Mode Conversion Caused by Surface Imperfections of a Dielectric Slab Waveguide

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(Manuscript received May 8, 1969)

This paper contains a perturbation theory which is applicable to the scattering losses suffered by guided modes of a dielectric slab waveguide as a consequence of imperfections of the waveguide wall. The development of the theory occupies the bulk of the paper. Numerical results appear in Sections VI and VIII to which a reader less interested in the theory is referred.

The theory allows us to conclude that random deviations of the waveguide wall in the order of 1 percent, for guides designed to guide an optical wave of $\lambda_0 = 1\mu$ *wavelength, can cause scattering losses of 10 percent per centimeter or 0.46 dB per centimeter. A systematic sinusoidal deviation of the waveguide wall can cause total exchange of energy from the lowest order to the first order guided mode in a distance of approximately 1 cm if the amplitude of the sinusoidal deviation from perfect straightness is only 0.5 percent of the thickness of the guide. An rms deviation of one of the waveguide walls of 9Å causes a radiation loss of 10 dB per kilometer (index difference 1 percent, guide width* 2.5μ*).*

I. INTRODUCTION

The problem of how to transmit laser light over large distances or carry it short distances inside the laboratory has renewed the interest in dielectric waveguides.[1–5] Such waveguides usually used in the form of clad fibers or as strips of a medium of larger dielectric constant embedded in another dielectric medium are capable, in principle, of guiding electromagnetic radiation. By proper dimensioning, a dielectric waveguide can be made to transmit only one guided mode. In this respect mode guidance by dielectric waveguides resembles mode guidance by hollow metallic waveguides. Hollow metallic tubes can be constructed to allow only one mode to propagate so that mode conversion (except for conversion to the reflected dominant mode) becomes impossible. Such truly single mode operation is impossible for dielectric waveguides since these guides can always lose electromagnetic energy to the continuous spectrum of unguided modes.

The possible solutions of Maxwell's equations for a dielectric waveguide consist of a discrete spectrum of a finite number of guided modes plus a continuum of waveguide modes.[6] The guided modes have field configurations which concentrate the electromagnetic energy inside and in the immediate vicinity of the structure. The continuum of unguided modes extends to infinite distances from the waveguide and consists of a superposition of incident and reflected waves. A convenient way of visualizing the physical significance of the continuum of unguided modes is as follows. If a plane wave is incident on the dielectric waveguide at an arbitrary angle, part of it penetrates the dielectric structure while some portion is reflected. The resulting superposition field of incident and reflected waves satisfies Maxwell's equations and the boundary conditions at the dielectric waveguide and as such can be viewed as a mode of the structure, but the energy of this mode is not concentrated near the waveguide and there are no specific restrictions on the projection of the propagation vector in the direction of the guide axis.

A perfect dielectric waveguide can transmit any of its guided modes without converting energy to any of the other possible guided modes or to the continuous spectrum. But any imperfection of the guide, such as a local change of its index of refraction or a deviation from perfect straightness or an imperfection of the interface between two regions with different index of refraction, couples the particular guided mode to all other guided modes as well as to all the modes of the unguided continuum. Imperfections of this type are unavoidable. They transfer energy from the desired guided mode to unwanted guided modes and the radiation field of the continuum of unguided modes, thus increasing the loss of the desired guided mode.

This paper gives a simple, approximate theory of the losses of dielectric waveguides, caused by imperfections of the boundary between the inner region of higher dielectric index and the surrounding outer region of the dielectric waveguide. Even though the method of analysis used here can be used to describe any arbitrary dielectric waveguide, we limit the discussion to a simple case. We describe the effects of mode conversion for a dielectric slab surrounded by vacuum, assuming for simplicity, that there is no variation of the dimensions or properties of the rod as well as the field distribution in one co-ordinate direction. The restriction of demanding $\partial/\partial_y = 0$ for one of the co-ordinates y is no limitation on the method of analysis but is imposed strictly for convenience. It simplifies the analysis considerably without drastically changing the conclusions. The tolerance requirements based on our analysis are rather stringent. They show the order of magnitude of the losses which can be expected from deviations from perfect geometry. Additional variations in the direction considered perfect in this paper is unlikely to improve any of the loss predictions.

