The Theory of Cylindrical Magnetic Domains

By A. A. THIELE

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The theory of cylindrical magnetic domains provides conditions governing the size and stability of circular cylindrical magnetic domains in plates of uniaxial magnetic materials together with an estimate of the range of applicability of these conditions. The results of the theory are directly applicable to the design of cylindrical domain devices. Computation to first and second order of the energy variation resulting from general small deviation in the domain shape from an initially circular shape yields the conditions governing domain size and stability. The physical origin of the various terms in the energy expansion is examined in detail. A graph from which many domain size and stability properties may be obtained summarizes the results of the energy variation calculation. The minimum theoretically attainable domain diameter is approximately $\sigma_w/\pi M_s^2$, *where* σ_w *is the wall energy density and* M_s *is the saturation magnetization. For domains to exist, the effective anisotropy field must be greater than* $4\pi M_s$.

I. INTRODUCTION

The recent development of a technique for the propagation of isolated magnetic domains in an arbitrary direction in anisotropic ferromagnetic thin films by P. C. Michaelis created a renewed interest in the use of domain propagation for device purposes.[1] The technique used by Michaelis for propagating domains along the easy axis is quite different from that used for propagation along the hard axis. During discussions on the possible application of these techniques, A. H. Bobeck, U. F. Gianola, R. C. Sherwood, and W. Shockley suggested that for general symmetrical domain propagation the direction of magnetization must lie normal to the plane of the film[2]. The recognition that rare earth orthoferrites have the required properties came in response to this suggestion.[3] Experimental work on the application of this type of domain motion device was then begun. Although at the present time this work has been largely concentrated on the orthoferrites, there exist other materials, such as the hexagonal ferrites and manganese bismuth, having the required properties.

The present work directs attention to structures in which the properties of the material used require the magnetization to lie normal to the surface of the plate. The modes of operation of devices constructed from such structures are classified according to the effect of wall motion coercivity. In the case of very high wall motion coercivity, the application of shaped applied fields determines the initial domain configuration which is then maintained by coercivity. For very low wall coercivity, on the other hand, the saturation magnetization, wall energy, plate thickness and bias field determine the domain size and shape. Between these two extremes, there is a continuum of intermediate modes. In either extremal mode, a complete set of operations (logic, memory, and transmission) may be performed.[4] The present work concerns only the low coercivity mode and specifically, right circular cylindrical domains in plates of uniform thickness and small variations therefrom. When observed by means of the Faraday effect, cylindrical domains have the appearance (particularly when in motion) of bubbles and therefore are colloquially referred to as "bubbles".

The present work largely treats the theory of cylindrical domains with experiments and applications being considered only briefly. Section II presents the domain model and mode of description. Section III contains the calculation of the energy derivatives used in the investigation of domain size and stability. Section IV contains an interpertation of the energy derivatives in terms of fields and potentials. Section V discusses the solution of the domain size and stability equations. Section VI discusses the range of validity of the domain model used in the previous sections. It is found that several assumptions implicit in the model are related, and a requirement on materials suitable for the production of circular domains is obtained. Appendix A contains a derivation the properties of certain elliptic integrals appearing in the theory of circular domains, Appendix B is a listing of the standard forms and series expansions of the magnetostatic force and stability functions, and Appendix C is a list of mathematical symbols.

II. THE DOMAIN MODEL AND MODE OF DESCRIPTION

Figure 1 shows the magnetic domain structure to be considered here.[5] The isolated magnetic domain is magnetized downward while the remainder of the plate is saturated upward. The domain will be

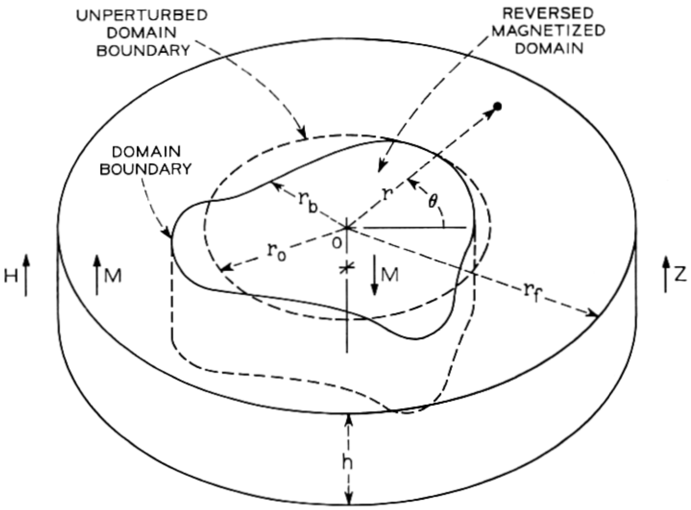

Fig. 1 — Magnetic domain configuration.

considered to be near circular. Examination of the variation of domain energy under a variation of domain shape from the assumed unperturbed shape yields domain stability. Once created, a cylindrical domain continues to exist if the magnetic configuration meets the conditions for stable equilibrium. The stability of a given configuration, however, does not guarantee that it can be produced. The generation of cylindrical domains is a separate problem which is not treated here.

2.1 *Description of the Domain*

A cylindrical (r, θ, z) coordinate system is placed at the center of the domain with its z-axis perpendicular to the plane of the plate. The plate is taken to have planar surfaces and a uniform thickness h. Only the case of a plate of infinite extent, $r_f = \infty$, is considered here. It is assumed that the material constraints allow the magnetization to lie only along the z-axis and the magnitude of the magnetization is independent of the local magnetic field. The boundary between the two regions of magnetization, the domain wall, is assumed to be independent of z (no wall bulging) and to have a width which is negligible in comparison to the domain radius. It is assumed that a wall energy density per unit area σ_w may be assigned independently of either the orientation or curvature of the wall. The assumptions about the detailed magnetic configuration (the magnetization magnitude and orientation and the wall energy and shape) are coupled by the material properties. Section VI contains a detailed discussion of the validity of these assumptions and the cylindrical wall assumption. Even though the foregoing assumptions appear quite drastic and restrictive, experimentally there does exist a region in which the results obtained under these assumptions are both accurate and useful.

The expansion

$$r_b(\theta) = \sum_{n=0}^{\infty} r_n \cos [n(\theta - \theta_n)] \tag{1}$$

of $r_b(\theta)$ in terms of the Fourier coefficients, r_n and θ_n, describes the domain shape in the plane. The n value is called the "rotational periodicity." The condition

$$| r_0 | \gg \sum_{n=1}^{\infty} n | r_n | \tag{2}$$

assures that the domain is near circular and that the function $r_b(\theta)$ is single valued and smooth.

It is convenient to introduce the finite variations of the r_n and θ_n, Δr_n and $\Delta \theta_n$, respectively, in order to describe small variations in domain size and shape from the strictly circular domain of radius r_0 $[r_b(\theta) = r_0]$. In terms of these variations, a small variation of the wall shape from $r_b(\theta) = r_0$ may be written as

$$r_b(\theta) = r_0 + \Delta r_0 + \sum_{n=1}^{\infty} \Delta r_n \cos [n(\theta - \theta_n - \Delta \theta_n)] \tag{3a}$$

where, by assumption,

$$| r_0 | \gg | \Delta r_0 | + \sum_{n=1}^{\infty} n | \Delta r_n |. \tag{3b}$$

Subject to the restrictions stated, equation (3) describes an arbitrary variation because of the completeness of the Fourier expansion.

The externally applied magnetic field, **H**, is taken to be spatially uniform and to lie in the positive z direction. (The presence of a component of the applied field in the plane of the plate has no effect to the approximation that the magnetization lies only along the z-axis.)

The assumed simple forms of the applied field and magnetic configurations permit the use of simple formal expressions for these quantities. The expression for the externally applied field is

$$\mathbf{H} = H\mathbf{i}_z \tag{4}$$

where H is a constant and $\mathbf{i}_z$ is the unit vector in the z-direction. The magnetization may be written in terms of the unit step function,

$$u(x) \equiv \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0, \end{cases} \tag{5}$$

as

$$\mathbf{M} = \mathbf{i}_z M_s \{1 - 2u[r_b(\theta) - r]\} u(z + \tfrac{1}{2}h) u(-z + \tfrac{1}{2}h). \tag{6}$$

2.2 *The Energy Variation*

The investigation of domain size and stability proceeds by computing the first and second variations of the total system energy with respect to the r_n and θ_n. The total energy of the domain is

$$E_T = E_W + E_H + E_M, \tag{7}$$

where E_W is the total wall energy, E_H is the interaction energy with the externally applied field, and E_M is the internal magnetostatic energy. The total wall energy, under the previously stated assumptions, is the product of the wall energy density σ_w and the wall area a:

$$E_W = \int_a \sigma_w \, da = h\sigma_w \int_0^{2\pi} \left\{ r_b^2(\theta) + \left[\frac{\partial r_b(\theta)}{\partial \theta} \right]^2 \right\}^{\frac{1}{2}} d\theta. \tag{8}$$

The interaction energy of the magnetization with the externally applied field is

$$E_H = -\int_V \mathbf{M} \cdot \mathbf{H} \, dV = -\int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} M_z H r \, dr \, d\theta \, dz, \tag{9}$$

and the internal magnetostatic energy is

$$E_M = \frac{1}{2} \int_V \int_{V'} \frac{\nabla \cdot \mathbf{M} \nabla' \cdot \mathbf{M}'}{| \mathbf{r} - \mathbf{r}' |} dV' \, dV$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{\dfrac{\partial M_z}{\partial z} \dfrac{\partial M_z'}{\partial z'}}{s} rr' \, dr \, d\theta \, dz \, dr' \, d\theta' \, dr' \tag{10a}$$

where

$$s^2 \equiv r^2 + r'^2 - 2rr' \cos(\theta - \theta') + (z - z')^2. \tag{10b}$$

In expressions (9) and (10), V indicates volume and primes indicate quantities in the second coordinate system used in describing the internal magnetostatic interaction.

The variation in the total energy when the r_n and θ_n are varied is

$$\triangle E_T = \sum_{n=0}^{\infty} \left[\left(\frac{\partial E_T}{\partial r_n} \right)_0 \triangle r_n + \left(\frac{\partial E_T}{\partial \theta_n} \right)_0 \triangle \theta_n \right]$$
$$+ \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\left(\frac{\partial^2 E_T}{\partial r_n \, \partial r_m} \right)_0 \triangle r_n \triangle r_m + 2 \left(\frac{\partial^2 E_T}{\partial r_n \, \partial \theta_m} \right)_0 \triangle r_n \triangle \theta_m \right.$$
$$\left. + \left(\frac{\partial^2 E_T}{\partial \theta_n \, \partial \theta_m} \right)_0 \triangle \theta_n \triangle \theta_m \right] + O_3 \tag{11}$$

where the subscript O refers to evaluation of the partial derivatives at the circular domain state, $r_b(\theta) = r_0$, and O_3 refers to terms of order three and higher in the combination of $\triangle r_n$ and $\triangle \theta_n$. The first partial derivatives of the energy, $(\partial E_T/\partial r_n)_0$ and $(\partial E_T/\partial \theta_n)_0$, are the generalized forces of the system, while the second derivatives of the total energy form the elements of the stiffness matrix.

Knowledge of the generalized forces and the stiffness matrix completely characterizes domain size and stability. It is shown in Section III that only the energy derivatives, $(\partial E_T/\partial r_0)_0$ and the $(\partial^2 E_T/\partial r_n^2)_0$, are non-zero when $r_b(\theta) = r_0$. The equation obtained by setting the only nonzero generalized force equal to zero is called the "force equation." The expansion (1) is a quasi-normal mode expansion since circular domains are completely metastable with respect to the θ_n and the stiffness matrix is diagonal with respect to the r_n.

III. CALCULATION OF THE ENERGY DERIVATIVES

3.1 *Derivatives of the Wall Energy*

The derivatives of the total wall energy are computed by substituting the wall shape expression (1) into the wall energy expression (8), noting that

$$\frac{\partial r_b}{\partial \theta} = -\sum_{n=1}^{\infty} n r_n \sin [n(\theta - \theta_n)] \tag{12}$$

and differentiating under the integral sign. There results

$$\frac{\partial E_W}{\partial r_n} = h\sigma_w \int_0^{2\pi} \left\{ r_b \cos [n(\theta - \theta_n)] - \frac{\partial r_b}{\partial \theta} n \sin [n(\theta - \theta_n)] \right\}$$
$$\cdot \left[r_b^2 + \left(\frac{\partial r_b}{\partial \theta} \right)^2 \right]^{-\frac{1}{2}} d\theta \tag{13a}$$

and

$$\frac{\partial^2 E_W}{\partial r_n\,\partial r_m} = h\sigma_w \int_0^{2\pi} \Bigg(\Big\{\cos[n(\theta - \theta_n)]\cos[m(\theta - \theta_m)] + nm\sin[n(\theta - \theta_n)]\sin[m(\theta - \theta_m)]\Big\}\left[r_b^2 + \left(\frac{\partial r_b}{\partial \theta}\right)^2\right]^{-\frac{1}{2}}$$

$$- \left\{r_b\cos[n(\theta - \theta_n)] - \frac{\partial r_b}{\partial \theta}\, n\sin[n(\theta - \theta_n)]\right\}$$

$$\cdot\left\{r_b\cos[m(\theta - \theta_m)] - \frac{\partial r_b}{\partial \theta}\, m\sin[m(\theta - \theta_m)]\right\}$$

$$\cdot\left[r_b^2 + \left(\frac{\partial r_b}{\partial \theta}\right)^2\right]^{-\frac{3}{2}}\Bigg)\, d\theta \qquad (13b)$$

with analogous expressions for

$$\partial E_W/\partial\theta_n\,, \qquad \partial^2 E_W/\partial\theta_n\,\partial\theta_m\,, \quad \text{and} \quad \partial^2 E_W/\partial r_n\,\partial\theta_m\,.$$

Evaluating equations (13) for a circular domain,

$$r_b(\theta) = r_0 \quad \text{and} \quad [\partial r_b(\theta)/\partial\theta] = 0,$$

the circular domain derivatives are

$$\left(\frac{\partial E_W}{\partial r_0}\right)_0 = 2\pi h\sigma_w \qquad (14a)$$

$$\left(\frac{\partial^2 E_W}{\partial r_n^2}\right)_0 = \frac{\pi}{r_0}\, h\sigma_w n^2, \qquad n \geqq 1 \qquad (14b)$$

and all of the first and second derivatives of the total wall energy not explicitly stated are zero.

