{12}$$

Hence the correlation function of equation (10) becomes, assuming $\sigma_m = \sigma_n$,

$$\rho_\epsilon(D \mid d_1 \, , \, \cdots \, , d_M) = \frac{\displaystyle\sum_{m=1}^M \sum_{n=1}^M \rho_{mn}(D; d_m - d_n)}{\displaystyle\sum_{m=1}^M \sum_{n=1}^M \rho_{mn}(0; d_m - d_n)}. \tag{13}$$

If all spacings between two adjacent antennas are equal, then $d_m - d_n = (m - n)d_1$ where d_1 is the distance between two adjacent antennas. We may let $d = d_1$, and simplify the notation of equation (13) to

$$\rho_\epsilon(D \mid d) = \frac{\displaystyle\sum_{m=1}^M \sum_{n=1}^M \rho_{mn}(D; d)}{\displaystyle\sum_{m=1}^M \sum_{n=1}^M \rho_{mn}(0; d)}. \tag{14}$$

Equation (14) shows that a normalized autocorrelation function of an M-branch predetection combined signal is a normalized extended autocorrelation function of second kind in terms of all individual normalized correlation functions between branches. We notice that

$$\rho_\epsilon(D \mid d) \leqq \rho_\epsilon(0 \mid d) = 1, \tag{15}$$

and as stated in Section II

$$\rho_{mn}(0; d) = \rho_{mn}(d). \tag{16}$$

We may also realize that

$$\rho_{12}(D; d) = \rho_{23}(D; d) = \rho_{34}(D; d) = \cdots$$

and

$$\rho_{13}(D; d) = \rho_{24}(D; d) = \rho_{35}(D; d) = \cdots \, . \tag{17}$$

Hence, equation (14) can be further simplified as

$$\rho_\epsilon(D \mid d)$$

$$= \frac{M\rho_{11} + (M-1)(\rho_{12} + \rho_{21}) + (M-2)(\rho_{13} + \rho_{31}) + \cdots + \rho_{1M} + \rho_{M1}}{M\overset{0}{\rho}_{11} + (M-1)(\overset{0}{\rho}_{12} + \overset{0}{\rho}_{21}) + (M-2)(\overset{0}{\rho}_{13} + \overset{0}{\rho}_{31}) + \cdots + \overset{0}{\rho}_{1M} + \overset{0}{\rho}_{M1}}, \tag{18}$$

where $\rho_{mn} = \rho_{mn}(D; d)$ and $\rho_{mn}^0(0; d)$ used in equation (18) are for simplicity (ρ_{mn} and ρ_{mn}^0 are derived in Appendix A). If we let the antenna spacing $d/\lambda = 0$, then $\rho_\epsilon(D \mid 0)$ from equation (18) represents the correlation function of two single-branch signals

$$\rho_\epsilon(D \mid 0) = J_0^2(\beta D) \tag{19}$$

which agrees with that in Ref. 6.

Several numerical calculations have been carried out for the following example: Two four-branch diversity receivers, each of them with fixed antenna spacing $d/\lambda = 0.5$ or $d/\lambda = 1.0$, are mounted on the roof of the mobile unit, as shown in Fig. 1. These two receivers are separated by a distance D/λ (D/λ varies from 0 to 4) for two cases, $\alpha = 0°$ and $\alpha = 90°$. The calculations of the extended correlation function $\rho_\epsilon(D \mid d)$ of these two signals, obtained from their respective receivers when the mobile unit is moving, are shown in Figs. 2 and 3. Both figures indicate the values of D/λ which give the least correlation between two signals. We also note that the correlations at $\alpha = 0°$ are higher than that at $\alpha = 90°$. Figures 2 and 3 can also represent the auto correlation of a signal received from a single four-branch diversity receiver which has its antenna spacing $d/\lambda = 0.5$ or 1.0 and moves on a street with a constant speed $V(D = Vt)$. The power spectrum of such a signal can be obtained by taking the Fourier transform of its autocorrelation function.

