Radiation Losses of Dielectric Waveguides in Terms of the Power Spectrum of the Wall Distortion Function

By DIETRICH MARCUSE

(Manuscript received July 23, 1969)

In an earlier paper I described a perturbation theory of the radiation losses of a dielectric slab waveguide. The statistical treatment of the radiation losses was based on the correlation function of the wall distortion. This paper discusses the results of the radiation loss theory in terms of the power spectrum of the function describing the thickness of the slab. We found that only those mechanical frequencies θ *of the power spectrum contribute to the radiation loss that fall into the range* $\beta_0 - \mathrm{k} < \theta < \beta_0 + \mathrm{k}$. ($\beta_0$ = *propagation constant of guided mode,* k = *free space propagation constant.) The mechanical frequencies near both end points of this mechanical frequency range contribute more to the radiation loss than the region well inside of this range.*

We also discuss the far-field radiation pattern caused by a strictly sinusoidal wall distortion.

I. INTRODUCTION

In an earlier paper I developed a perturbation theory of the mode conversion effects between guided modes and of the radiation losses of a given guided mode caused by deviations from perfect straightness of the waveguide wall.[1] For simplicity, the discussion had been limited to a waveguide in the form of an infinitely extended dielectric slab.

The statistical discussion had been based on the description of the wall distortion by means of a correlation function. In Ref. 1 an exponential correlation function had been assumed. However, it has been established that the shape of the correlation function has little influence on the radiation losses.

It is possible to base the discussion of radiation losses not on correlation functions, but on the mechanical power spectrum of the wall distortion function. This study provides information as to how the various mechanical frequencies of the wall distortion function contribute to the radiation losses.

The analysis of Ref. 1 was based on the use of radiation modes of the dielectric slab which represent standing waves in directions transverse to the propagation direction of the guided modes. The question naturally arises how a superposition of these standing waves can result in radiation flowing away from the rod. This question is answered by examining the far field radiation pattern caused by a sinusoidal distortion of one wall of the dielectric waveguide. This paper gives the relation between the length of the mechanical period, the wavelength of the guided mode, and the direction of the main lobe of the radiation.

II. RADIATION LOSS AND POWER SPECTRUM

The amplitudes of the modes of the continuous spectrum were derived in Ref. 1, equations (65) and (69). We have

$$g_e(\rho, L) = \frac{Lk^2}{2i(\pi)^{\frac{1}{2}}}(n_g^2 - 1)\frac{\rho(\cos\kappa_0 d \cos\sigma d)[\varphi(\theta) - \psi(\theta)]}{\left[\beta\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)(\rho^2\cos^2\sigma d + \sigma^2\sin^2\sigma d)\right]^{\frac{1}{2}}} \tag{1}$$

for the even modes, and

$$g_0(\rho, L) = \frac{Lk^2}{2i(\pi)^{\frac{1}{2}}}(n_g^2 - 1)\frac{\rho(\cos\kappa_0 d \sin\sigma d)[\varphi(\theta) + \psi(\theta)]}{\left[\beta\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)(\rho^2\sin^2\sigma d + \sigma^2\cos^2\sigma d)\right]^{\frac{1}{2}}} \tag{2}$$

for the odd modes. The functions

$$\varphi(\theta) = \frac{1}{L}\int_0^L [f(z) - d]e^{-i\theta z}\,dz, \tag{3}$$

$$\psi(\theta) = \frac{1}{L}\int_0^L [h(z) + d]e^{-i\theta z}\,dz, \tag{4}$$

with

$$\theta = \beta_0 - \beta \tag{5}$$

are the Fourier transforms of the wall distortion functions $f(z) - d$ and $h(z) + d$. [$x = f(z)$ is the boundary of the dielectric-air interface, $x = d$ describes the wall of the perfect guide, and $x = h(z)$ is the distorted boundary near $x = -d$.]