3.2 *Derivatives of the Applied Field Interaction Energy*

The applied field interaction energy is evaluated by substituting the formal expressions for the applied field (4) and the magnetic configuration (6) into the applied field interaction expression (9), changing the order of integration, and integrating.

$$E_H = -M_s H \int_0^{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \{1 - 2u[r_b(\theta) - r]\} \times u(z + \tfrac{1}{2}h)u(-z + \tfrac{1}{2}h)r\, dz\, dr\, d\theta \qquad (15a)$$

$$= hM_s H\left[\int_0^{2\pi} r_b^2(\theta)d\theta\right] - \text{constant}. \qquad (15b)$$

The infinite constant is independent of the r_n and θ_n and does not contribute to the derivatives. Differentiating yields

$$\frac{\partial E_H}{\partial r_n} = 2hM_sH \int_0^{2\pi} r_b \cos [n(\theta - \theta_n)] d\theta \tag{16a}$$

and

$$\frac{\partial^2 E_H}{\partial r_n \, \partial r_m} = 2hM_sH \int_0^{2\pi} \cos [n(\theta - \theta_n)] \cos [m(\theta - \theta_m)] \, d\theta \tag{16b}$$

with analogous expressions for

$$\partial E_H/\partial \theta_n , \qquad \partial^2 E_H/\partial r_n \, \partial \theta_m , \quad \text{and} \quad \partial^2 E_H/\partial \theta_n \, \partial \theta_m .$$

Evaluation of equation (16) for $r_b(\theta) = r_0$ yields

$$\left(\frac{\partial E_H}{\partial r_0}\right)_0 = 4\pi r_0 hM_sH, \tag{17a}$$

$$\left(\frac{\partial^2 E_H}{\partial r_0^2}\right)_0 = 4\pi hM_sH, \tag{17b}$$

$$\left(\frac{\partial^2 E_H}{\partial r_n}\right)_0 = 2\pi hM_sH, \qquad n \geqq 1, \tag{17c}$$

and all the other first and second derivatives of the applied field interaction energy are zero.

3.3 *Derivatives of the Internal Magnetostatic Energy*

The formal expression for the internal magnetostatic energy is obtained by substituting the expression for the magnetic configuration (6) into expression (10). In dealing with the self-interaction energy, it is necessary to use two coordinate systems: an unprimed system and a primed system. Throughout the following calculation functions of the spatial coordinates (r, θ, and z) are written with primes whenever they are of the primed coordinates. Thus $\mathbf{M}$, when considered as a function of the primed coordinates, is written $\mathbf{M}'$. The subscripted r_n and θ_n are independent parameters and are never primed.

The calculation begins with the evaluation of $\partial M_z/\partial z$ by differentiating expression (6) and noting that

$$\frac{d}{dx} u(x) = \delta(x) \tag{18}$$

where $\delta(x)$ is the Dirac delta function. Then

$$\frac{\partial M_z}{\partial z} = M_s kg \tag{19a}$$

where

$$k[r, r_b(\theta)] \equiv 1 - 2u[r_b(\theta) - r] \tag{19b}$$

and

$$g(z) \equiv \delta\left(z + \frac{h}{2}\right) - \delta\left(-z + \frac{h}{2}\right). \tag{19c}$$

After changing the order of integration, the expression for the internal magnetostatic energy becomes

$$E_M = \frac{1}{2} M_s^2 \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{kk'gg'rr'}{s} dz\, dz'\, d\theta\, d\theta'\, dr\, dr'. \tag{20}$$

The factor $g(z)g(z')/s$ contains the z and z' dependence of this integral. From expression (19c) it can be seen that this factor consists of four terms. Application of the transformation $(z, z') \rightarrow (-z, -z')$ to two of the terms under the integral sign combines these four terms into two terms. Making the transformation $(z, z') \rightarrow (z, \mathsf{z})$, where

$$\mathsf{z} \equiv z - z', \tag{21}$$

on the remaining terms and carrying out the integration over z yields the expression for the internal magnetostatic energy in terms of an integral over surface magnetic charges. This expression is

$$E_M = M_s^2 \mathsf{Z} \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \frac{kk'rr'}{s} d\theta\, d\theta'\, dr\, dr' \tag{22}$$

where Z is an operator defined by

$$\mathsf{Z}\{\ \} \equiv \int_{-\infty}^\infty d\mathsf{z}[\delta(\mathsf{z}) - \delta(\mathsf{z} - h)]\{\ \} \tag{23}$$

$$s^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta') + \mathsf{z}^2. \tag{24}$$

The factor kk' contains the r_n and θ_n dependence of the integral so that the derivatives of E_M may be calculated by replacing this factor by its derivatives under the integral. Evaluating the first derivatives yields

$$\frac{\partial kk'}{\partial r_n} = k \frac{\partial k'}{\partial r_b(\theta')} \frac{\partial r_b(\theta')}{\partial r_n} + k' \frac{\partial k}{\partial r_b(\theta)} \frac{\partial r_b(\theta)}{\partial r_n}$$

$$= k \frac{\partial k'}{\partial r_b} \cos[n(\theta' - \theta_n)] + k' \frac{\partial k}{\partial r_b} \cos[n(\theta - \theta_n)] \tag{25a}$$

$$\frac{\partial kk'}{\partial \theta_n} = k \frac{\partial k'}{\partial r_b(\theta')} \frac{\partial r_b(\theta')}{\partial \theta_n} + k' \frac{\partial k}{\partial r_n(\theta)} \frac{\partial r_b(\theta)}{\partial \theta_n}$$

$$= -k \frac{\partial k'}{\partial r_b} n r_n \sin [n(\theta' - \theta_n)] - k' \frac{\partial k}{\partial r_n} n r_n \sin [n(\theta - \theta_n)]. \quad (25b)$$

Substituting these derivatives into the integral and exchanging the primed and unprimed r and θ. The first term becomes identical to the second. The derivatives of the internal magnetic interaction energy are then

$$\frac{\partial E_M}{\partial r_n} = 2M_s^2 \mathsf{Z} \int_0^\infty \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} \frac{1}{s} k' \frac{\partial k}{\partial r_b} \cdot \cos [n(\theta - \theta_n)] r' r \, d\theta' \, dr' \, d\theta \, dr \quad (26a)$$

$$\frac{\partial^2 E_M}{\partial r_n \, \partial r_m} = 2M_s^2 \mathsf{Z} \int_0^\infty \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} \frac{1}{s} \cdot \left\{ k' \frac{\partial^2 k}{\partial r_b^2} \cos [n(\theta - \theta_n)] \cos [m(\theta - \theta_m)] + \frac{\partial k'}{\partial r_b} \frac{\partial k}{\partial r_b} \cdot \cos [n(\theta - \theta_n)] \cos [m(\theta' - \theta_m)] \right\} r' r \, d\theta' \, dr' \, d\theta \, dr \quad (26b)$$

with analogous expressions for

$$\partial E_M / \partial \theta_n , \qquad \partial^2 E_M / \partial r_n \, \partial \theta_m , \quad \text{and} \quad \partial^2 E_M / \partial \theta_n \, \partial \theta_m .$$

For circular domains ($r_b = r_0$) the factors k, k', $\partial k / \partial r_b$, and $\partial k' / \partial r_b$ are independent of θ and θ' so that the integrands are periodic in θ' with periodicity 2π. The range of integration of θ' may therefore be changed from $[0, 2\pi]$ to $[\theta, 2\pi + \theta]$ so that after making the transformation $(\theta, \theta') \rightarrow (\theta, \zeta)$ where

$$\zeta \equiv \theta' - \theta, \quad (27)$$

the range of integration of both θ and ζ is again $[0, 2\pi]$. Note that now

$$s^2 = r^2 + r'^2 - 2rr' \cos \zeta + z^2 \quad (28)$$

depends only on ζ.

Using trigonometric identities, the integrands of the integrals for the various derivatives are written as a sum of terms each of which is the product of a factor depending only on θ and a factor depending on ζ. Carrying out the integration over θ yields, for $r_b(\theta) = r_0$,

$$\left(\frac{\partial E_M}{\partial r_0} \right)_0 = 4\pi M_s^2 \mathsf{Z} \int_0^\infty \int_0^\infty \int_0^{2\pi} \frac{1}{s} k' \frac{\partial k}{\partial r_b} r' r \, d\zeta \, dr' \, dr, \quad (29a)$$

$$\left(\frac{\partial^2 E_M}{\partial r_0^2}\right)_0 = 4\pi M_s^2 \mathsf{Z} \int_0^\infty \int_0^\infty \int_0^{2\pi} \frac{1}{s}\left(k' \frac{\partial^2 k}{\partial r_b^2} + \frac{\partial k'}{\partial r_b}\frac{\partial k}{\partial r_b}\right) r' r \, d\zeta \, dr' \, dr, \tag{29b}$$

$$\left(\frac{\partial^2 E_M}{\partial r_n^2}\right)_0 = \frac{1}{2} 4\pi M_s^2 \mathsf{Z} \int_0^\infty \int_0^\infty \int_0^{2\pi} \frac{1}{s} \cdot\left(k' \frac{\partial^2 k}{\partial r_b^2} + \frac{\partial k'}{\partial r_b}\frac{\partial k}{\partial r_b} \cos n\zeta\right) r' r \, d\zeta \, dr' \, dr, \quad n > 0, \tag{29c}$$

while all the remaining first and second derivatives are zero. Note that by inspection of these integrals and the definitions of k, k', and r_b that

$$\left(\frac{\partial^2 E_M}{\partial r_0^2}\right)_0 = \frac{\partial}{\partial r_0}\left(\frac{\partial E_M}{\partial r_0}\right)_0 \tag{30a}$$

and

$$\left(\frac{\partial^2 E_M}{\partial r_n^2}\right)_0 = \frac{1}{2}\frac{\partial}{\partial r_0}\left(\frac{\partial E_M}{\partial r_0}\right)_0 - \frac{1}{2} 4\pi M_s^2 \mathsf{Z} \int_0^\infty \int_0^\infty \int_0^{2\pi} \frac{\partial k'}{\partial r_b}\frac{\partial k}{\partial r_b}\frac{(1 - \cos n\zeta)}{s} \cdot r' r \, d\zeta \, dr' \, dr, \qquad n > 0. \tag{30b}$$

Noting that from expressions (18) and (19b)

$$\frac{\partial k}{\partial r_b} = -2\,\delta(r - r_b) \tag{31}$$

and using the definition of the Z operator given in expression (24), expression (29a) may be integrated with respect to r and z, and the second term of expression (30b) may be integrated with respect to r, r', and z. The result after some rearrangement is

$$\left(\frac{\partial E_M}{\partial r_0}\right)_0 = -(2\pi h^2)(4\pi M_s^2) F(2r_0/h), \tag{32a}$$

$$\left(\frac{\partial^2 E_M}{\partial r_0^2}\right)_0 = -(4\pi h)(4\pi M_s^2)\frac{\partial F(2r_0/h)}{\partial(2r_0/h)}, \tag{32b}$$

$$\left(\frac{\partial^2 E_M}{\partial r_n^2}\right)_0 = -(2\pi h)(4\pi M_s^2)\frac{\partial F(2r_0/h)}{\partial(2r_0/h)} + (h)(4\pi M_s^2)\frac{2r_0}{h}\left[L_n\left(\left(\frac{h}{2r_0}\right)^2\right) - L_n(0)\right] \tag{32c}$$

where renaming r' to r and using expression (19b)

$$F\left(\frac{2r_0}{h}\right) \equiv \frac{2r_0}{\pi h^2}[2B(r_0, r_0, h) - 2B(r_0, r_0, 0) - B(r_0, \infty, h) + B(r_0, \infty, 0)], \tag{33a}$$

$$B(r_0, r_f, z) \equiv \int_0^{\pi} \int_0^{r_b} (\rho^2 + z^2)^{-\frac{1}{2}} r \, dr \, d\zeta, \tag{33b}$$

$$\rho^2 = r_0^2 + r^2 - 2r_0 r \cos \zeta, \tag{33c}$$

and where

$$L_n[(z/2r_0)^2] \equiv \int_0^{\pi} [(z/2r_0)^2 + \tfrac{1}{2}(1 - \cos \zeta)]^{-\frac{1}{2}}(1 - \cos n\zeta) \, d\zeta. \tag{34}$$

The L_n functions are reduced to standard elliptic integral form, and power series expansions are obtained for both large and small values of the argument in Appendix A. The B function is integrated once after displacing the origin of the cylindrical coordinate system from 0 to 0′ as is shown in Fig. 2. The transformation connecting the (r, ζ) and (ρ, φ) coordinate systems is

$$\rho \sin \varphi = r \sin \zeta \tag{35a}$$

$$\rho \cos \varphi = r \cos \zeta - r_0 . \tag{35b}$$

After the transformation

$$B(r_0, r_f, z) = \begin{cases} \displaystyle\int_0^{\pi} \int_0^{r=r_f} \frac{\rho \, d\rho \, d\varphi}{(\rho^2 + z^2)^{\frac{1}{2}}}, \; r_0 < r_f & \text{(36a)} \\ \displaystyle\int_{\pi/2}^{\pi} \int_0^{r_0=r_f} \frac{\rho \, d\rho \, d\varphi}{(\rho^2 + z^2)^{\frac{1}{2}}}, \; r_0 = r_f . & \text{(36b)} \end{cases}$$

Equations (36a) and (36b) are integrated to obtain, in either case,

$$B(r_0, r_f, z) = \int_{\zeta=0}^{\zeta=\pi} (\rho_b^2 + z^2)^{\frac{1}{2}} \, d\varphi - \int_{\zeta=0}^{\zeta=\pi} |z| \, d\varphi \tag{37}$$

where ρ_b is the value of ρ along the boundary $r = r_f$. For $r_f = r_0$, $\rho_b = -2r_0 \cos \varphi$ so that

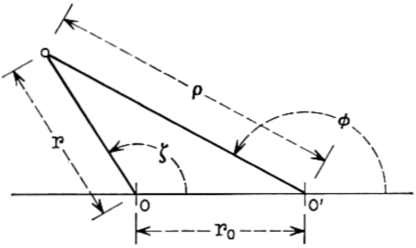

Fig. 2 — The (ζ, r) and (ρ, φ) coordinate systems.

$$B(r_0, r_0, h) = \int_{\pi/2}^{\pi} [h^2 + (2r_0)^2 \cos^2 \varphi]^{\frac{1}{2}} \, d\varphi - \frac{\pi}{2} \mid h \mid \tag{38a}$$

and

$$B(r_0, r_0, 0) = 2r_0 . \tag{38b}$$

The remaining two terms of equation (33a) must be evaluated as a limit

$$\begin{aligned} \lim_{r_f \to \infty} [B(r_0, r_f, 0) - B(r_0, r_f, h)] \\ &= \lim_{r_f \to \infty} \int_0^{\pi} - [(\rho_b^2 + h^2)^{\frac{1}{2}} - \rho_b] \, d\varphi + \pi \mid h \mid \\ &= \pi \mid h \mid \end{aligned} \tag{39}$$

since ρ_b approaches infinity when r_f approaches infinity. Combining these results yields

$$F(2r_0/h) = \frac{2}{\pi} (2r_0/h)^2 \left\{ \int_0^{\pi/2} [(h/2r_0)^2 + \sin^2 \varphi]^{\frac{1}{2}} \, d\varphi - 1 \right\} \tag{40}$$

and

$$\frac{\partial F(2r_0/h)}{\partial (2r_0/h)} = (h/2r_0) \left\{ 2F(2r_0/h) - \frac{2}{\pi} \int_0^{\pi/2} [(h/2r_0)^2 + \sin^2 \varphi]^{-\frac{1}{2}} \, d\varphi \right\} . \tag{41}$$

Appendix B lists the standard elliptic integral form of the force function F and power series expansions for large and small values of the argument. In Fig. 3 the force function is plotted as a function of the domain diameter measured in units of the plate thickness

$$d/h = 2r_0/h. \tag{42}$$

The stability functions S_n, also shown on this plot, are defined in Section 5.1.