3.2 *Derivation of the Correlation Function of a Signal Envelope Received from a Directional Antenna Array*

Signal reception from a directional antenna array with M antenna elements has been also suggested as a means of overcoming multipath

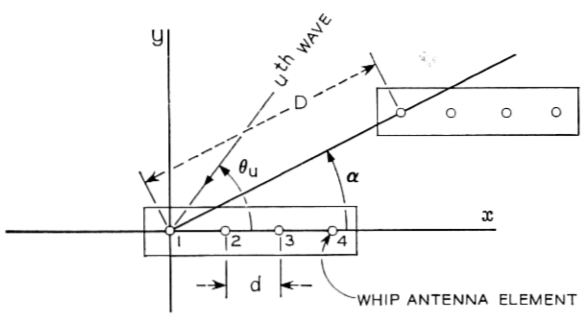

Fig. 1—Coordinate system of a M-branch diversity mobile radio receiver ($M = 4$ branches).

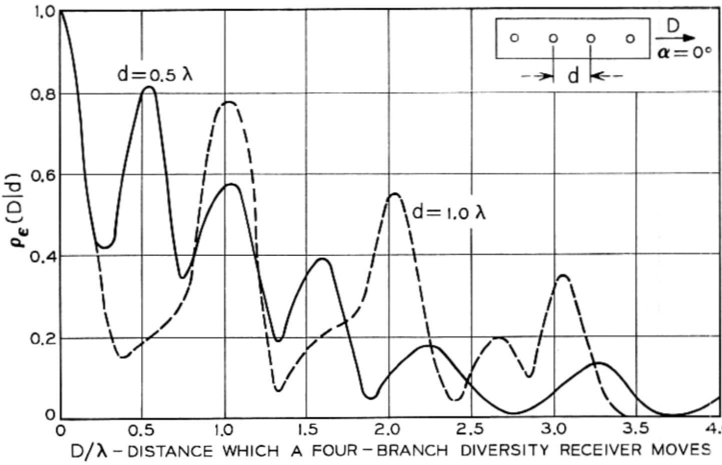

Fig. 2 — Normalized autocorrelation function of a four-branch diversity receiver moving at $\alpha = 0°$.

fading in mobile radio propagation.[10–12] The derivation of the correlation function of this signal envelope is as follows.

Suppose that the same kind of signal which consists of multipath vertical polarized waves as mentioned in Section 3.1 is received by a directional M-antenna array. The M antennas are spaced by d_1, d_2,

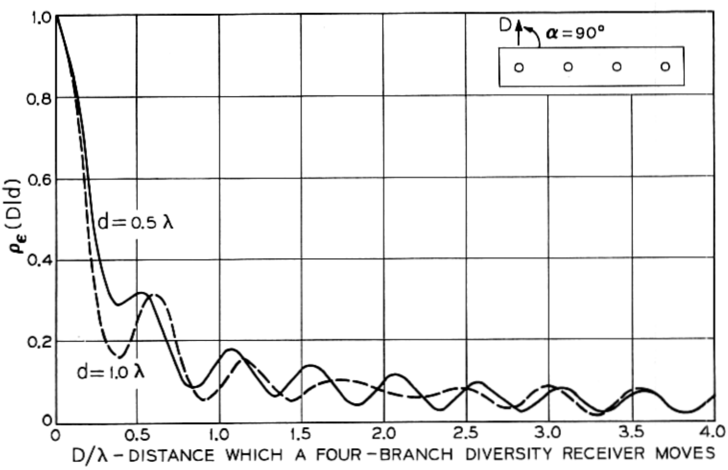

Fig. 3 — Normalized autocorrelation function of a four-branch diversity receiver moving at $\alpha = 90°$.

$d_3, \cdots, d_M$ respectively from an arbitrary common point. After the antenna array is moved by a distance D, (see Fig. 4) the received signal envelope ϵ which is the amplitude of the sum of M individual signals can be expressed as[13]

$$\begin{aligned}\epsilon(D; d_1, d_2, d_3, \cdots, d_M) \\ &= | s_1(D; d_1) + s_2(D; d_2) + \cdots + s_M(D; d_M) | \\ &= \left| \sum_1^M s_m(D; d_m) \right| \\ &= | X(D; d_1, d_2, \cdots, d_M) + jY(D; d_1, d_2, \cdots, d_M) |, \end{aligned} \tag{20}$$

where s_m is a complex variable which represents the amplitude and the phase of an individual signal. X and Y are the real and imaginary parts of the total signal.