The meaning of the constants appearing in equations (1) to (5) is:

β_0 = propagation constant of guided mode (propagating in z-direction),
β = component of the propagation constant of the continuum mode in z-direction,
k = propagation constant in free space,
L = length of guide section with wall distortions,
n_g = dielectric constant of slab,

$$\rho = (k^2 - \beta^2)^{\frac{1}{2}} \tag{6}$$

$$\sigma = (n_g^2 k^2 - \beta^2)^{\frac{1}{2}}, \tag{7}$$

$$\kappa_o = (n_g^2 k^2 - \beta_0^2)^{\frac{1}{2}}, \tag{8}$$

$$\gamma_0^2 = (\beta_0^2 - k^2)^{\frac{1}{2}}. \tag{9}$$

The y-component of the electric radiation field caused by the wall distortions is given by

$$E_y = \int_0^\infty [g_e(\rho, L)\mathcal{E}_e(\rho, z) + g_0(\rho, L)\mathcal{E}_0(\rho, z)]\, d\rho. \tag{10}$$

The functions $\mathcal{E}_e$ and $\mathcal{E}_o$ are the even and odd radiation modes. The ratio of scattered power to incident guided mode power is obtained from

$$\frac{\Delta P}{P} = \int_{-k}^{k} (|\, g_e(\rho, L)\,|^2 + |\, g_0(\rho, L)\,|^2)\, \frac{\beta}{\rho}\, d\beta. \tag{11}$$

For simplicity we assume that one wall of the slab is perfect

$$h(z) = -d, \tag{12}$$

so that

$$\psi(\theta) = 0, \tag{13}$$

the relative scattering loss, follows from equations (1), (2), and (10)

$$\frac{\Delta P}{P} = \int_{-k}^{k} \frac{1}{d^2}\, L\, |\, \varphi(\theta)\,|^2\, I(\beta)\, d\beta \tag{14a}$$

with

$$I(\beta) = \frac{(kd)^4}{4\pi}(n_g^2 - 1)^2 \frac{\cos^2 \kappa_0 d}{\beta_0 d + \dfrac{\beta_0}{\gamma_0}} (\rho d) \left[\frac{\cos^2 \sigma d}{(\rho d)^2 \cos^2 \sigma d + (\sigma d)^2 \sin^2 \sigma d} + \frac{\sin^2 \sigma d}{(\rho d)^2 \sin^2 \sigma d + (\sigma d)^2 \cos^2 \sigma d}\right]. \tag{14b}$$

Since $\varphi(\theta)$ is the Fourier component of the wall distortion function its absolute square value

$$| \varphi(\theta) |^2 \tag{15}$$

is the "power spectrum" of $f(z) - d$. It is apparent from equation (14) that $\Delta P/P$ depends on the power spectrum of the wall distortion function. Incidentally, equation (14) is not a statistical expression, but holds for a specific dielectric slab waveguide. We entered the power spectrum in the combination $L | \varphi |^2$ in equation (14) since this combination is independent of L for a randomly varying function $f(z) - d$.

Equation (14) allows us immediately to determine the range of mechanical frequencies θ which contribute to the radiation loss. The integral in equation (14) is extended from $-k$ to k, the β range of continuous radiation modes. The range of mechanical frequencies contributing to the scattering loss is therefore given by

$$\beta_0 - k < \theta < \beta_0 + k. \tag{16}$$

This is an important result since it states that those parts of the power spectrum which lie outside of the range, equation (16), do not contribute to radiation loss.

This last statement must not be misconstrued to mean that a waveguide with a sinusoidal wall distortion extending over length L

$$f(z) = d + a \sin \theta' z \qquad 0 \leqq z \leqq L \tag{17}$$

with θ' lying outside the range of equation (16) does not lose power by radiation. The power spectrum of equation (17) is

$$| \varphi(\theta) |^2 = \left[\frac{a}{L} \frac{\sin (\theta' - \theta) \dfrac{L}{2}}{\theta' - \theta} \right]^2 . \tag{18}$$

A term with $\theta' + \theta$ in the denominator has been neglected in equation (18). The accuracy of this approximation improves with increasing values of L.

It is apparent from equation (18) that $| \varphi(\theta) |^2$ has non-vanishing values for $\theta \neq \theta'$ so that there is some small contribution to radiation loss even if θ' lies outside of the range of equation (16).

However, if we consider the limit $L \to \infty$ we can approximate the power spectrum, equation (18), by a δ-function:

$$\lim_{L \to \infty} | \varphi(\theta) |^2 = \frac{\pi a^2}{2L} \delta(\theta - \theta'). \tag{19}$$

In this special case the expression (14a) for the scattered power becomes

$$\frac{\Delta P}{P} = \frac{\pi}{2}\left(\frac{a}{d}\right)^2 I(\beta_0 - \theta'). \tag{20}$$

The scattering from a dielectric waveguide with a wall distortion function whose power spectrum is a δ-function is proportional to $I(\beta_0 - \theta')$.