IV. THE ENERGY VARIATION—ORIGIN OF TERMS

Summing the results of the last section according to expression (7), the total energy variation expression (11) is

$$\begin{aligned} \Delta E = {} & [2\pi h \sigma_w + 4\pi r_0 h M_s H - (2\pi h^2)(4\pi M_s^2) F(2r_0/h)] \, \Delta r_0 \\ & + \frac{1}{2} \left[4\pi h M_s H - (4\pi h)(4\pi M_s^2) \frac{\partial F(2r_0/h)}{\partial (2r_0/h)} \right] (\Delta r_0)^2 \\ & + \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{\pi}{r_0} h \sigma_w n^2 + 2\pi h M_s H - (2\pi h)(4\pi M_s^2) \frac{\partial F(2r_0/h)}{\partial (2r_0/h)} \right. \\ & \left. + (h)(4\pi M_s^2) \frac{2r_0}{h} [L_n((h/2r_0)^2) - L_n(0)] \right\} (\Delta r_n)^2 + O_3 \end{aligned} \tag{43}$$

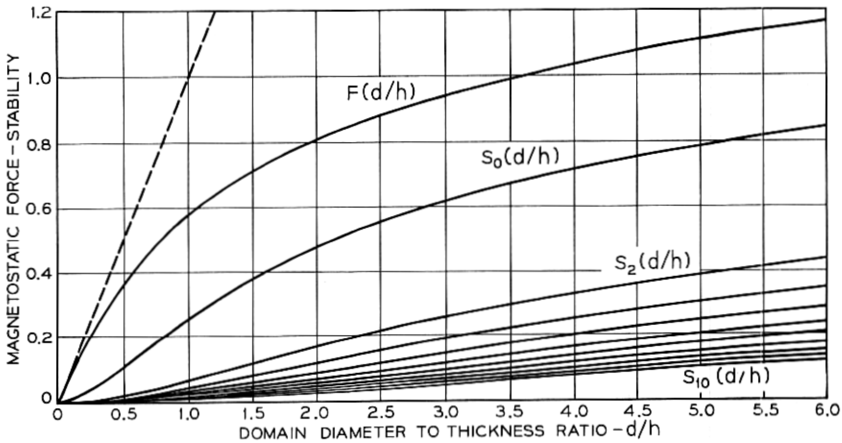

Fig. 3 — The magnetostatic radial force function F and stability functions, $S_0 - S_{10}$, $S_1 = 0$, as functions of domain diameter to thickness ratio, d/h.

where F is defined by expression (33) and plotted in Fig. 3, the L_n are defined by expression (34), and all terms not explicitly stated are equal to zero. The remainder of this section treats the physical origin of the terms in the energy variation expression (43).

4.1 *The Generalized Forces*

The coefficients of the linear variation terms are the negatives of the generalized forces. All forces except the r_0 force are identically zero, which for a circular domain is a consequence of the rotational symmetry of the system. The first term in the coefficient of Δr_0 is the product of the wall energy density σ_w and the rate of change of wall area with respect to r_0, $2\pi h$. The second term is the product of the external field interaction energy density $2M_sH$ and the rate of change of domain volume with respect to r_0, $2\pi h r_0$. The third term is the rate of change of the internal magnetostatic energy with respect to r_0.

The internal magnetostatic force may be identified in expression (43) and using expressions (32), (33), and (39) may be written in the form

$$-\left(\frac{\partial E_M}{\partial r_0}\right)_0 = 2\pi h^2(4\pi M_s^2)F(2r_0/h) \tag{44a}$$

$$= (2\pi r_0 h)(2M_s)\left\{4\pi M_s + \frac{4M_s}{h}\left[\int_0^{2\pi}\int_0^{r_0}\frac{r\,dr\,d\zeta}{(\rho^2+h^2)^{\frac{1}{2}}} - \int_0^{2\pi}\int_0^{r_0}\frac{r\,dr\,d\zeta}{\rho}\right]\right\} \tag{44b}$$

where ρ^2 is given in expression (33c). In expression (44b) the first factor in parentheses is the domain wall area, the second is the change in magnetization at the wall as the wall moves, and the quantity in braces has the form of an H field whose origin will now be interpreted by superposition of sources. The internally produced field arises from the superposition of the internal field of a plate uniformly magnetized normal to its surface with magnetization magnitude M_s and two disks of magnetic charge of uniform magnetic surface charge density $\pm 2M_s$ and radius r_0. The first term within the braces is thus the demagnetizing field of the infinite plate of uniform magnetization. The second term is the difference in magnetostatic potential between a point on the edge of a disk of magnetic surface charge of uniform density $4M_s$ and a point removed a distance h from this point in a direction normal to the plane of the charge disk divided by the distance h. This is just the z-averaged z-component of the field produced along the wall by the two charge disks since

$$\langle H_z \rangle_{av} \equiv \frac{1}{h} \int_0^h H_z \, dz = \frac{1}{h} \int_0^h - \frac{\partial \Omega}{\partial z} \, dz = -\frac{\Omega(h) - \Omega(0)}{h} \tag{45}$$

where Ω denotes the magnetostatic scalar potential and z is measured from the edge of the disk. Comparing expressions (44a) and (44b), the total internally produced z-averaged z-component of the magnetic field along the domain wall is

$$\langle H_{M_z} \rangle_{av} = -(4\pi M_s)(h/2r_0)F(2r_0/h) \tag{46}$$

so that the total force per unit wall area (averaged over z) is

$$-\frac{1}{2\pi r_0 h} \left(\frac{\partial E_T}{\partial r_0} \right)_0 = -\frac{\sigma_w}{r_0} - 2M_s(H + \langle H_{M_z} \rangle_{av}). \tag{47}$$

The first term is the product of the wall energy density and the wall curvature and always corresponds to an inward directed force. The second term is the change of magnetization at the moving domain wall times the z-averaged z-component of the total field at the wall. [The problem may initially be set up using this fact (Ref. 2, pp. 1922–1925).] The properties of the force function will now be examined in some detail. From expressions (43) or (44a) the first order variation in internal magnetostatic energy, when r_0 is varied, is

$$\Delta E_M = -2(\pi h^2)(4\pi M_s^2)F(2r_0/h)\Delta r_0 . \tag{48}$$

The plot of F in Fig. 3, and the expansions for large and small values of the argument show that the force function is everywhere positive, is monotonic increasing, and has a negative second derivative. Since the force function is everywhere positive, the internal magnetic interaction energy at all times acts in such a way as to expand the domain. Section 4.2 treats the effect of the slope and curvature properties of the force curve on domain size and stability. Substituting the expansion of the force function for small values of the argument (138d) into expression (48) produces the energy variation for small values for r_0/h,

$$\Delta E_M = \{2\pi h r_0 2M_s(-4\pi M_s) + (4\pi M_s^2)16r_0^2 - (\pi h^2)(4\pi M_s^2)2[\tfrac{1}{4}(2r_0/h)^3 - \tfrac{3}{64}(2r_0/h)^5 + \cdots]\}\Delta r_0 . \tag{49}$$

(In the remainder of this section frequent reference will be made to the properties of F and the L_n given in Appendices A and B.) The interaction of the magnetization with the existing field from the infinite dipole sheet produces the first term in expression (49). This may be seen by comparison with expression (47) and observing that the field internally generated in the infinite dipole sheet with no reversals is $-4\pi M_s$. In Fig. 3 a dashed line through the origin with numerical slope one represents this term and forms the small r_0/h asymptotic of F.

The second term in expression (49) is the only thickness independent term in the expansion and therefore must be identical to the variation of self-energy of the two disks of magnetic charge which form the ends of the reversal when r_0 is varied. Since the interaction with the infinite charge sheet and the self-energy of the disk have been taken into account, the remaining terms are the mutual interaction of the magnetic charge disks.

For large r_0/h, an energy expansion in terms of h/r_0 is appropriate. Substituting the expansion of the force function for large values of the argument (138c) into expression (48) yields

$$\Delta E_M = -h^2(4\pi M_s^2)\left\{[1 + \tfrac{3}{16}(h/2r_0)^2 + 0_4] + [2 - \tfrac{1}{4}(h/2r_0)^2 + 0_4] \ln \left| 4\frac{2r_0}{h} \right| \right\}\Delta r_0 . \tag{50}$$

This expansion obscures the identity of both the infinite sheet magnetic field term and the charge plate self-energy term so that a local (to the wall) magnetic energy lowering per unit line length description appears appropriate. However, the energy reduction per unit line length to lowest order in $2r_0/h$ is

$$\frac{E_M\ (\text{domain}) - E_M\ (\text{uniform magnetization})}{2\pi r_0}$$

$$= -\frac{h^2}{\pi}\,(4\pi M_s^2)\,\ln\left|\frac{4}{e^{\frac{1}{2}}}\frac{2r_0}{h}\right| \qquad (51)$$

so that the energy lowering *per unit line length* for the domain of infinite diameter is infinite. [Equation (51) is obtained by integrating equation (50) to lowest order. The integration constant is determined to be zero by term by term integration of the expansions of F for large and small values of the argument and comparing at $2r_0/h = 1$.] The conclusion that the energy lowering per unit line length for an isolated straight line reversal may also be obtained by considering the energy lowering in a strip reversal when the strip width approaches infinity. The author's intention at the outset of this entire calculation was to calculate the numerical value of this magnetic energy reduction per unit wall length. The internal magnetic interaction, however, retains just enough of its global character when the domain is very large so that no finite limiting value for this energy reduction exists.

The internally generated magnetic field at the wall of the domain, for large r_0/h is obtained from expression (46). To lowest order it is

$$\langle H_{M_z}\rangle_{\text{av}} = -\frac{(4\pi M_s)}{\pi}\,\frac{h}{2r_0}\,\ln\left|4e^{\frac{1}{2}}\frac{2r_0}{h}\right| \qquad (52)$$

which approaches zero as the diameter approaches infinity as it must, since for an infinite straight line magnetization reversal, symmetry requires that the z-component of the field be zero along the reversal.

4.2 *The Stiffness Matrix*

The second variation of the energy with respect to the Fourier coefficients describing the domain determines the stability of the domain. Since the stiffness of the domain with respect to externally applied forces is proportional to the coefficient of the bilinear form which is the second variation of the energy, the matrix formed by these coefficients is called the stiffness matrix. The stiffness matrix is composed of three independent submatrices. The second derivatives of the energy with respect to the Fourier amplitudes form the radial stiffness matrix; the second derivatives of the energy with respect to the Fourier phases form the angular stiffness matrix; and the derivatives of the energy with respect to one Fourier amplitude and one Fourier phase form the mixed stiffness matrix. The derivative of the energy with respect to r_n and r_m

are called the (n, m) radial stiffness matrix element, with similar notation for the other submatrices.

All derivatives not explicitly exhibited in expression (43) are zero. Thus, the angular stiffness matrix and the mixed stiffness matrix are zero and the radial stiffness matrix is diagonal so that the system is completely metastable with respect to angle and the amplitudes are normal modes of the system for small amplitudes.

The (0, 0) radial stiffness matrix element is simply the derivative of the negative of the radial generalized force so that no further discussion of it is necessary. It should be noted that the derivative of the internal magnetostatic term with respect to wall position is not directly related to the radial field or potential at the wall since the derivative used in computing the radial field at the wall must be taken with the wall position held fixed.

4.2.1 *The Radial Stiffness Matrix Elements for* $n \geqq 1$

The diagonal radial stiffness matrix elements, for $n \geqq 1$, are the sum of four terms in expression (43). The first term, which always has a stabilizing effect, is the increase in total wall energy due to the lengthening of the wall caused by the deviation from a strictly circular shape. Imposing a sinusoidal variation of amplitude Δs onto a straight line produces a relative increase in length of

$$\frac{s + \Delta s}{s} = 1 + \left(\frac{\pi \Delta r_n}{\lambda_n}\right)^2 + \cdots . \tag{53}$$

The corresponding wavelength in expression (43) is

$$\lambda_n = \frac{2\pi r_0}{n}. \tag{54}$$

The wall energy term in expression (43),

$$\Delta E_{Wn} = \sigma_w 2\pi r_0 h \left(\frac{\pi \Delta r_n}{\lambda_n}\right)^2 , \tag{55}$$

is thus the product of the wall energy density, the wall area, and the variation in wall area per unit area. Notice that the relative variation in wall length or area is independent of the wall curvature, $1/r_0$, to lowest order in the amplitude of the variation.