If the spacings between adjacent antennas are equal, then antenna m and antenna n are separate dyb $d_m - d_n = (m - n)d$. Therefore X and Y of equation (20) are functions of D and d only. Suppose that all d's are constants, the autocorrelation function of signal envelope ϵ can be obtained by using the equation:[14]

$$\rho_\epsilon(D \mid d) \doteq \frac{\langle X_1(0; d) X_2(D; d)\rangle_{\text{av}}^2 + \langle X_1(0; d) Y_2(D; d)\rangle_{\text{av}}^2}{\langle X_1^2(0; d)\rangle_{\text{av}}^2} \tag{21}$$

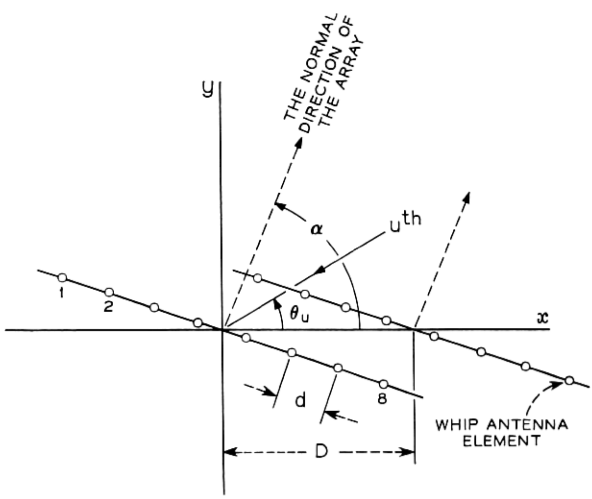

Fig. 4 — Coordinate system of a broadside directional antenna array ($M = 8$ elements).

provided X and Y are gaussian variables, all $\langle X_m \rangle_{av}$ are zeros, and $\langle X_m^2 \rangle_{av}$ and $\langle Y_m^2 \rangle_{av}$ are equal, where $m = 1$ or 2. These facts are shown in Appendix B. If the antenna spacing $d/\lambda = 0$, then $\rho_\epsilon(D \mid 0)$ actually represents the correlation between two single-branch signals, which agrees with equation (19) and Ref. 6.

The normalized correlation function of a signal received from a broadside directional antenna array is

$$\rho_\epsilon(D \mid d) = \frac{1}{4} \frac{\left\{ \sum_{m=1}^{K} \sum_{n=1}^{K} [J_0(A_1) + J_0(B_1) + J_0(A_2) + J_0(B_2)] \right\}^2}{\left[\sum_{m=1}^{K} \sum_{n=1}^{K} [J_0(A_0) + J_0(B_0)] \right]^2} \tag{22}$$

where

$$K = \frac{M}{2} \qquad \text{for} \quad M \text{ is even}$$

$$= \frac{M+1}{2} \quad M \quad \text{for is odd,}$$

and A_1, B_1, A_2, and B_2 are shown in equation (48). A_0 and B_0 are shown in equation (49).

Several numerical calculations have been carried out for the following example: Two eight-element broadside antenna arrays, each of them with fixed antenna spacing $d/\lambda = 0.5$ or $d/\lambda = 1.0$, are mounted on the roof of the mobile unit. These two arrays are separated by a distance D/λ (D/λ varies from 0 to 4) for two cases $\alpha = 0°$ and $\alpha = 90°$. The calculations of the extended correlation function $\rho_\epsilon(D \mid d)$ between two signals received from their respective arrays when the mobile unit is moving are shown in Figs. 5 and 6. Both figures indicate the values of D/λ which have the least correlation between two signals. The extended correlation curve of $d/\lambda = 0.5$ is quite different from that $d/\lambda = 1.0$ in both figures. The curve of $d/\lambda = 0.5$ in Fig. 5 shows that the high correlation and low correlation are about 0.25λ apart; however, this phenomenon does not appear for $d/\lambda = 0.5$, but rather for $d/\lambda = 1.0$ in Fig. 6. It can be explained as follows. For the directional antenna array with spacing $d = \lambda/2$, most of the energy is contained in the two major broadside lobes, while for the directional antenna array with antenna spacing $d = \lambda$, most of the energy is contained in the two major end-fire lobes. As the vehicle moves, strong standing waves may occur when the major antenna lobes lie in line with the motion of the vehicle, such as for the case $\alpha = 0°$ and $d = \lambda/2$; or the case $\alpha = 90°$ and $d = \lambda$.