The function $I(\beta)$ is plotted in Fig. 1 for $n_g = 1.01$, $kd = 8.0$, and $\beta_0 d = 8.041$. The scattering caused by a wall distortion with a δ-function spectrum (a sinusoidal wall distortion of infinite length) is nearly independent of the value of $\beta = \beta_0 - \theta'$ over most of the β-range. There are two sharp peaks at $\beta \approx k$ and $\beta \approx -k$. The physical reasons for the sharp increase in loss at these values is easy to understand if we consider the direction of the radiation pattern as a function of θ'. We show in Section III [equation (35)] that the angle α between the waveguide and the main radiation lobe is given by

$$\cos\alpha = \frac{\beta}{k} = \frac{\beta_0 - \theta'}{k}. \tag{21}$$

The two peaks of the function $I(\beta)$, or correspondingly of the radiation loss, are associated with

$$\alpha \approx 0 \quad \text{and} \quad \alpha \approx \pi. \tag{22}$$

This shows that the radiation loss is high when the radiation pattern is

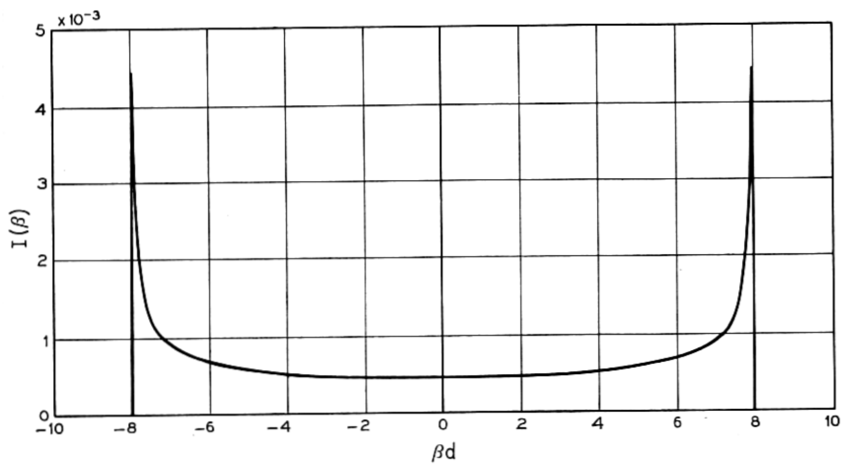

Fig. 1 — Graphical representation of the function $I(\beta)$ [eq. (14b)]. $n_g = 1.01$, $kd = 8.0$, $\beta_0 d = 8.041$.

directed very nearly parallel to the surface of the waveguide. The radiation modes gain more power if the guided mode can interact with them over a longer distance. An observation of this loss peak is reported in Ref. 2.

A power spectrum with sharp peaks much like that of equation (18) or (19) is not likely to occur for dielectric waveguides with random imperfections of the dielectric interface. It is much more reasonable to expect that such waveguides may have spectral distributions which are nearly independent of θ over a certain range of θ values. In the limit of a "white" spectrum,

$$|\varphi(\theta)|^2 = \text{constant}, \tag{23}$$

the scattering loss is proportional to the integral over the function $I(\beta)$ shown in Fig. 1. The two peaks contribute very little to this integral. Numerical integration of $I(\beta)$ of Fig. 1 including and excluding the peaks resulted in the values:

$$\int_{-8}^{8} I(\beta)\, d\beta = 0.011, \int_{-7.8}^{7.8} I(\beta)\, d\beta = 0.0096,$$

$$\text{and} \quad \int_{-7.5}^{7.5} I(\beta)\, d\beta = 0.0087.$$

This result is reassuring for the use of the perturbation theory which was used to derive equation (14). The perturbation theory is based on the assumption that power is converted from the guided mode to the radiation field but that no power is converted back from the radiation field to the guided mode. This approximation is certain to yield better results if the radiation pattern is directed away from the rod. In other words, the perturbation theory will work poorest in the region of the peaks of Fig. 1. However, for spectra that do not particularly favor the regions of these peaks, the contribution of those regions (which at the same time give the least reliable results) to the total radiation loss is only slight.