The second order change in volume of the domain interacting with the externally applied field produces the second term in the radial stiffness matrix elements while the rate of change of the internal magnetostatic forces at the wall produces the third term. The sum of the second and third terms is one-half the (0, 0) radial stiffness matrix element. This factor of one-half relates to the fact that a variation of Δr_n, $n \geqq 1$ produces only one-half the mean square variation $r_b(\theta)$ as is produced by an equal variation in r_0. This shape-independent, second-order variation in energy arises from the variation in the generalized forces, or fields at the wall, when the domain radius is varied. [See also the steps leading to expression (30b).]

4.2.2 *Translation Invariance*

The requirement of translation invariance in the infinite plate completely determines the (1, 1) radial stiffness matrix element. Consider a cylindrical domain of radius r_0 with a cylindrical coordinate system placed at its center. Under a displacement of the coordinate system of magnitude s in the $\theta = \pi$ direction, the description of the boundary in the new coordinate system is

$$r_b(\theta) = r_0 - \frac{1}{4}\frac{s^2}{r_0} + s\cos\theta + \frac{1}{4}\frac{s^2}{r_0}\cos 2\theta + 0_4 . \tag{56a}$$

Thus, to second order in s, term by term comparison with definition (3) yields

$$\Delta r_0 = -\frac{1}{4}\frac{s^2}{r_0}, \qquad \Delta r_1 = s, \quad \text{and} \quad \Delta r_2 = \frac{1}{4}\frac{s^2}{r_0}. \tag{56b, c, d}$$

The formal change in energy under this displacement (11) is

$$\Delta E = \left(\frac{\partial E}{\partial r_0}\right)_0 \left(-\frac{s^2}{r_0}\right) + \frac{1}{2}\left(\frac{\partial^2 E}{\partial r_1^2}\right)_0 s^2 + 0_3 . \tag{57}$$

Obtaining $(\partial E/\partial r_0)_0$ and $(\partial^2 E/\partial r_1^2)_0$ from expression (43), and substituting expressions (84), (85), (86), (100), and (138a) verifies that

$$\left(\frac{\partial^2 E_T}{\partial r_1^2}\right)_0 = \frac{1}{2r_0}\left(\frac{\partial E_T}{\partial r_0}\right)_0. \tag{58}$$

The coefficient of s^2 in expression (57) is thus zero as required by translation invariance, and further the (1, 1) stiffness matrix element is zero whenever the total radial generalized force is zero.

4.2.3 *The Magnetostatic Stiffness Terms*

The interpretation of the radial stiffness matrix elements for the higher n values is now considered. As in the case of the generalized forces, examination of the expansions for small r_0 allows the self-interaction energy of the two charge disks which make up the ends of the domain to be separated from the mutual interaction of these charges. The variation in the internal magnetostatic energy due to a variation in some r_n for a circular domain is in general from expression (43)

$$\triangle E_{Mn} \equiv 4\pi M_s^2 \left\{ -(\pi h) \frac{\partial F(2r_0/h)}{\partial(2r_0/h)} + r_0 \left[L_n\left(\frac{h^2}{4r_0^2}\right) - L_n(0) \right] \right\} (\triangle r_n)^2, \qquad n \geqq 1. \qquad (59)$$

Separating the h independent and h dependent terms of the power series representation in powers of $2r_0/h$ uniquely separates the above expression into two parts, one part representing the self-interaction of the charge disks and the other representing the mutual interaction of these disks. The h independent terms then represent the self-interaction forces of the charge disks and the h dependent terms represent the mutual interaction forces. In the expansion of L_n for large $(h/2r_0)^2$, expression (129a), all terms of $L_n[(h/2r_0)^2]$ are h dependent. Using the large $(h/2r_0)^2$ expansion of F, expression (138d), and the expressions for $L_n(0)$, (115) and (116), the thickness independent part of expression (59) is

$$\triangle E_{Mn}(\text{Self}) = 4\pi M_s^2 r_0 \left(8 - 4 \sum_{j=1}^{n} \frac{1}{2j-1} \right) (\triangle r_n)^2, \qquad n \geqq 1. \qquad (60)$$

This energy variation contains a term which results from the variation in the overall size of the disks of charge as well as the shape dependent terms. The size variation term will now be identified and subtracted out so that the shape dependent part of the self-interaction energy may be seen explicitly. From expression (49) and the discussion following it, the ratio of the variation in energy of two isolated disks to the variation in disk area is $(4\pi M_s^2)(16r_0/2\pi)$. The variation in disk area for a variation in r_n for $n \geqq 1$ is $(\pi/2)(\triangle r_n)^2$ so that the change in self-energy of the two disks, other than that due to their mutual interaction or change in overall size, is

$$\triangle E_{Mn}(\text{Self-Shape}) = \begin{cases} 0, & n = 1 \qquad (61a) \\ -(4\pi M_s^2) 4r_0 \left(\sum_{j=2}^{n} \frac{1}{2j-1} \right) (\triangle r_n)^2, & n > 1. \qquad (61b) \end{cases}$$

It is not surprising that the variation of r_1 produces no shape related energy change, since from expression (56) this variation is to lowest order a displacement with a size change coming in second order. It is seen that the terms which remain after cancellation all come from $L_n(0)$.

In expression (59) the first term is independent of n and the $-4\pi M_s^2 r_0 L_n(0)(\Delta r_n)^2$ term has been identified with the variation in the self-energy of the charge disks. The term $4\pi M_s^2 r_0 L_n[(h/2r_0)^2](\Delta r_n)^2$ must therefore contain all of the shape dependent part of the charge disk mutual interaction energy. This term also contains a contribution due to the variation in the total amount of charge and contribution due to the shape independent, general smearing out of the charge distribution. Since the second order change in the total amount of charge is independent of n for $n \geqq 1$, these two contributions may be removed from the mutual interaction energy variation by replacing L_n by $L_n - L_\infty$. The remaining mutual interaction energy variation is specifically due to the shape of the variation. This energy variation is

$$\Delta E_{Mn}(\text{Mutual-Shape})$$
$$= (4\pi M_s^2) r_0 \left[L_n\left(\frac{h^2}{4r_0^2}\right) - L_\infty\left(\frac{h^2}{4r_0^2}\right) \right] (\Delta r_n)^2, \qquad n \geqq 1$$
$$= (4\pi M_s^2)\, r_0 \left[M_{n,2n+1}\left(\frac{2r_0}{h}\right)^{2n+1} + M_{n,2n+3}\left(\frac{2r_0}{h}\right)^{2n+3} + \cdots \right],$$
$$n \geqq 1 \qquad (62)$$

where the final form is obtained using the expansion for L_n, equation (131), and the $M_{n,m}$ are the constants of the expansion. The interaction energy of planar multipoles of order n and higher has the form of equation (62), as it must since the variation in the charge distribution for each n may be expressed in terms of such multipoles.

The variation in internal magnetostatic energy due to a variation of r_n, in the infinite sheet, for large $2r_0/h$, to lowest order in $h/2r_0$, is

$$\Delta E_{Mn} = (4\pi M_s^2)\left(\frac{h^2}{4r_0}\right)$$
$$\cdot \left[-2n^2 \ln \left| 4\,\frac{2r_0}{h} \right| - 2n^2 - 2 + (4n^2 - 1) \sum_{j=1}^{n} \frac{1}{2j - 1} \right] (\Delta r_n)^2,$$
$$n \geqq 1 \qquad (63)$$

using equation (59), the large $2r_0/h$ expansion of F (138c) and of $L_n(h^2/4r_0^2) - L_n(0)$, (105), (116) and (125).

The charge-disk self-interaction energy is not evident in this expansion because it is exactly cancelled by the leading term of the mutual interaction energy. In contrast to the energy reduction per unit line length for a straight line reversal in an infinite sheet (which has no finite value), it is possible in the case of this variation of the domain structure to compute the energy variation per unit line length. In terms of the wavelength of the variation λ, defined in expressions (54) or (127) and in the limit of $r_0 \to \infty$, the total variation in energy when r_n is varied is [using expressions (43) and (55) and the limit (128)]

$$\frac{\Delta E_T}{2\pi r_0} = \left\{ \left[\pi^2 \sigma_w / h - (4\pi M_s^2)\pi \ln \left| \frac{4e\lambda}{\pi h} \right| \right] \frac{h^2}{\lambda^2} + O_4\left(\frac{h}{\lambda}\right) \right\} (\Delta r_n)^2. \quad (64)$$

Comparison of expression (64) with expression (53) shows that the magnetostatic energy variation per unit line length for a circle of infinite diameter is the product of the magnetostatic energy density constant, the variation in line length, and the logarithm of a maximum effective interaction distance, $4e\lambda/\pi$. (The maximum effective interaction distance for the magnetostatic energy lowering per unit line length is proportional to r_0.) Hagedorn has computed the magnetostatic energy variation per unit line length for the case of a sinusoidal variation imposed on an infinite straight line reversal.[6] The calculation was carried out by considering the energy variation produced by a sinusoidal applied to a strip domain pattern in the limit of infinite strip width. The result of this calculation is

$$\Delta E_M/(\text{unit length}) = -(4\pi M_s^2)\pi \ln \mid \lambda/(2.111h) \mid (h/\lambda)^2 (\Delta r)^2, \quad (65)$$

which differs from the result for the infinite circle by the constant inside the logarithm.

4.3 *Summary*

The physical origin of terms of the energy variation has thus been traced in the limiting cases of both large and small r_0/h. In either of these limiting cases, it is thus possible to develop intuition with regard to the behavior of the domains. Since, as has been shown, the interpretation of the meaning of the energy terms in the limiting cases is qualitatively different, the development of intuition in the transition region is quite difficult. In many device applications this transition region is the preferred region of operation, making the use of analytical and numerical methods a necessity.

V. THE SIZE AND STABILITY OF CYLINDRICAL DOMAINS

The energy variation expansion (43) in principle contains all cylindrical domain size and stability information. This section treats briefly the use of this expression in the determination of domain size and stability. The only non-zero generalized force in expression (43) is the uniform radial force. When this force is set equal to zero (the force equation), the system is in equilibrium. Thus the condition that the system be in equilibrium provides, given a material and plate thickness, an equation relating domain size and the applied field. The location of the zeros in expression (43) (all terms not explicitly exhibited are zero) shows that the system is completely metastable with respect to angle and that the radial stiffness matrix is diagonal. The radial amplitudes are thus quasi-normal modes, and the study of stability reduces to the study of the stability of the individual radial amplitudes.

5.1 *Normal Form of the Energy Expansion*

Before proceeding with the discussion, it is appropriate to introduce some new notation and to rearrange the energy variation expansion into what will be called normal form. Since the stiffness matrix is of interest only when the domain is in equilibrium, the applied field H is eliminated from it using the force equation. The geometrical dependences of the various magnetostatic stability terms are then combined and normalized to the wall stiffness term by defining the "stability functions" as

$$S_0(d/h) \equiv F(d/h) - d \frac{\partial}{\partial d} F(d/h) \tag{66a}$$

and

$$S_n(d/h) \equiv -\frac{1}{n^2 - 1} \left\{ S_0(d/h) + \frac{1}{2\pi} (d^2/h^2)[L_n(h^2/d^2) - L_n(0)] \right\}, \quad n \geqq 2. \tag{66b}$$

The S_1 function is undefined or may be taken to be zero since translation invariance in the infinite plate requires that the (1, 1) stiffness matrix element the identically zero whenever the generalized radial force is zero, as is assumed to be the case here. The S_n functions are plotted in Fig. 3 up to S_{10}; they are given in standard elliptic integral form together with power series expansions for large and small values of the argument in Appendix B. The domain diameter, $d = 2r_0$ represents domain size in this section. The normal form of the energy expansion is written as a function of the ratios of the three fundamental lengths of the system: the plate thickness h, the domain diameter d, and the "characteristic length" defined by

$$l \equiv \frac{\sigma_w}{4\pi M_s^2}. \tag{67}$$

The characteristic length depends only on the type of material used.

Dividing the energy variation expansion (43) by the normalizing energy $2(4\pi M_s^2)(\pi h^3)$ and introducing the notation of the preceding paragraph, the normal form of the energy expansion results:

$$\frac{\Delta E_T}{2(4\pi M_s^2)(\pi h^3)} = \left[\frac{l}{h} + \frac{d}{h}\frac{H}{4\pi M_s} - F\left(\frac{d}{h}\right)\right]\frac{\Delta r_0}{h} + \frac{1}{2}\left\{-\left(2\,\frac{h}{d}\right)\left[\frac{l}{h} - S_0\left(\frac{d}{h}\right)\right]\left(\frac{\Delta r_0}{h}\right)^2 + \sum_{n=2}^{\infty}(n^2 - 1)\left(\frac{h}{d}\right)\left[\frac{l}{h} - S_n\left(\frac{d}{h}\right)\right]\left(\frac{\Delta r_n}{h}\right)^2\right\} + O_3 . \quad (68)$$

In expression (68) the coefficient $-[l/h + (d/h)(H/4\pi M_s) - F(d/h)]$ is the normalized radial force. Setting this force equal to zero yields the normalized force equation. The remaining bracketed quantities $[l/h - S_n(d/h)]$ are proportional to the diagonal elements of the stiffness matrix, and are called "stability coefficients." For uniform radial variation, the stability coefficient has the opposite sign from the (0, 0) element of the radial stiffness matrix; thus this stability coefficient is negative whenever the domain is stable. For the other r_n variations, on the other hand, the stability coefficient has the same sign as the corresponding element in the stiffness matrix, and these stability coefficients are positive whenever the domain is stable.

5.2 *Graphical Solution of the Force Equation*

A graphical solution to the force equation

$$\frac{l}{h} + \frac{d}{h}\frac{H}{4\pi M_s} - F\left(\frac{d}{h}\right) = 0 \quad (69)$$

may be obtained by constructing a straight line on Fig. 3 whose intercept with the vertical axis is l/h and whose numerical slope is $H/4\pi M_s$. The intersections of this straight line with the F curve are then the solutions to the force equation.