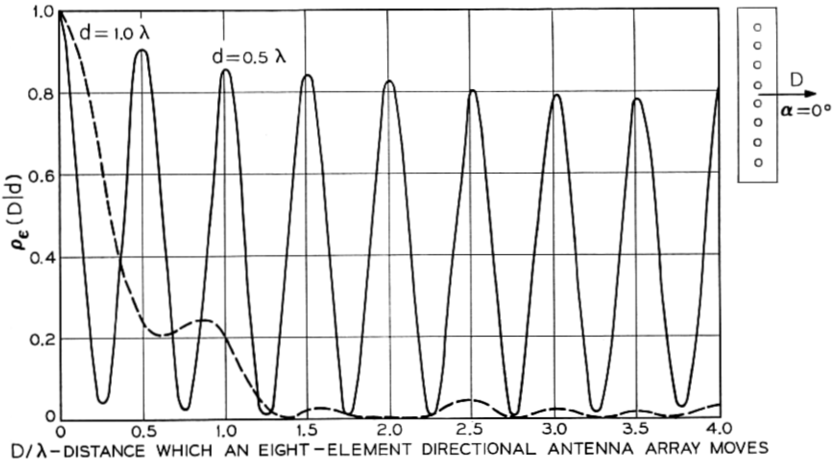

Fig. 5 — Normalized autocorrelation function of an eight-element directional antenna array pointing at $\alpha = 0°$.

The autocorrelations obtained from these standing waves, then, become oscillatory in nature, as we would expect.

Figures 5 and 6 can also represent the autocorrelation of a signal received from an eight-element broadside antenna array which has its antenna spacing $d/\lambda = 0.5$ or 1.0 and moves on a street with a constant

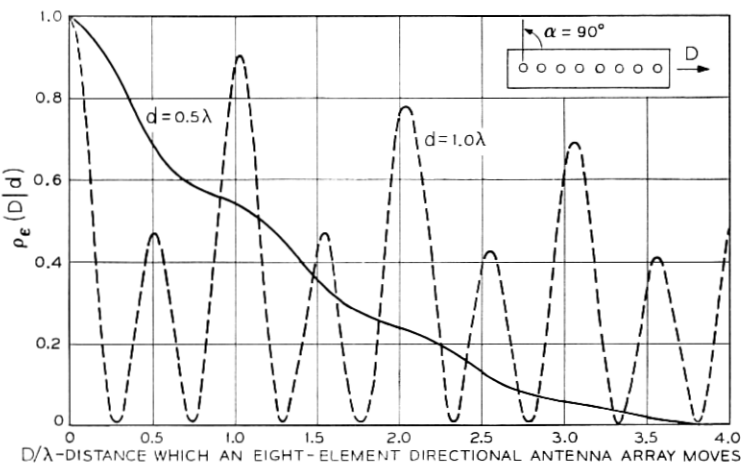

Fig. 6 — Normalized autocorrelation function of an eight-element directional antenna array pointing at $\alpha = 90°$.

speed V ($D = Vt$). The power spectrum of such a signal can be obtained by taking the Fourier transform of its autocorrelation function as we mentioned in Section 3.1.