II. TE MODES OF A DIELECTRIC SLAB

Let us consider the transverse electric modes of the dielectric slab of Fig. 1. True to the simplifying assumption discussed in Section I, we assume

$$\frac{\partial}{\partial_y} = 0 \tag{1}$$

with y being the co-ordinate perpendicular to the x and z directions, but parallel to the slab. The only nonvanishing field components are E_y, H_x, and H_z.

Leaving the z and time dependence

$$e^{i(\omega t - \beta z)} \tag{2}$$

understood, we obtain the following modes of the ideal structure as a solution of Maxwell's equations satisfying the boundary conditions.

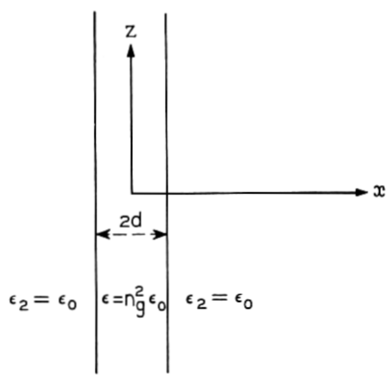

Fig. 1 — Geometry of a dielectric slab waveguide.

2.1 *Even Guided Modes*

For even guided modes

$$\mathcal{E}_y = A^{(e)} \cos \kappa x \quad \text{for} \quad |x| \leqq d, \tag{3a}$$

$$\mathcal{E}_y = A^{(e)} \cos \kappa d e^{-\gamma(x-d)} \quad \text{for} \quad x \geqq d, \tag{3b}$$

$$\mathcal{H}_x = -\frac{i}{\omega\mu}\frac{\partial \mathcal{E}_y}{\partial z}, \tag{4}$$

$$\mathcal{H}_z = \frac{i}{\omega\mu}\frac{\partial \mathcal{E}_y}{\partial x}, \tag{5}$$

The field component $\mathcal{E}_y$ satisfies the wave equation

$$\frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2} + n_0^2 k^2 \mathcal{E}_y = 0. \tag{6}$$

The value of the index of refraction n_0 is different inside and outside of the dielectric slab. For simplicity, we assume

$$n_0 = 1 \quad \text{for} \quad |x| > d. \tag{7}$$

The other constants are related as follows

$$k^2 = \omega^2 \epsilon_0 \mu_0 , \tag{8}$$

$$\kappa = (n^2 k^2 - \beta^2)^{\frac{1}{2}}, \tag{9}$$

$$\gamma = (\beta^2 - k^2)^{\frac{1}{2}}. \tag{10}$$

The propagation constant β is obtained as a solution of the eigenvalue equation

$$\tan \kappa d = \frac{\gamma}{\kappa}. \tag{11}$$

The mode amplitude A can be expressed in terms of the power P carried by the mode.

$$P = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} (-\mathcal{E}_y \mathcal{H}_x^*)\, dx = \frac{\beta}{\omega\mu} \int_0^{\infty} |\mathcal{E}_y|^2\, dx. \tag{12}$$

P is the power per unit length (unit length in y-direction) flowing along the z-axis. We obtain for the amplitude coefficient

$$A^{(e)2} = \frac{2\omega\mu}{\beta d + \dfrac{\beta}{\gamma}} P. \tag{13}$$

2.2 *Even Modes of the Continuum*

The continuum of unguided modes of even symmetry is given by the equations:

$$\mathcal{E}_y = B^{(e)} \cos \sigma x \quad \text{for} \quad |x| \leqq d, \tag{14a}$$

$$\mathcal{E}_y = C^{(e)} e^{i\rho x} + D^{(e)} e^{-i\rho x} \quad \text{for} \quad x \geqq d. \tag{14b}$$

The other field components follow again from equations (4) and (5) and $\mathcal{E}_y$ is a solution of equation (6). The constants are related to each other by the equations

$$\sigma = (n^2 k^2 - \beta^2)^{\frac{1}{2}}, \tag{15}$$

$$\rho = (k^2 - \beta^2)^{\frac{1}{2}}. \tag{16}$$

The radial propagation constant ρ can assume all values from 0 to ∞. The continuous mode spectrum starts at $\beta = k$ and continuous to $\beta = 0$ at which point we have $\rho = k$. Larger values of ρ are obtained for imaginary values of β corresponding to modes of the continuum exhibiting a cutoff behavior.