As was stated in Section 5.1, (*i*) the force function has a positive first derivative and negative second derivative for all nonzero values of its argument, (*ii*) it is zero and has a first derivative of unity when its argument is zero, and (*iii*) it becomes logarithmic for large values of its argument. From these properties and examination of Fig. 3, several properties of the solutions to the force equation may be appreciated. For negative values of the applied fields, there is only one solution to the force equation. Examination of the sign of the radial force which results when the diameter is varied about the solution diameter while all other variables are held fixed shows that this solution is unstable. For small positive applied fields, there are two solutions to the force equation, the larger diameter solution being radially stable, the other radially unstable. However, a radially stable solution does not guarantee that the system is stable with respect to all possible deformations, and this must be investigated separately. As the applied field is increased, the two solutions move closer together until they coalesce. When the applied field is increased beyond this point, there are no solutions. Since the function F is asymptotic to a straight line through the origin having unit slope, the solutions will always vanish for a value of the applied field which is greater than $4\pi M_s$. Stable isolated cylindrical domains thus exist only in the presence of an applied field having magnitude between zero and $4\pi M_s$ and polarity tending to collapse the domain.

5.3 *Graphical Determination of Domain Stability*

The stability coefficients are determined graphically by constructing a horizontal line at height l/h on the force stability graph. Metastability for each normal mode of deformation occurs at the intersection of this line with the corresponding stability function. Since the stability functions are monotonic, the diameter of metastability of each normal mode of deformation is uniquely defined and forms the boundary between the regions of stability and unstability. The circular domain will be stable with respect to all variations when its diameter is greater than the radial metastability diameter and less than the metastability diameter for a variation with a rotational periodicity of two. The normal variations with rotational periodicity two are referred to as "elliptical" deformations. When the domain is stable with respect to elliptical deformation, it is necessarily stable with respect to the variations of higher spatial frequency since the stability functions of higher spatial frequency lie progressively (with respect to n) below the elliptical stability function. The radial stability function S_0 and the elliptical stability function S_2 thus form the boundary of the region of total cylindrical domain stability. Therefore, given the magnetic material type and plate thickness, the range of stable domain diameters and the corresponding applied fields may be determined with the aid of these functions.

5.4 *Minimum Domain Diameter*

For any given value of l/h, the minimum domain diameter is the collapse diameter determined by S_0. The domain diameter measured in units of the characteristic length is $d/l = (d/h)/(l/h)$ which is the inverse of the numerical slope of a line drawn in Fig. 3 from the origin to the operating point. The line of maximum slope, which both passes through the origin and contacts the S_0 curve at at least one point, thus determines the smallest domain diameter attainable in a given material. The coordinates of this contact point are $d/h \approx 1.2$ and $l/h \approx 0.3$, so that the minimum attainable domain diameter is

$$d_{\min} \approx 4l. \tag{70}$$

VI. RANGE OF VALIDITY OF THE MODEL AND THE QUALITY FACTOR

At the present time, no quantitative evaluation of the range of validity of the domain structure model used here has been carried out. The qualitative discussion given here, it is hoped, will provide the reader with an appreciation of the magnitude of the effects produced by the relaxation of the various constraints artificially imposed by the model and the dependence of these effects on the system parameters. It has been assumed that domain walls are cylindrical, have zero width, and have a definite energy per unit area which is independent of wall orientation or curvature, and that the magnetization lies perpendicular to the surface of the plate. Section 6.1 treats the effect of the relaxation of the cylindrical wall approximation only. In Section 6.2, the other assumptions are all shown to be coupled using the simplest uniaxial material model. A single dimensionless material parameter q, which complements the characteristic length l in characterizing circular domain materials, is used to express the results obtained from the simplest material model.

6.1 *The Cylindrical Wall Approximation*

The discussion of the cylindrical wall approximation uses the coordinate system and domain configuration of Fig. 1 except that the walls are allowed to curve as shown in Fig. 4. The radius function, $r_b(\theta, z)$, is determined by the requirement that it minimize the total energy. The Euler equation which results from this two dimensional field variational problem is an integro-differential equation similar to those which appear in Hartree self-consistent field calculations. No solution of this equation, numerical or otherwise, has been attempted or is contemplated at the present time. The Euler equation consists of terms arising from: the wall energy, the interaction of the magnetization with the applied field, the self-interaction of the magnetostatic charges at the surface of the plate, the self-interaction of the charges produced by the slope of the

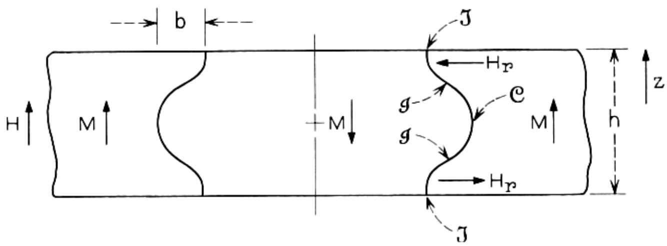

Fig. 4 — Cross section of a noncylindrical, near-circular, domain.

domain wall, and the mutual interaction of the plate surface charges with the domain wall magnetic charges. Boundary conditions (obtained from the appropriate transversality condition[7]) require that the wall surface be perpendicular to the plate surface at all intersection points (𝔍 in Fig. 4). Physically (since in the model used here the crystal is assumed to be strain free) the surface cannot interact with the domain wall, and therefore the wall must intersect the surface at right angles.

Although it is not clear that domains having a roughly conical shape are ruled out, it will be assumed that the domain has reflection symmetry through the central plane of the plate and that the radius is a function of z only, $r_b(z)$. In this case the wall must be vertical at the central plane as indicated at ℭ in the figure so that the single parameter b represents the magnitude of the wall bulging. Since the Euler equation requires the curve to be smooth, there must be an inflection point, 𝔤, between ℭ and 𝔍. The wall area, and thus total wall energy, is a quadratic increasing function of the wall curvature so that the concentration of the curvature at the center and ends of the wall, produced by the transversality and symmetry conditions, tends to reduce the wall bulging.

The radial field at the domain wall from the charges at the surface of the plate is directed as shown in Fig. 4. The effect of the interaction of the magnetostatic charges due to the slope of the wall with the radial component of the field from the surface charges is destabilizing for either positive or negative bulging. This interaction produces a negative quadratic term in the total energy. However, at the plate surface, where the magnitude of the radial field is greatest, the transversality condition requires that the charge density produced by the wall slope is zero so that the magnitude of this negative term is small. The z component of the field from the charges on the surface of the plate determines the direction of bulging. (The applied field, being uniform by assumption, need not be considered.) Along an initially cylindrical wall the internal field is everywhere directed, so as to make the domain expand, and attains its greatest magnitude at the center plane of the plate. It therefore provides a linear term in the total energy which tends to bulge the wall in the positive direction as shown in Fig. 4.

Thus, for near cylindrical walls, the bulging is determined by the interaction of this force (tending to bulge the wall) with the wall energy (acting to stabilize the wall) and the radial field (acting to destabilize the wall). The self-interaction of the wall charges enters only as a higher-order term. It should be noted that the transversality condition acts both to strengthen the stabilizing term and weaken the destabilizing term.

The relevant dimensionless wall energy for the wall bulging problem is $l/h = \sigma_w/(h4\pi M_s^2)$. Wall bulging is expected to decrease with increasing wall energy. A second independent effect related to l/h may be appreciated by inspection of the S_0 and S_2 curves in Fig. 3. It can be seen from Fig. 3, equation (68), and the discussion following it that, since the S_0 and S_2 curves bound the region in which stable circular domains exist, d/h must increase with increasing l/h. By symmetry, the z-component of the internally generated magnetic field at a cylindrical wall is zero for a domain of infinite diameter and clearly increases monotonically as the domain diameter to thickness ratio decreases. Thus, as the plate is made thicker, the bulging force becomes stronger and the stabilizing force becomes weaker. Since several independent effects cooperate to increase bulging with increasing plate thickness, the onset may be quite rapid when it does occur. Domain collapse data taken at $d/h \approx 1$ is in good agreement with predictions made on the basis of equation (68) and Fig. 3.[8] This then provides some indication that the cylindrical wall approximation remains valid at this thickness.

6.2 *The Quality Factor*

The discussion of the approximations other than the cylindrical wall approximation uses a polar (M_s, η, ν) coordinate system where η is the polar angle and ν is the aximuthal angle to specify the orientation of M_s (See Fig. 5). The polar axis is taken to be the z-axis of the preceding sections. The domain wall is taken to be planar with its position and orientation specified by a plane at its center. The axis through the origin in the direction of the wall normal is denoted by ξ. The position of the central wall plane is denoted by ξ_0. The orientation angles of the wall normal are denoted by ν_w and η_w, (see Fig. 5).

In the simplest uniaxial material whose easy axis is the z-axis the magnetic energy density for a planar wall is (Ref. 9, pp. 189–192)

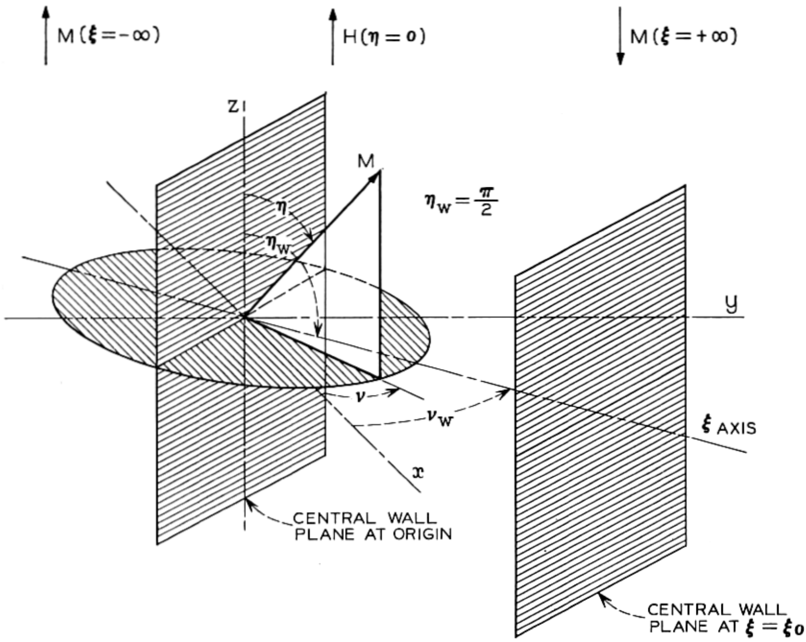

Fig. 5 — Coordinate system for specification of domain walls.

$$\rho_E = A\left[\left(\frac{\partial\eta}{\partial\xi}\right)^2 + \sin^2\eta\left(\frac{\partial\nu}{\partial\xi}\right)^2\right] + K_u\sin^2\eta - \mathbf{H}\cdot\mathbf{M} + 2\pi\frac{1}{\xi^2}(\mathbf{M}\cdot\xi)^2 \tag{71}$$

where A is the exchange energy density coefficient, K_u is the anisotropy energy density coefficient, $\mathbf{H}$ is the externally applied field, and the last term is obtained by integrating $\nabla\cdot\mathbf{B} = 0$. For a uniformly magnetized material in the absence of applied or internal fields, this expression reduces to $\rho_E = K_u \sin^2 \eta$ which has absolute minima at $\eta = 0$ and $\eta = \pi$. The z-axis is thus the easy axis as is required for consistency with the preceding sections.

The anisotropy energy density coefficient is sometimes expressed in terms of the effective anisotropy field $H_a \equiv 2k_u/M_s$. The quality factor is now defined as the dimensionless anisotropy energy coefficient or dimensionless anisotropy field

$$q \equiv \frac{K_u}{2\pi M_s^2} = \frac{H_a}{4\pi M_s}. \tag{72}$$

6.2.1 *The Nucleation Field*

When a bias field H is applied in the positive z-direction and demagnetizing fields are neglected the energy density is

$$\rho_E = K_u \sin^2 \eta - HM_s \cos \eta \tag{73}$$

which, for $H < H_a$, has a local minimum at magnetization orientation $\eta = \pi$ and an absolute minimum for $\eta = 0$. When $H > H_a$, only the minimum at $\eta = 0$ remains. In a perfect crystal the effective anisotropy field is thus the field at which the magnetization becomes unstable with respect to reorientation (assuming it is initially oriented in the negative z-direction). If a reorienting field is applied locally (local but over a region whose dimensions are much greater than a wall width so that the effect of exchange forces can be neglected), then H_a is the total local field required for the nucleation of a domain at that locality. If the nucleation field H_N is understood in this sense, then in a perfect crystal q is the nucleation field measured in units of $4\pi M_s$:

$$\frac{H_N}{4\pi M_s} = q. \tag{74}$$

In an imperfect crystal $H_N/4\pi M_s$ may be either larger or smaller than q. If it is larger, the material may be expected to have a high wall motion coercivity.

6.2.2 *Susceptibility*

When a transverse bias field $H_t(\eta = \pi/2)$ is applied and demagnetizing fields are neglected, the energy density becomes

$$\rho_E = K_u \sin^2 \eta - H_t M_s \sin \eta \tag{75}$$

which has stable magnetization orientations

$$\eta = \begin{cases} \sin^{-1}\left(\dfrac{H_t}{H_a}\right) = \sin^{-1}\left(\dfrac{H_t}{4\pi M_s}\dfrac{1}{q}\right), & H_t < H_a; \\ \dfrac{\pi}{2}, & H_t \geqq H_a. \end{cases} \tag{76}$$

The transverse susceptibility is therefore

$$\chi_t = \frac{\partial M_t}{\partial H_t} = \begin{cases} \dfrac{1}{4\pi q} & (H_t < H_a) \\ 0 & (H_t \geqq H_a) \end{cases} \tag{77}$$

where $M_t = M_s \sin \eta$ is the component of the magnetization in the direction of H_t. Thus, the susceptibility to tipping of the magnetization by a transverse field is inversely proportional to q.