IV. CONCLUSION

The derivation of a general correlation function of two random variables, each of them involving two parameters, has been obtained. The terms "extended correlation function of first kind" and "extended correlation function of second kind" have been defined. The application of the extended correlation function is demonstrated. The correlation function of a diversity signal and the correlation function of a directional antenna array signal are derived with the help of the extended correlation function in this paper. Several numerical calculations have also been carried out. From these correlation functions we can obtain the least correlations between two signals under certain circumstances. Also, we can obtain the power spectra by taking the Fourier transform of these correlation functions. Thus, it seems likely that these functions will find general application.

V. ACKNOWLEDGMENT

The author is indebted to C. L. Mallows and M. J. Gans with whom previous discussions have been very helpful. He also wishes to thank W. C. Jakes, Jr., for his many suggestions.

APPENDIX A

Finding Normalized Cross Correlation Functions From Individual Branch Signals of a Predetection Diversity Combining Receiver

It is easy to show that the signal from branch m in equation (5) is

$$r_m = \left| \sum_{1}^{N} A_u \exp\left[+j\beta D \cos(\theta_u - \alpha) + j(m-1)\beta d \cos \theta_u\right] \right|$$

$$= | X_m + jY_m |, \tag{23}$$

where

$$A_u = R_u + jS_u ,$$

$$X_m = \sum_{u=1}^{N} R_u \cos \phi_u + S_u \sin \phi_u , \tag{25}$$

$$Y_m = \sum_{u=1}^{N} S_u \cos \phi_u - R_u \sin \phi_u , \tag{26}$$

$$-\phi_u = \beta D \cos(\theta_u - \alpha) + (m - 1)\beta d \cos\theta_u \qquad (27)$$

(R_u and S_u are independent gaussian amplitudes with zero mean and unit variance). The diversity receiver is located at the point (D, α) in polar coordinate. The distance D, the angle α, and the arrival of uth wave at angle θ_u are shown in Fig. 1. We assume the N waves are uniformly distributed in angle. Now we can average the product of two components of two branches—branch m and branch n—as

$$\begin{aligned}\langle X_m(D_1)X_n(D_1 + D)\rangle_{av} &= \langle X_m(0)X_n(D)\rangle_{av} \\ &= NE\{\cos[BD\cos(\theta_u - \alpha) - (m - n)\beta d\cos\theta_u]\} \\ &= N[J_0(a)J_0(b) - 2J_2(a)J_2(b) + 2J_4(a)J_4(b) \\ &\qquad - 2J_6(a)J_6(b) + \cdots] \\ &= NJ_0(a^2 + b^2)^{\frac{1}{2}},\end{aligned} \qquad (28)$$

where[15]

$$a = \beta D\cos\alpha - (m - n)\beta d, \qquad (29)$$

$$b = \beta D\sin\alpha, \qquad (30)$$

$$a^2 + b^2 = (BD)^2 + (m - n)^2(\beta d)^2 - 2(m - n)\beta^2 Dd\cos\alpha, \quad (31)$$

and

$$\begin{aligned}\langle X_m(D_1)Y_n(D_1 + D)\rangle_{av} &= \langle X_m(0)Y_n(D)\rangle_{av} \\ &= NE\{\sin[\beta D\cos(\theta_u - \alpha) - (m - n)\beta d\cos\theta_u]\} \\ &= 0.\end{aligned} \qquad (32)$$

Also

$$\langle X_m^2(D_1)\rangle = \langle Y_m^2(D_1)\rangle = N. \qquad (33)$$

Substituting equations (28), (32), and (33) into the following equation[6,14]

$$\rho_{mn}(D; d) \doteqdot \frac{\langle X_m(0; d)X_n(D; d)\rangle_{av}^2 + \langle X_m(0; d)Y_n(D; d)\rangle_{av}^2}{\langle X_m^2(0; d)\rangle_{av}^2}. \qquad (34)$$