III. THE FAR FIELD RADIATION PATTERN

The far field pattern of the radiation field (that is excited by the lowest order even guided mode traveling in the dielectric slab with sinusoidal perturbation of one wall) can easily be calculated from equation (10). The even and odd radiation modes were given in Ref. 1 (for $|x| > d$)

$$\mathcal{E}_y^{(e)} = \left[\frac{2\omega\mu P}{\pi\beta(\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d)}\right]^{\frac{1}{2}}$$

$$\times [\rho \cos \rho(|x| - d) \cos \sigma d - \sigma \sin \rho(|x| - d) \sin \sigma d] e^{i(\omega t - \beta z)} \quad (24)$$

$$\mathcal{E}_y^{(0)} = \frac{x}{|x|}\left[\frac{2\omega\mu P}{\pi\beta(\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d)}\right]^{\frac{1}{2}}$$

$$\times [\rho \cos \rho(|x| - d) \sin \sigma d + \sigma \sin \rho(|x| - d) \cos \sigma d] e^{i(\omega t - \beta z)}. \quad (25)$$

With $\psi(\theta) = 0$ and

$$\varphi(\theta) \approx \frac{a}{iL} \exp\left[i(\theta' - \theta)\frac{L}{2}\right] \frac{\sin (\theta' - \theta)\frac{L}{2}}{\theta' - \theta} \quad (26)$$

and with the help of equations (1) and (2) we get from equation (10)

$$E_y = -\frac{ak^2}{(2)^{\frac{1}{2}}\pi}(\omega\mu P)^{\frac{1}{2}}(n_g^2 - 1)\frac{\cos \kappa_0 d}{\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)^{\frac{1}{2}}}$$

$$\times \int_0^\infty \frac{\rho}{\beta}\left\{\frac{\cos \sigma d[\rho \cos \rho(x - d) \cos \sigma d - \sigma \sin \rho(x - d) \sin \sigma d]}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d}\right.$$

$$\left. + \frac{\sin \sigma d[\rho \cos \rho(x - d) \sin \sigma d + \sigma \sin \rho(x - d) \cos \sigma d]}{\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d}\right\}$$

$$\times \exp\left[i(\theta' - \theta)\frac{L}{2}\right]\frac{\sin (\theta' - \theta)\frac{L}{2}}{\theta' - \theta} \times e^{i(\omega t - \beta z)}\, d\rho. \quad (27)$$

In the far field with $x \to \infty$ and $z \to \infty$ (but L finite) we can obtain an approximate solution of the integral in equation (27) by the method of stationary phase.[3] The sine and cosine functions of argument $\rho(x - d)$ can be expressed as sums of exponential functions. The most important terms of the integrand of equation (27) are, therefore, of the form

$$\exp [-i(\beta z \pm \rho x)]. \quad (28)$$

This exponential term is an extremely rapidly varying function of ρ as $x \to \infty$ and $z \to \infty$. All other terms in the integrand vary slowly by comparison. According to the method of stationary phase the contribution to the integral comes predominantly from a region that is determined by

$$\frac{\partial}{\partial \rho}(\beta z \pm \rho x) = 0. \quad (29)$$

With the help of equation (6), equation (29) leads to the condition

$$\frac{x}{z} = \pm\frac{\rho_0}{\beta} \tag{30}$$

or

$$\rho_0 = k \sin \alpha \tag{31a}$$

$$\beta = k \cos \alpha \tag{31b}$$

with

$$\cos \alpha = \frac{z}{(x^2 + z^2)^{\frac{1}{2}}} = \frac{z}{r}. \tag{32}$$

For $x > 0$ and $z > 0$ only the $+$ sign in equation (30) is possible. This is an important point. It shows that even though the radiation modes, equations (24) and (25), represent standing wave patterns in x-direction only, the outward traveling part of the decomposition of the standing wave into traveling waves makes a contribution to the radiation field, equation (27).

All terms of the integrand with the exception of equation (28) can be taken out of the integral. The remaining integration can be carried out using the expansion

$$\beta z + \rho x = k(x \sin \alpha + z \cos \alpha) - \frac{1}{2}\frac{z}{k \cos^3 \alpha}(\rho - \rho_0)^2 + \cdots$$

$$\cdot \int_0^\infty e^{-i(\beta z + \rho x)}\, d\rho = (1 + i)(\pi)^{\frac{1}{2}} \frac{(k)^{\frac{1}{2}} \cos \alpha}{(r)^{\frac{1}{2}}} e^{-ik(x \sin \alpha + z \cos \alpha)}. \tag{33}$$