6.2.3 *Wall Energy and Wall Width*

Consider now a planar Bloch wall, $\boldsymbol{\xi}\cdot\mathbf{M} = 0$, between two regions whose magnetization at points far from the wall lies along the two easy directions, $\eta = 0$ and π, and again assume that there are no applied fields or fields produced by boundary surfaces. Under these conditions, the magnetic configuration is determined by minimization of the wall energy per unit surface area which in this case is (Ref. 9, pp. 189–192)

$$\sigma_w = \int_{-\infty}^{\infty} \left[A\left(\frac{\partial \eta}{\partial \xi}\right)^2 + K_u \sin^2 \eta \right] d\xi. \tag{78}$$

Carrying out the minimization results in

$$\sigma_w = 4(AK_u)^{\frac{1}{2}} \tag{79}$$

for

$$\xi - \xi_0 = \frac{1}{\pi} l_w \log \tan \left(\frac{\eta}{2}\right) \tag{80}$$

where

$$l_w \equiv \pi\left(\frac{A}{K_u}\right)^{\frac{1}{2}} \tag{81}$$

is the wall width. The definition of wall width is somewhat arbitrary since the wall extends over all space. In this case, following page 191 of Ref. 9, it is chosen so that the magnetization would complete its entire rotation of π radians in a length l_w if the entire rotation took place at its maximum rate, the rate at the center of the wall.

The ratio of the characteristic length, equation (67), to the wall width is

$$\frac{l}{l_w} = \frac{2}{\pi} q, \tag{82a}$$

so that the ratio of the minimum domain diameter, equation (70), to the wall width is

$$\frac{d_{\min}}{l_w} = \frac{4l}{l_w} = \frac{8}{\pi} q. \tag{82b}$$

The approximation of zero wall width thus improves as q becomes larger.

The approximation that the wall energy is independent of wall curvature is clearly related to the wall width. At large distances from the planar wall equation (80) becomes

$$| \eta - \eta_0 | = 2\exp(-\pi \, | \xi - \xi_0 | /l_w) \tag{83}$$

where η_0 is the appropriate equilibrium orientation of the magnetization at a distance far removed from the wall. Such an exponential relation will hold for the approach to any stable equilibrium orientation in the presence of isotropic exchange. The change in energy of the wall due to overlapping of the tails of the wall as the wall is curved is clearly related to q, becoming larger as q becomes smaller. In order to solve for the dependence of the wall energy on curvature it is necessary to solve the entire (including magnetostatics) micromagnetics problems.[10]

6.2.4 *Summary*

The preceding results may be summarized by noting that the higher the q value, the more closely the simple uniaxial model obeys the constraints of the domain model used in the previous sections. It is clear that, for domains of the type considered to exist at all, q must be greater than one. For device operation, q should probably have a value greater than two.

VII. CONCLUSIONS

The theory of cylindrical magnetic domains yields conditions which predict the size and stability of these domains and provides an estimate of the range of applicability of the model used. The results of theory appear to be accurate in a range useful in the construction of circular domain devices.

The domains considered are isolated right circular cylinders in plates of uniaxial magnetic material of uniform thickness cut so that the plate normal is parallel to the easy axis. The first and second order energy variations which result from a general small deviation from the strictly circular shape determine domain size and stability. The energy method was chosen in preference to the magnetostatic field method because of the uniformity it provides in accounting for the forces in both the equilibrium and stability problems. The integrals arising from the energy method are interpreted physically in terms of fields and interacting charges. The physical interpretation of the integrals is quite different in the limiting cases of very large or very small domains. The integrals are related to special cases of the fields of uniformly charged disks computed by C. Snow and tabulated by N. B. Alexander and A. C.

Downing.[11,12] The present work obtains the needed properties of the integrals (expansions, recursion relations, and others) directly from the definitions.

When the energy variation is described in terms of a Fourier decomposition of the domain radius function, only the generalized force corresponding to a change in domain size is non-zero and the stiffness matrix is completely metastable with respect to angle (phase) and diagonal with respect to the Fourier amplitudes. Since the Fourier amplitude stiffness matrix elements are all found to be distinct, the description is unique and may be described as a quasi-normal mode description.

The normal mode description is summarized by a single graph from which many domain properties may be determined by construction. Cylindrical domains exist only in the presence of a bias field directed so as to tend to collapse the domains and having a magnitude between 0 and $4\pi M_s$. The uniform radial collapse of the domain and the run-out of the domain into an initially elliptical shape bound in the region of stability. The minimum attainable domain diameter in a given material is $d_{\min} \approx 4l$ occurring a plate thickness of $\sim 4l$. It is estimated that the cylindrical wall approximation begins to become doubtful at a plate thickness greater than $4l$. In order for cylindrical domains to exist, $H_a \approx 4\pi M_s$ and in general approximations such as the approximation of zero wall width become more accurate for $H_a \gg 4\pi M_s (d_{\min}/l_w = 8H_a/4\pi^2 M_s$ where l_w is the wall width).

It is interesting to note that since stable cylindrical domains of a definite size exist in the total absence of wall motion coercivity and may be freely moved, they form a relative, easily observable, classical model for illustrating several particle-field concepts. They may be considered a two-dimensional particle which is produced as a singularity of finite extent in an underlying three-dimensional field (the magnetization). Cylindrical domains are particularly useful for demonstrating the concept of identical particles since, while it is possible to put identifying marks on domain locations, it is not possible to mark individual domains. (Cylindrical domains do exist in two species which may be distinguished by the direction of rotation of the spins in the domain wall.[13] All attempts to observe this difference up to the present time have been unsuccessful.)

VIII. ACKNOWLEDGMENTS

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APPENDIX A

Integrals of Cylindrical Domain Theory

This appendix contains the reduction to standard form of the elliptic integrals which arise in the theory of cylindrical domains in plates of infinite extent and power series expansions of these integrals. All the properties of the functions obtained here are used in either the physical interpretation of the energy variation expansion or in generating the numerical values of the force and stability functions.

It is convenient to define functions U and V which appear repeatedly in cylindrical domain theory. The elliptic integrals which appear in the final results of the theory appear only in the forms U and V, U being a function of only the complete elliptic integral of the second kind and V being only a function of the complete elliptic integral of the first kind. Because of the form of the U and V functions, it has proven easier to obtain the needed properties, (such as the series expansions) directly from the integral definitions rather than deducing them from the tabulated properties of elliptic integrals.

The latter half of this appendix treats the properties of the L_n functions. A recursion relation is obtained and used to reduce the L_n to functions of U and V. Power series expansions of the L_n are obtained directly from the definition (34).

A.1 *Definition of the U and V Functions*

The functions are defined in the alternate forms

$$U(x) \equiv \int_0^{\pi} [x + \tfrac{1}{2}(1 - \cos \alpha)]^{\frac{1}{2}} \, d\alpha \tag{84a}$$

$$= 2 \int_0^{\pi/2} (x + \sin^2 \beta)^{\frac{1}{2}} \, d\beta \tag{84b}$$

$$= 2 \int_0^{\pi/2} (x + 1 - \sin^2 \gamma)^{\frac{1}{2}} \, d\gamma \tag{84c}$$

$$= 2(x + 1)^{\frac{1}{2}} E\left(\frac{1}{1 + x}\right) \tag{84d}$$

and

$$V(x) \equiv \int_0^{\pi} [x + \tfrac{1}{2}(1 - \cos \alpha)]^{-\frac{1}{2}} \, d\alpha \tag{85a}$$

$$= 2 \int_0^{\pi/2} (x + \sin^2 \beta)^{-\frac{1}{2}} \, d\beta \tag{85b}$$

$$= 2 \int_0^{\pi/2} (x + 1 - \sin^2 \gamma)^{-\frac{1}{2}} \, d\gamma \tag{85c}$$

$$= 2(x + 1)^{-\frac{1}{2}} K\left(\frac{1}{1 + x}\right) \tag{85d}$$

where the dummy variables are related by $\beta = \alpha/2$ and $\gamma = \pi/2 - \alpha/2$ and where K and E are the complete elliptic integrals of the first and second kind respectively. The argument of the elliptic integrals is the parameter m of Abramowitz and Stegun.[14] The parameter m is equal to the parameter k^2 of Jahnke and Emde or Groebner and Hofreiter.[15,16]

A.2 *Differential Equations and the Power Series Expansion of U and V*

From the definitions (84) and (85)

$$\frac{dU}{dx} = \tfrac{1}{2} V. \tag{86}$$

The differential equations obeyed by U and V are

$$\left[(x^2 + x) \frac{d^2}{dx^2} + \frac{1}{4}\right] U(x) = 0 \tag{87a}$$

and

$$\left[(x^2 + x) \frac{d^2}{dx^2} + (2x + 1) \frac{d}{dx} + \frac{1}{4}\right] V(x) = 0. \tag{87b}$$

The U differential equation is verified by substituting in the defining relation (84a) and then reducing the resulting equation to the indentity

$$0 = \int_0^{\pi} \frac{d}{d\alpha} \frac{\sin \alpha}{[x + \frac{1}{2}(1 - \cos \alpha)]^{\frac{1}{2}}} \, d\alpha \tag{88a}$$

$$= \int_0^{\pi} \frac{x \cos \alpha + \frac{1}{2} \cos \alpha - \frac{1}{4} - \frac{1}{4} \cos^2 \alpha}{[x + \frac{1}{2}(1 - \cos \alpha)]^{\frac{3}{2}}} \, d\alpha. \tag{88b}$$

The V equation is then easily obtained by differentiation of the U equation.

The roots of the indicial equations of these equations are separated by 1 [they are 0 and 1 in equation (87a) and -1 and 0 in equation (87b)] so that the series expansion of U is of the form

$$U(x) = \sum_{j=0}^{\infty} U_j x^j \quad \text{where} \quad U_j = U_j' + U_j'' \tfrac{1}{2} \ln \left| \frac{16}{x} \right| \tag{89a, b}$$

and

$$V(x) = \sum_{j=0}^{\infty} V_j x^j \quad \text{where} \quad V_j = V_j' + V_j'' \tfrac{1}{2} \ln \left| \frac{16}{x} \right| . \tag{90a, b}$$

The form of the logarithmic terms has been chosen with some foresight. Substitution of the expansions into the differential equations and comparing coefficients gives the recursion relations

$$U_{j+1}'' = -\frac{(j - \frac{1}{2})^2}{j(j+1)} U_j'', \qquad j \geqq 1, \tag{91a}$$

$$U_{j+1}' = -\frac{j - \frac{1}{2}}{j(j+1)} \left[(j - \tfrac{1}{2}) U_j' - \frac{(j + \frac{1}{4})}{j(j+1)} U_j'' \right], \qquad j \geqq 1, \tag{91b}$$

$$V_{j+1}'' = -\left(\frac{j + \frac{1}{2}}{j+1} \right)^2 V_j'', \qquad j \geqq 0, \tag{92a}$$

$$V_{j+1}' = -\frac{j + \frac{1}{2}}{(j+1)^2} \left[\left(j + \frac{1}{2} \right) V_j' - \frac{1}{2} \frac{1}{j+1} V_j'' \right], \qquad j \geqq 0. \tag{92b}$$

The starting values of V_j', V_j'', U_j' and U_j'' are determined directly from the integral definitions of the functions (84) and (85) and the differential equation (86) relating U and V. This is quite straightforward except for V which must be expanded

$$V(x) = 2 \int_0^{\epsilon} \frac{d\beta}{(x + \beta^2)^{\frac{1}{2}}} + 2 \int_{\epsilon}^{\pi/2} \frac{d\beta}{\sin \beta} + O\left(x^2, \epsilon^2, \frac{x}{\epsilon^2}\right) \tag{93a}$$

and evaluated as a limit

$$\lim_{x \to 0} V(x) = \ln \left| \frac{16}{x} \right| . \tag{93b}$$

The limiting value of V may also be obtained quite easily from equation (85d) and the tabulated properties of the complete elliptic integral of the first kind.[15] The expansions of U and V are thus

$$U(x) = \left(2 + \frac{1}{2} x + \frac{3}{32} x^2 - \frac{3}{64} x^3 + \frac{665}{24576} x^4 + \cdots \right)$$

$$+\left(0 + x - \frac{1}{8}x^2 + \frac{3}{64}x^3 - \frac{25}{1024}x^4 + \cdots\right)\frac{1}{2}\ln\left|\frac{16}{x}\right| \quad (94)$$

and

$$V(x) = \left(0 + \frac{1}{2}x - \frac{21}{64}x^2 + \frac{185}{768}x^3 + \cdots\right)$$
$$+\left(2 - \frac{1}{2}x + \frac{9}{32}x^2 - \frac{25}{128}x^3 + \cdots\right)\frac{1}{2}\ln\left|\frac{16}{x}\right|. \quad (95)$$

A.3 *Expansion of U and V in Terms of Inverse Powers*

Making a Taylor series expansion of equations (84b) and (85b) respectively and integrating yields the expansions of U and V in terms of the inverse powers of the argument

$$U(x) = 2x^{\frac{1}{2}}\int_0^{\pi/2}(1 + x^{-1}\sin^2\beta)^{\frac{1}{2}}\,d\beta$$
$$= \sum_{j=0}^{\infty}\frac{-2}{2j-1}\frac{(2j)!}{(-4)^j(j!)^2}x^{-(j-\frac{1}{2})}\int_0^{\pi/2}\sin^{2j}\beta\,d\beta$$
$$= \pi\sum_{j=0}^{\infty}\frac{-1}{(2j-1)(-16)^j}\left[\frac{(2j)!}{(j!)^2}\right]^2 x^{-(j-\frac{1}{2})}. \quad (96a)$$
$$= \pi x^{\frac{1}{2}}\left[1 + \frac{1}{4}x^{-1} - \frac{3}{64}x^{-2} + \frac{5}{256}x^{-3} + \cdots\right] \quad (96b)$$

and

$$V(x) = 2x^{-\frac{1}{2}}\int_0^{\pi/2}(1 + x^{-1}\sin^2\beta)^{-\frac{1}{2}}\,d\beta$$
$$= \sum_{j=0}^{\infty}2\frac{(2j)!}{(-4)^j(j!)^2}x^{-(j+\frac{1}{2})}\int_0^{\pi/2}\sin^{2j}\beta\,d\beta$$
$$= \pi\sum_{j=0}^{\infty}\frac{1}{(-16)^j}\left[\frac{(2j)!}{(j!)^2}\right]^2 x^{-(j+\frac{1}{2})}. \quad (97a)$$
$$= \pi x^{-\frac{1}{2}}\left[1 - \frac{1}{4}x^{-1} + \frac{9}{64}x^{-2} - \frac{25}{256}x^{-3} + \cdots\right]. \quad (97b)$$

A.4 *Definition of the L_n Functions*

The L_n functions are defined in expression (34) by

$$L_n(x) \equiv \int_0^{\pi}\frac{(1 - \cos n\alpha)\,d\alpha}{[x + \frac{1}{2}(1 - \cos\alpha)]^{\frac{1}{2}}}, \qquad x \geqq 0,\ n \geqq 0 \quad (98a)$$

or with the change of variable $\beta = \alpha/2$

$$L_n(x) = 4 \int_0^{\pi/2} \frac{\sin^2 n\beta}{(x + \sin^2 \beta)^{\frac{1}{2}}} \, d\beta, \qquad x \geq 0, n \geq 0. \tag{98b}$$

It can be seen directly from equation (98b) that for a fixed value of n

$$\frac{dL_n(x)}{dx} < 0, \qquad L_n(\infty) = 0, \qquad \text{and} \quad L_0(x) = 0. \qquad \text{(99a, b, c)}$$

From definitions (84b), (85b), and (98b)

$$L_1(x) = 2[U(x) - xV(x)]. \tag{100}$$

The higher L functions are determined by means of a recursion relation.