Then we obtain the final result

$$\rho_{mn}(D; d) \doteqdot J_0^2[(\beta D)^2 + (m - n)^2(\beta D)^2 - 2(m - n)^2 Dd\cos\alpha]^{\frac{1}{2}} \qquad (35)$$

and

$$\rho_{mn}(d) = \rho_{mn}(0; d) = J_0^2[(m - n)\beta d]. \tag{36}$$

We also can show the relations

$$\rho_{12} = \rho_{23} = \rho_{34} ; \qquad \rho_{21} = \rho_{32} = \rho_{43} ,$$

$$\rho_{13} = \rho_{24} = \rho_{35} ; \qquad \rho_{31} = \rho_{42} = \rho_{53} ,$$

$$\rho_{mn} \neq \rho_{nm} \quad \text{for} \quad m \neq n,$$

$$\rho_{mn}(D; d) = \rho_{nm}(-D; d).$$

APPENDIX B

Finding a Normalized Correlation Function From a Real Part and an Imaginary Part of a Signal Received From a Directional Antenna Array

It is easy to show that a signal consisting of N multipath vertical polarized waves received from an equal-spaced directional antenna array at a distance D from a reference position is[16]

$$E_z(D; d) = \sum_{u=1}^{N} A_u\{1 + \exp(j\psi) + \exp(j2\psi) + \exp(j3\psi) + \cdots + \exp[j(M - 1)\psi]\} \cdot \exp(j\beta D \cos\theta_u), \tag{37}$$

where A_u was defined in equation (24),

- $\psi = \beta d \sin(\alpha - \theta_u) + \delta$,
- d is antenna spacing between two antennas,
- M is the number of elements,
- α is the normal direction of the array,
- δ is the relative phase between antennas,
- D is the distance measured from the coordinate origin to the center position of antenna array. (The center position of the antenna array is assumed always on the axis, that is, at the position $(D, 0)$.), and
- θ_u is the angle of arrival of the uth wave and is assumed to be uniformly distributed.

The coordinate system of a directional antenna array is shown in Fig. 6. Since the spacings between antennas are equal, we can let the phase refer to the center point of the array. Then equation (37) can be simplified by combining the first term and the Mth term, the second term and the $(M - 1)$th term and so forth.[13] The result becomes

$$E_z(D; d) = \sum_{u=1}^{N} A_u \exp (j\beta D \cos \theta_u) \cdot \left[2 \cos \left(\frac{M-1}{2} \psi \right) + 2 \cos \left(\frac{M-3}{2} \psi \right) + \cdots + 2Q \right], \tag{38}$$

where

$$Q = 1 \quad \text{if} \quad M = \text{odd}$$
$$= \cos (\tfrac{1}{2}\psi) \quad \text{if} \quad M = \text{even}.$$

Equation (38) can be separated into a real part and an imaginary part as

$$E_z(D; d) = X + jY$$

and

$$\epsilon(D; d) = | E_z(D; d) | = (X^2 + Y^2)^{\frac{1}{2}}, \tag{39}$$

where

$$X = 2 \sum_{m=1}^{K} \sum_{u=1}^{N} [R_u \cos (\beta D \cos \theta_u) - S_u \sin (\beta D \cos \theta_u)] \cdot \cos \left(\frac{M + 1 - 2m}{2} \psi \right)$$
$$= 2 \sum_{m=1}^{K} x_m \tag{40}$$

$$Y = 2 \sum_{m=1}^{K} \sum_{u=1}^{N} [R_u \sin (\beta D \cos \theta_u) + S_u \cos (\beta D \cos \theta_u)] \cdot \cos \left(\frac{M + 1 - 2m}{2} \psi \right)$$
$$= 2 \sum_{m=1}^{K} y_m , \tag{41}$$

where

$$K = \frac{M}{2} \quad \text{if} \quad M \text{ is even}$$
$$= \frac{M+1}{2} \quad \text{if} \quad M \text{ is odd}. \tag{42}$$

Since R_u and S_u are independent gaussian variables, it is easy to realize that all x_m and y_m are independent gaussian variables. Hence X and Y are also gaussian variables. The mean values of X and Y are zeros, and the mean squares of X and Y are the same. Therefore equation (21) can be applied. The following term in equation (21) can be replaced by

$$\langle X_1(0; d)X_2(D; d)\rangle_{av} = 4\left[\sum_{m=1}^{K} x_m(0; d)\right]\left[\sum_{m=1}^{K} x_m(D; d)\right]$$
$$= 4 \sum_{m=1}^{K} \sum_{n=1}^{K} \langle x_m(0; d)x_n(D; d)\rangle_{av} . \tag{43}$$