The far field is therefore obtained in the form

$$E_y = \frac{1}{(\pi)^{\frac{1}{2}}} \exp\left(i\frac{\pi}{4}\right) ak^{\frac{3}{2}}(\omega\mu P)^{\frac{1}{2}}(n_g^2 - 1)\frac{\cos \kappa_0 d}{\left(\beta_0 d + \dfrac{\beta_0}{\gamma_0}\right)^{\frac{1}{2}}}$$

$$\cdot \frac{\rho_0^2 \sin 2\sigma_0 d - i\rho_0\sigma_0 \cos 2\sigma_0 d}{(\rho_0^2 + \sigma_0^2)\sin 2\sigma_0 d - 2i\rho_0\sigma_0 \cos 2\sigma_0 d}\,\frac{\sin\,(\theta' - \theta)\dfrac{L}{2}}{\theta' - \theta}$$

$$\cdot \exp\left[i(\theta' - \theta)\frac{L}{2}\right] e^{i\rho_0 d} \frac{1}{(r)^{\frac{1}{2}}} e^{i[\omega t - k(x \sin \alpha + z \cos \alpha)]}. \tag{34}$$

The index zero was added to σ to indicate that it must be evaluated from equations (7) and (8) using ρ_0 of equation (31a).

Equation (34) reveals several important features of the far field of radiation. This field is essentially a plane wave traveling in the direction of α ($\tan \alpha = x/z$, and x and z are the coordinates of the point of observation).

The field intensity is inversely proportional to the square root of the distance r from (the sinusoidally distorted) waveguide section. The dependence on distance is inversely proportional to $(r)^{\frac{1}{2}}$ rather than r because the waveguide is infinitely extended in y-direction (see Ref. 1).

The main radiation lobe occurs at the maximum value of $[\sin (\theta' - \theta)L/2]/(\theta' - \theta)$ that is at $\theta = \theta'$ or from equations (5) and (31b) at

$$\cos \alpha_m = \frac{\beta_0 - \theta'}{k} \tag{35}$$

(β_0 = propagation constant of guided mode).

The width of the main lobe depends on the length L of the sinusoidally distorted waveguide section. The difference in angle between the peak of the lobe and the first null determines the half width of the main lobe

$$\Delta\alpha = \frac{2\pi}{Lk \sin \alpha} \text{ for } \alpha \neq 0. \tag{36a}$$

The width of the main radiation lobe is inversely proportional to L. The lobe is narrowest for $\alpha = \pi/2$ and becomes wider as α decreases toward zero. If the peak of the main lobe is at $\alpha = 0$, we obtain

$$\Delta\alpha = \left(\frac{4\pi}{Lk}\right)^{\frac{1}{2}} \text{ for } \alpha = 0. \tag{36b}$$

The peak amplitude of the main radiation lobe is not strongly dependent on α. The increase in radiated power in forward direction ($\alpha = 0$) which is apparent from Fig. 1 is caused by the broadening of the radiation lobe with decreasing angle.

IV. CONCLUSION

The radiation loss of dielectric waveguides caused by deviations from perfect straightness of the waveguide walls depends on the "power spectrum" of the wall deviation function. A sinusoidal wall perturbation gives rise to radiation into a particular direction in space. Each Fourier component of the Fourier expansion of the wall distortion function is responsible for radiation into a particular direction. The width of the radiation lobes is wide for scattering directions parallel to the rod so that those Fourier components responsible for forward and backward scattering contribute more to the radiation loss than those causing scattering in other directions. However, this preferential loss behavior is not very pronounced, so that the Fourier components responsible for forward and backward scattering contribute only a small amount of the total radiation loss caused by a broad power spectrum.

The coupling between two guided modes of the dielectric waveguide is also governed by equation (5). Only one component of the power spectrum of the wall distortion function influences the coupling between two guided modes, while the entire range of mechanical frequencies, equation (16), determines the radiation loss.

The general predictions of this theory have been experimentally verified. Microwave experiments on a periodically corrugated teflon rod have shown that the radiation losses are negligibly small if the period of the corrugation is such that θ lies outside of the interval indicated by equation (16).[2] However, if θ falls inside of the interval, equation (16), considerable radiation losses do occur. The peak of the radiation losses shown in Fig. 1 and the direction and width of the radiation lobes have also been observed in agreement with this theory.

REFERENCES

1. Marcuse, D., "Mode Conversion Caused by Surface Imperfections of a Dielectric Slab Waveguide," B.S.T.J., this issue, pp. 3187–3215.
2. Marcuse, D., and Derosier, R. M., "Mode Conversion Caused by Diameter Changes of a Round Dielectric Waveguide," B.S.T.J., this issue, pp. 3217–3232.
3. Mathews, J., and Walker, R. L., "Mathematical Methods of Physics," New York: W. A. Benjamin, 1965, pp. 85–86.