A.5 *The* L_n *Recursion Relation*

The L_n recursion relation is

$$L_{n+1}(x) = \frac{1}{2n+1} [4n(2x+1)L_n(x) - (2n-1)L_{n-1}(x) - 8nxV(x)], \qquad n \geqq 1. \tag{101}$$

The recursion relation is verified by substituting in the definitions of L_n and V, equations (98a) and (85a), and reducing the resulting equation to the identity

$$0 = \int_0^\pi \frac{d}{d\alpha} \{\sin n\alpha[x + \tfrac{1}{2}(1 - \cos\alpha)]^{\frac{1}{2}}\} \, d\alpha \tag{102a}$$

$$\int_0^\pi \frac{n \cos n\alpha[x + \frac{1}{2}(1 - \cos\alpha)] + \frac{1}{4}\sin n\alpha \sin\alpha}{[x + \frac{1}{2}(1 - \cos\alpha)]^{\frac{1}{2}}} \, d\alpha. \tag{102b}$$

The initial functions $L_0(x)$ and $L_1(x)$ are given by equations (99c) and (100). Note that for large values of x the recursion relation is unstable for increasing n.

A.6 *Power Series Expansion of the* L_n *Function*

The function $L_0(x)$ is identically zero, equation (99c). The series expansion for $L_1(x)$, obtained from equations (100), (94), and (95), is

$$L_1(x) = \left(4 + x - \frac{13}{16}x^2 + \frac{9}{16}x^3 - \frac{5255}{12288}x^4 + \cdots\right) + \left(0 - 2x + \frac{3}{4}x^2 - \frac{15}{32}x^3 + \frac{175}{512}x^4 + \cdots\right)\frac{1}{2}\ln\left|\frac{16}{x}\right|. \tag{103}$$

From the recursion relation (101) and the initial functions (99c) and

(100) the general form of $L_n(x)$ is

$$L_n(x) = u_n(x)U(x) + v_n(x)V(x) \tag{104}$$

where $u_n(x)$ and $v_n(x)$ are polynominals of order n or less in x. Because of the form of $U(x)$ and $V(x)$, expressions (89) and (90), an expansion of the form

$$L_n(x) = \sum_{j=0}^{\infty} L_{n,j}x^j \tag{105a}$$

where

$$L_{n,j} = L'_{n,j} + L''_{n,j}\,\tfrac{1}{2}\ln\left|\frac{16}{x}\right| \tag{105b}$$

clearly exists.

Expressions for either the coefficients in the polynominals $u_n(x)$ and $v_n(x)$ or the $L_{n,j}$ may be determined in closed form by similar methods. It has, however, proven more useful to use the recursion relation directly when the complete expression of the form of expression (104) is desired and the expansion (105) when a power series is desired.

To obtain the $L_{n,j}$ the expansion (105) is substituted in the recursion relation (101) and coefficients of x are compared to obtain a hierarchy, in j, of recursion relations, each member of the hierarchy being factorable and depending only on the preceding member. These recursion relations are then factored and successively summed.

The coefficient of x^j is

$$L_{n+1,j} = \frac{1}{2n+1}\{4nL_{n,j} - (2n-1)L_{n-1,j} + 8n[L_{n,j-1} - V_{j-1}]\}, \qquad n \geqq 0 \tag{106a}$$

where

$$L_{n,-1} = 0 \quad \text{and} \quad V_{-1} = 0. \tag{106b, c}$$

With the definition

$$Q_{n,j} \equiv (2n-1)[L_{n,j} - L_{n-1,j}] \tag{107}$$

the second order recursion relation (106) factors into two first order recursion relations

$$Q_{n+1,j} = Q_{n,j} + 8n(L_{n,j-1} - V_{j-1}), \qquad n \geqq 0 \tag{108a}$$

and

$$L_{n+1,j} = L_{n,j} + \frac{1}{2n+1} Q_{n+1,j}, \qquad n \geqq 1. \tag{108b}$$

The recursion relations for $j = 0$ and $j = 1$ will now be summed. From expressions (99c) and (105a)

$$L_{0,j} = 0 \tag{109}$$

so that using expression (107)

$$Q_{1,j} = L_{1,j} . \tag{110}$$

For $j = 0$ using expressions (106b) and (106c), the recursion relation (108b) becomes simply

$$Q_{n+1,0} = Q_{n,0} . \tag{111}$$

By inspection of expression (103) the initial value of $Q_{n,0}$ is

$$Q_{1,0} = L_{1,0} = 4 \tag{112}$$

so that from expression (111)

$$Q_{n,0} = 4, \qquad n \geqq 1. \tag{113}$$

The recursion relation (108b) thus becomes

$$L_{n,0} = L_{n-1,0} + \frac{4}{2n-1}, \qquad n \geqq 0 \tag{114}$$

which with the initial value of expression (109) may be summed to yield

$$L_{n,0} = \begin{cases} 0, & n = 0, \quad (115a) \\ 4 \sum_{j=1}^{n} \frac{1}{2j-1}, & n > 0. \quad (115b) \end{cases}$$

From the form of the expansion (105) it can be seen that

$$L_n(0) = L_{n,0} , \tag{116}$$

so that with (99a, b)

$$-4 \sum_{j=1}^{n} \frac{1}{2j-1} \leqq L_n(x) - L_n(0) \leqq 0, \qquad n \geqq 1. \tag{117}$$

For evaluating the $Q_{n,1}$ and $L_{n,1}$ sums, two relations are needed:

$$\sum_{j=1}^{n} \sum_{k=1}^{j} \frac{1}{2k-1} = \sum_{k=1}^{n} \sum_{j=k}^{n} \frac{1}{2k-1}$$

$$= \sum_{k=1}^{n} \frac{n-k+1}{2k-1}$$

$$= \tfrac{1}{2}(2n + 1)\left(\sum_{k=1}^{n} \frac{1}{2k - 1}\right) - \frac{n}{2} \tag{118}$$

and similarly

$$\sum_{j=1}^{n} j \sum_{k=1}^{j} \frac{1}{2k - 1} = \frac{1}{8}(2n + 1)^2 \sum_{k=1}^{n} \frac{1}{2k - 1} - \frac{1}{8} n^2. \tag{119}$$

Evaluation of the sums of expressions (118) and (119) is analogous to integrating $x^n \log x$ where $n = 0$ and 1. With sufficient patience the sums can clearly be carried out for any finite n.

For $j = 1$, expression (108a) becomes [using expressions (95), (109), and (115)]

$$Q_{n+1,1} = Q_{n,1} + 8n\left[4 \sum_{j=1}^{n} \frac{1}{2j - 1} - \ln \left| \frac{16}{x} \right|\right], \qquad n \geqq 1 \tag{120}$$

where the initial function is [using expressions (103), (105), and (110)]

$$Q_{1,1} = L_{1,1} = 1 - \ln \left| \frac{16}{x} \right|. \tag{121}$$

Summing [using expression (119)] yields

$$\begin{aligned} Q_{n+1,1} &= 1 - \ln \left| \frac{16}{x} \right| + \sum_{k=1}^{n} 8k\left[4 \sum_{j=1}^{k} \frac{1}{2j - 1} - \ln \left| \frac{16}{x} \right|\right] \\ &= -(2n + 1)^2 \ln \left| \frac{16}{x} \right| + 4(2n + 1)^2 \sum_{k=1}^{n} \frac{1}{2k - 1} - (4n^2 - 1), \\ &\qquad n \geqq 1. \end{aligned} \tag{122}$$

The $L_{n,1}$ recursion relation is then

$$L_{n+1,1} = L_{n,1} + (2n + 1)\left[-\ln \left| \frac{16}{x} \right| + 4 \sum_{k=1}^{n} \frac{1}{2k - 1}\right] - 2n + 1, \qquad n \geqq 1 \tag{123}$$

which may be summed using the initial value of expression (121) and the sums (118) and (119):

$$\begin{aligned} L_{n+1,1} &= 1 - \ln \left| \frac{16}{x} \right| \left[1 + \sum_{k=1}^{n} (2k + 1)\right] \\ &\quad + \sum_{k=1}^{n} 4(2k + 1) \sum_{j=1}^{k} \frac{1}{2j - 1} + \sum_{k=1}^{n} (-2k + 1) \end{aligned} \tag{124}$$

$$L_{n,1} = \begin{cases} 0, & n = 0. \quad (125a) \\ -2n^2\left(\frac{1}{2}\ln\left|\frac{16}{x}\right|\right) - 2n^2 + (4n^2 - 1)\sum_{k=1}^{n}\frac{1}{2k-1}, & n \geqq 1. \quad (125b) \end{cases}$$

It is clear that the procedure leading to expressions (115) and (125) may be carried onward to lead to an expansion of the form

$$L_n(x) - L_n(0) = +\sum_{j=1}^{\infty}\left[\sum_{k=1}^{j}\left(L^{(1)}(j, k)\tfrac{1}{2}\ln\left|\frac{16}{x}\right| + L^{(2)}(j, k) + L^{(3)}(j, k)\sum_{m=1}^{k}\frac{1}{2m-1}\right)n^{2k}\right]x^j \tag{126}$$

where the $L^{(p)}(j, k)$ are functions of j and k only. It is clear that in the limit $x \rightarrow 0$, $n \rightarrow \infty$ an expansion in terms of

$$\left(\frac{\lambda}{h}\right)^2 = \frac{\pi^2}{xn^2} \tag{127}$$

may be made where λ/h is introduced as the finite expansion parameter. Replacing the sum in expression (125) by its approximating integral yields

$$\lim_{x\rightarrow 0}\,[L_n(x) - L_n(0)] = -2\pi^2\frac{h^2}{\lambda^2}\ln\left|\frac{4e\lambda}{\pi h}\right| + O_4\left(\frac{h}{\lambda}\right) \tag{128a}$$

where

$$n = \pi h x^{-\frac{1}{2}}\lambda^{-1}. \tag{128b}$$

A.7 *Expansions of* L *in Terms of Inverse Powers*

The expansion of L in terms of inverse powers of the argument is obtained by Taylor expansion of expression (98b) and integrating. This yields

$$\begin{aligned} L_n(x) &= 4x^{-\frac{1}{2}}\int_0^{\pi/2}\frac{\sin^2 n\beta}{(1 + x^{-1}\sin^2\beta)^{\frac{1}{2}}}\,d\beta \\ &= 4\sum_{j=0}^{\infty}\frac{(2j)!}{(-4)^j(j!)^2}\,x^{-(j+\frac{1}{2})}\int_0^{\pi/2}\sin^2 n\beta\,\sin^{2j}\beta\,d\beta, \\ &\qquad\qquad n \geqq 1 \end{aligned} \tag{129a}$$

where for example

$$L_1(x) = \pi\sum_{j=0}^{\infty}\frac{(2j+1)}{(j+1)}\frac{1}{(-16)^j}\left[\frac{(2j)!}{(j!)^2}\right]^2 x^{-(j+\frac{1}{2})}. \tag{129b}$$

By considering the Fourier decomposition of $\sin^j \beta$ it can be seen that

$$\int_0^{\pi/2} \sin^2 n\beta \sin^{2j} \beta \, d\beta = \frac{1}{2} \int_0^{\pi/2} \sin^{2j} \beta \, d\beta, \qquad n > j$$

$$= \frac{\pi}{4} \frac{(2j)!}{4^j (j!)^2}, \qquad n > j \tag{130}$$

(independent of n) so that

$$L_0(x) = 0,$$

$$L_n(x) = \pi \sum_{j=0}^{n-1} \frac{1}{(-16)^j} \left[\frac{(2j)!}{(j!)^2}\right]^2 x^{-(j+\frac{1}{2})} + O[x^{-(n+\frac{1}{2})}]), \qquad n > 1 \tag{131a}$$

or identifying with expression (97)

$$L_n(x) = V(x) + O[x^{-(n+\frac{1}{2})}], \qquad n > 1. \tag{131b}$$

Evaluating expression (129a) directly in those cases in which expressions (129b) or (131a) cannot be used yields

$$L_0(x) = 0, \tag{132a}$$

$$L_1(x) = \pi x^{-1/2} - \frac{3\pi}{8} x^{-3/2} + \frac{15\pi}{64} x^{-5/2} + O(x^{-7/2}) \tag{132b}$$

$$L_2(x) = \pi x^{-1/2} - \frac{\pi}{4} x^{-3/2} + \frac{15\pi}{128} x^{-5/2} + O(x^{-7/2}), \tag{132c}$$

$$L_n(x) = \pi x^{-1/2} - \frac{\pi}{4} x^{-3/2} + \frac{9\pi}{64} x^{-5/2} + O(x^{-7/2}). \tag{132d}$$

A.8 *The Gaussian Transformation*

In the neighborhood of $x = 1$, the convergence of the power series in x or x^{-1} is rather slow. Either the gaussian or Landen transformations may be used to transform the U, V, or L_1 functions into a region of rapid convergence.[16] In the present case, the gaussian transformation is preferred since it does not introduce incomplete elliptic integrals as does the Landen transformation.