The term $\langle X_1(0; d) Y_2(D; d)\rangle$ also can be obtained, and is equal to equation (43), by replacing x_n by y_n. Then equation (20) becomes

$$\rho_\epsilon(D/d) = \frac{\left[\sum_{m=1}^{K} \sum_{n=1}^{K} \langle x_m(0; d)x_n(D; d)\rangle_{av}\right]^2 + \left[\sum_{m=1}^{K} \sum_{n=1}^{K} \langle x_m(0; d)y_n(D; d)\rangle_{av}\right]^2}{\left[\sum_{m=1}^{K} \sum_{n=1}^{K} \langle x_m(0; d)x_n(0; d)\rangle_{av}\right]^2}, \tag{44}$$

where K is shown in equation (42), and

$$\langle x_m(0; d)x_n(D; d)\rangle_{av} = \frac{N}{2} \langle \cos (\beta D \cos \theta_u) \cdot \{ \cos [(M + 1 - m - n)\psi] + \cos [(m - n)\psi]\}\rangle_{av} ,$$
$$\langle x_m(0; d)y_n(D; d)\rangle_{av} = \frac{N}{2} \langle \sin (\beta D \cos \theta_u) \cdot \{ \cos [(M + 1 - m - n)\psi] + \cos [(m - n)\psi]\}\rangle_{av} , \tag{45}$$

and

$$\psi = \beta d \sin (\alpha - \theta_u) + \delta.$$

Now we may consider only a broadside directional antenna array, that is, $\delta = 0$. Then the following terms can be derived:[15]

$$\langle \cos (a \cos \theta_u) \cdot \cos [b \sin (\alpha - \theta_u)]\rangle_{av}$$
$$= \tfrac{1}{2}\langle \cos [(a + b \sin \alpha)\cos \theta_u - b \cos \alpha \sin \theta_u] + \cos [(a - b \sin \alpha) \cos \theta_u + b \cos \alpha \sin \theta_u]\rangle_{av}$$
$$= \tfrac{1}{2}[J_0(A) + J_0(B)], \tag{46}$$

where

$$A = (a^2 + 2ab \sin \alpha + b^2)^{\frac{1}{2}},$$
$$B = (a^2 - 2ab \sin \alpha + b^2)^{\frac{1}{2}},$$
$$\langle \sin (a \cos \theta_u) \cos [b \sin (\alpha - \theta_u)] \rangle = 0. \tag{47}$$

Inserting the general formulas equation (46) and equation (47) into equation (45), it becomes

$$\langle x_m(0; d) x_n(D; d) \rangle_{\text{av}} = \frac{N}{2} [J_0(A_1) + J_0(B_1) + J_0(A_2) + J_0(B_2)]$$
$$\langle x_m(0; d) y_n(D; d) \rangle_{\text{av}} = 0 \tag{48}$$

where

$$\left.\begin{matrix} A_1 \\ B_1 \end{matrix}\right\} = \beta[D^2 \pm 2Dd(M + 1 - m - n) \sin \alpha + d^2(M + 1 - m - n)^2]^{\frac{1}{2}}$$

$$\left.\begin{matrix} A_2 \\ B_2 \end{matrix}\right\} = \beta[D^2 \pm 2Dd(m - n) \sin \alpha + d^2(m - n)^2]^{\frac{1}{2}}.$$

From equation (48), we can deduce the results

$$\begin{aligned} \langle x_m(0; d) x_n(0; d) \rangle_{\text{av}} & \\ &= N\{J_0[\beta d(M + 1 - m - n)] + J_0[\beta d(m - n)]\} \\ &= N[J_0(A_0) + J_0(B_0)] \end{aligned} \tag{49}$$

and

$$\langle x_m^2(0; d) \rangle = N\{J_0[\beta d(M + 1 - 2m)] + 1\}. \tag{50}$$

Then substituting equations (48) and (49) into equation (44), we complete the derivation of a normalized correlation function of a signal received from a broadside directional antenna array.

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