The result of the gaussian transformation is

$$x_1 = 4x^{\frac{1}{2}}(1 + x)^{\frac{1}{2}}[(1 + x)^{\frac{1}{2}} + x^{\frac{1}{2}}]^2, \tag{133a}$$

or inversely

$$x = \frac{x_1^2}{4(1 + x_1)^{\frac{1}{2}}[(1 + x_1)^{\frac{1}{2}} + 1]^2} \tag{133b}$$

for the argument and

$$V(x) = 2TV(x_1), \tag{134}$$

$$TU(x) = U(x_1) - \frac{x_1}{2} V(x_1), \tag{135}$$

$$TL_1(x) = L_1(x_1) + \frac{2x_1}{1 + T^2} V(x_1), \tag{136}$$

for the functions where

$$T \equiv (1 + x_1)^{\frac{1}{4}} = (1 + x)^{\frac{1}{2}} + x^{\frac{1}{2}}. \tag{137}$$

APPENDIX B

The Force and Stability Functions

This appendix is a compilation of expressions for the force F and stability S_n functions. Each of the functions is written in terms of the U and V or L_n functions of Appendix A. Expressions in terms of the complete elliptic integrals of the first and second kind (denoted by K and E respectively) permit the use of tables[14] or numerical computation using the Landen transformation or the gaussian transformation.[16] The gaussian transformation is used in Section A.8. The power series expansions provided are necessary in obtaining numerical values of the functions for either very large or very small values of the argument and also provide the asymptotic forms of the functions. The argument of the functions is the domain diameter to thickness ratio, $d/h = 2r_0/h$.

B.1 *The Force Function*

The force function is written in terms of U by comparing the form of F, expression (40), and the form of U, expression (84b),

$$F\left(\frac{d}{h}\right) = \frac{1}{\pi}\left(\frac{d}{h}\right)^2\left[U\left(\frac{h^2}{d^2}\right) - 2\right]. \tag{138a}$$

This expression is written in terms of the complete elliptic integral of the second kind using expression (84d)

$$F\left(\frac{d}{h}\right) = \frac{2}{\pi}\left(\frac{d}{h}\right)^2\left[\left(1 + \frac{h^2}{d^2}\right)^{\frac{1}{2}} E[(1 + h^2/d^2)^{-1}] - 1\right]. \tag{138b}$$

It is expanded about $h/d = 0$ using expression (94)

$$F\left(\frac{d}{h}\right) = \frac{1}{\pi}\left\{\left[\frac{1}{2} + \frac{3}{32}\left(\frac{h}{d}\right)^2 - \frac{3}{64}\left(\frac{h}{d}\right)^4 + \frac{665}{24576}\left(\frac{h}{d}\right)^6 + \cdots\right]\right.$$

$$+ \left[1 - \frac{1}{8}\left(\frac{h}{d}\right)^2 + \frac{3}{64}\left(\frac{h}{d}\right)^4 - \frac{25}{1024}\left(\frac{h}{d}\right)^6 + \cdots\right] \ln \left| 4\frac{d}{h} \right| \Bigg\}. \tag{138c}$$

Additional terms may be generated using expressions (89) and (91). It is expanded about $d/h = 0$ using expression (96b)

$$F\left(\frac{d}{h}\right) = \frac{d}{h} - \frac{2}{\pi}\left(\frac{d}{h}\right)^2 + \frac{1}{4}\left(\frac{d}{h}\right)^3 - \frac{3}{64}\left(\frac{d}{h}\right)^5 + \frac{5}{256}\left(\frac{d}{h}\right)^7 + \cdots . \tag{138d}$$

Additional terms may be generated using expression (96a).

B.2 *The Radial Stability Function*

The radial stability function is written in terms of U and V using the definition of the radial stability function [equation (66a)], the expression for F [equation (138a)], and comparing the derivative of F [equation (41)] with the form of V [equation (85b)],

$$S_0\left(\frac{d}{h}\right) = -\frac{1}{\pi}\left(\frac{d}{h}\right)^2\left[U\left(\frac{h^2}{d^2}\right) - \left(\frac{h}{d}\right)^2 V\left(\frac{h^2}{d^2}\right) - 2\right], \tag{139a}$$

or in terms of L_1 using the expression for L_1 [equation (100)] and the expressions for $L_1(0)$ [equations (115) and (116)]

$$S_0\left(\frac{d}{h}\right) = -\frac{1}{2\pi}\left(\frac{d}{h}\right)^2\left[L_1\left(\frac{h^2}{d^2}\right) - L_1(0)\right]. \tag{139b}$$

Expression (139a) is written in terms of the complete elliptic integrals of the first and second kind using expressions (84d) and (85d),

$$S_0\left(\frac{d}{h}\right) = -\frac{2}{\pi}\left(\frac{d}{h}\right)^2\left[\left(1 + \frac{h^2}{d^2}\right)^{\frac{1}{2}} E[(1 + h^2/d^2)^{-1}] - \left(\frac{h}{d}\right)^2\left(1 + \frac{h^2}{d^2}\right)^{-\frac{1}{2}} K[(1 + h^2/d^2)^{-1}] - 1\right]. \tag{139c}$$

The expansion about $h/d = 0$ is obtained using expressions (139b) and (103).

$$S_0\left(\frac{d}{h}\right) = \frac{1}{\pi}\Bigg\{\left[-\frac{1}{2} + \frac{13}{32}\left(\frac{h}{d}\right)^2 - \frac{9}{32}\left(\frac{h}{d}\right)^4 + \frac{5255}{24576}\left(\frac{h}{d}\right)^6 + \cdots\right] + \left[1 - \frac{3}{8}\left(\frac{h}{d}\right)^2 + \frac{15}{64}\left(\frac{h}{d}\right)^4 - \frac{175}{1024}\left(\frac{h}{d}\right)^6 + \cdots\right] \ln \left| \frac{4d}{h} \right| \Bigg\}. \tag{139d}$$

Additional terms may be generated using expressions (139a) and (89) through (92). The expansion about $d/h = 0$ is obtained using expressions (139b), (132b), and (103) to obtain L_1 (0)

$$S_0\left(\frac{d}{h}\right) = \frac{2}{\pi}\left(\frac{d}{h}\right)^2 - \frac{1}{2}\left(\frac{d}{h}\right)^3 + \frac{3}{16}\left(\frac{d}{h}\right)^5 - \frac{15}{128}\left(\frac{d}{h}\right)^7 + \cdots . \tag{139e}$$

Additional terms may be generated using expression (129b).

B.3 *The Elliptical Stability Function*

From the general definition of the S_n of expression (66b) the elliptical stability function is

$$S_2\left(\frac{d}{h}\right) = -\frac{1}{3}\left[S_0\left(\frac{d}{h}\right) + \frac{1}{2\pi}\left(\frac{d}{h}\right)^2\left\{L_2\left(\frac{h^2}{d^2}\right) - L_2(0)\right\}\right]. \tag{140a}$$

Using expression (139b) for S_0, the L_n recursion relation (101) to reduce L_2 to L_1, and V and (103) to obtain $L_1(0)$, S_2 is written in terms of L_1 and V:

$$S_2\left(\frac{d}{h}\right) = \frac{1}{18\pi}\left(\frac{d}{h}\right)^2\left\{4 - \left[1 + 8\left(\frac{h}{d}\right)^2\right]L_1\left(\frac{h^2}{d^2}\right) + 8\left(\frac{h}{d}\right)^2 V\left(\frac{h^2}{d^2}\right)\right\}. \tag{140b}$$

The function L_1 is then eliminated using expression (100) to obtain the expression in terms of U and V:

$$S_2\left(\frac{d}{h}\right) = \frac{1}{9\pi}\left(\frac{d}{h}\right)^2\left\{2 - \left[1 + 8\left(\frac{h}{d}\right)^2\right]U\left(\frac{h^2}{d^2}\right) + \left[5\left(\frac{h}{d}\right)^2 + 8\left(\frac{h}{d}\right)^4\right]V\left(\frac{h^2}{d^2}\right)\right\} \tag{140c}$$

which then is written in terms of the complete elliptic integrals of the first and second kind using expression (84d) and (85d):

$$S_2\left(\frac{d}{h}\right) = \frac{1}{9\pi}\left(\frac{d}{h}\right)^2\left\{2 - \left[2 + 16\left(\frac{h}{d}\right)^2\right]\left(1 + \frac{h^2}{d^2}\right)^{+\frac{1}{2}} E[(1 + h^2/d^2)^{-1}] + \left[10\left(\frac{h}{d}\right)^2 + 16\left(\frac{h}{d}\right)^4\right]\left(1 + \frac{h^2}{d^2}\right)^{-\frac{1}{2}} K[(1 + h^2/d^2)^{-1}]. \right. \tag{140d}$$

The expansion about $h/d = 0$ is obtained using expressions (140b), (95), and (103):

$$S_2\left(\frac{d}{h}\right) = \frac{1}{\pi}\left\{\left[-\frac{11}{6} - \frac{17}{96}\left(\frac{h}{d}\right)^2 + \frac{53}{288}\left(\frac{h}{d}\right)^4 - \frac{2929}{24576}\left(\frac{h}{d}\right)^6 + \cdots\right] + \left[1 + \frac{5}{8}\left(\frac{h}{d}\right)^2 - \frac{35}{192}\left(\frac{h}{d}\right)^4 + \frac{105}{1024}\left(\frac{h}{d}\right)^6 + \cdots\right] \ln\left|\frac{4d}{h}\right|\right\}. \tag{140e}$$

Additional terms may be generated using expressions (140c) and (89) through (92). The expansion about $d/h = 0$ is obtained using expressions (140b), (97b), and (132b):

$$S_2\left(\frac{d}{h}\right) = \frac{2}{9\pi}\left(\frac{d}{h}\right)^2 - \frac{1}{48}\left(\frac{d}{h}\right)^5 + \frac{5}{256}\left(\frac{d}{h}\right)^7 + \cdots . \tag{140f}$$

Additional terms may be generated using expressions (140b), (97a), and (129b).

B.4 *The General Stability Functions*

Using the definition (66b) and the expression for S_0 [equation (139b)], the S_n are written in terms of the L_n as

$$S_n\left(\frac{d}{h}\right) = -\frac{1}{n^2-1}\frac{1}{2\pi}\left(\frac{d}{h}\right)^2\left[L_n\left(\frac{h^2}{d^2}\right) - L_1\left(\frac{h^2}{d^2}\right) - L_n(0) + L_1(0)\right], \qquad n \geqq 2. \tag{141a}$$

The leading term of the expansion about $h/d = 0$ is obtained using expressions (103), (105), (115), (116), (125), and (126):

$$S_n\left(\frac{d}{h}\right) = \frac{1}{\pi}\left[\ln\left|\frac{4d}{h}\right| - \frac{4n^2-1}{2(n^2-1)}\sum_{j=1}^{n}\frac{j}{2j-1} + \frac{2n^2+1}{2(n^2-1)}\right] + O_2\left(\frac{nh}{d}\right), \qquad n \geqq 2. \tag{141b}$$

The expansion about $d/h = 0$ is obtained using expressions (115), (116), (129b), and (131a):

$$S_n\left(\frac{d}{h}\right) = \frac{1}{n^2-1}\left\{\frac{2}{\pi}\left(\frac{d}{h}\right)^2\sum_{j=2}^{n}\frac{1}{2j-1} + \frac{1}{2}\sum_{j=1}^{n-1}\frac{j}{j+1}\frac{1}{(-16)^j}\cdot\left[\frac{(2j)!}{(j!)^2}\right]^2\left(\frac{d}{h}\right)^{2j+3} + O_{2n+3}\left(\frac{d}{h}\right)\right\}, \qquad n \geqq 2. \tag{141c}$$

APPENDIX C

Symbol List

Numbers in parentheses are defining equations or figures.

A	exchange constant (71)
a	area
d	mean domain diameter, $2r_0$ (42)
$E(x)$	complete elliptic integral of the second kind ($x = m = k^2$)
E_H	energy due to applied field (9)
E_M	internal magnetostatic energy (10)

E_T total energy (7)
E_W total wall energy (8)
$F(x)$ generalized radial force function (33, 138, Fig. 3)
$\mathbf{H}$ magnetic field vector
H uniform applied field
H_a anisotropy field
H_N nucleation field (74)
$\langle H_z \rangle_{av}$ z-averaged z-component of magnetic field (45)
h plate thickness (Figs. 1, 4)
$K(x)$ complete elliptic integral of the first kind ($x = m = k^2$)
K_u uniaxial anisotropy constant (71)
$L_n(x)$ integrally defined function (34, 98)
l characteristic length, $\sigma_w/4\pi M_s^2$ (67)
l_w wall width (81)
$\mathbf{M}$ magnetization vector
M_s saturation magnetization
n rotational periodicity (1)
O_k terms of order k
q quality factor, $K_u/2\pi M_s^2 = H_a/4\pi M_s$ (72)
r cylindrical coordinate (Figs. 1, 2)
r_f plate radius (Fig. 3)
r_n nth radial Fourier amplitude (1)
r_0 mean domain radius (1)
$S_n(x)$ nth infinite plate stability function (66, 139, 140, 141)
s distance between interacting magnetic charges (10b, 24, 28)
$U(x)$ integrally defined function (84)
$u(x)$ unit step function (5)
V volume
$V(x)$ integrally defined function (85)
z cylindrical coordinate (Figs. 1, 4, 5)
Z operator (23)
z $z - z'$ (21)
$\triangle E$ variation in energy (11, 43)
$\triangle r_n$ variation in r_n (3)
$\triangle \theta_n$ variation in θ_n (3)
$\delta(x)$ dirac delta function
ζ $\theta' - \theta$ (27, Fig. 2)
η polar azimuthal angle (Fig. 5)
η_w polar angle of wall normal (Fig. 5)
θ cylindrical coordinate (Fig. 1)
θ_n nth Fourier phase angle (1)

λ_n wavelength of nth variation (54)
ν azimuthal angle (Fig. 5)
ν_w azimuthal angle of wall normal (Fig. 5)
ξ_0 wall displacement vector (Fig. 5)
ρ coordinate in displaced cylindrical coordinate system (35, Fig. 2)
σ_w wall energy density
φ coordinate in displaced cylindrical coordinate system (35, Fig. 2)
χ_t transverse susceptibility
Ω magnetostatic potential

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