The boundary conditions do not lead to an eigenvalue equation for β but they determine $C^{(e)}$ and $D^{(e)}$ in relation to $B^{(e)}$.

$$C^{(e)} = \tfrac{1}{2} B^{(e)} e^{-i\rho d}\left(\cos \sigma d + i \frac{\sigma}{\rho} \sin \sigma d\right), \tag{17}$$

$$D^{(e)} = C^{(e)*}, \tag{18}$$

(the asterisk indicates the complex conjugate quantity).

The normalization of the modes of the continuum involves a δ-function. Instead of equation (12) we use

$$P\, \delta(\rho - \rho') = \frac{\beta}{\omega\mu} \int_0^\infty \mathcal{E}_y(\rho)\, \mathcal{E}_y^*(\rho')\, dx. \tag{19}$$

With this normalization we get

$$B^{(e)2} = \frac{2\omega\mu P}{\pi\beta\left(\cos^2 \sigma d + \dfrac{\sigma^2}{\rho^2} \sin^2 \sigma d\right)}. \tag{20}$$

2.3 *Odd Guided Modes*

In a manner similar to that for obtaining the preceding equations we obtain the equations for the odd guided modes

$$\mathcal{E}_y = A^{(0)} \sin \kappa x \quad \text{for} \quad x \leqq d, \tag{21a}$$

$$\mathcal{E}_y = A^{(0)} \sin \kappa d e^{-\gamma(x-d)} \quad \text{for} \quad x \geqq d. \tag{21b}$$

Equations (4) through (10) apply to the odd modes unaltered. The eigenvalue equation is given by

$$\tan \kappa d = -\frac{\kappa}{\gamma}, \tag{22}$$

and the mode normalization is

$$A^{(0)^2} = \frac{2\omega\mu}{\beta d + \dfrac{\beta}{\gamma}} P. \tag{23}$$

2.4 *Odd Modes of the Continuum*

As in Section 2.3 we obtain the equations for the odd modes of the continuum

$$\mathcal{E}_y = B^{(0)} \sin \sigma x \quad \text{for} \quad |x| \leqq d, \tag{24a}$$

$$\mathcal{E}_y = C^{(0)} e^{i\rho x} + D^{(0)} e^{-i\rho x} \quad \text{for} \quad x \geqq d, \tag{24b}$$

$$C^{(0)} = \tfrac{1}{2} B^{(0)} e^{-i\rho d}\left(\sin \sigma d - i \frac{\sigma}{\rho} \cos \sigma d\right), \tag{25}$$

$$D^{(0)} = C^{(0)*}, \tag{26}$$

$$B^{(0)^2} = \frac{2\omega\mu P}{\pi\beta\left(\sin^2 \sigma d + \dfrac{\sigma^2}{\rho^2} \cos^2 \sigma d\right)}. \tag{27}$$

All these modes are orthogonal to one another. The even modes are orthogonal to all the odd modes, the guided modes are orthogonal to all the modes of the continuum, and all guided modes as well as all modes of the continuum are orthogonal among each other. The orthogonality of the modes of the continuum among each other was already expressed by equation (19). Labeling the discrete modes by indices and dropping the vector component label y we can express the orthogonality of the discrete modes by the equation

$$P\, \delta_{nm} = \frac{\beta_m}{2\omega\mu} \int_{-\infty}^{\infty} \mathcal{E}_n \mathcal{E}_m^* \, dx. \tag{28}$$

III. MODE COUPLING CAUSED BY IMPERFECTIONS

We want to study the losses which the lowest order guided mode suffers because of imperfections of the waveguide wall. A dielectric waveguide with wall imperfections is shown in Fig. 2.

Fig. 2 — Dielectric slab waveguide with wall distortions.

The waveguide with wall imperfections is mathematically described by a refractive index distribution

$$n^2(x, z) = n_0^2(x, z) + \Delta n^2(x, z). \tag{29}$$

The index distribution

$$n_0^2(x, z) = \begin{cases} n_g^2 & |x| < d \\ 1 & |x| > d \end{cases} \tag{30}$$

describes the ideal dielectric waveguide whose TE modes were given in the Section II. The additional term Δn^2 describes how the guide deviates from its perfect shape. Consider a deviation shown in Fig. 3. The corresponding distribution Δn^2 is (n_g = index of refraction of the dielectric material of the guide)

$$\Delta n^2 = \begin{cases} 0 \begin{cases} x < d & \text{if} \quad d < f(z) \\ x < f(z) & \text{if} \quad d > f(z) \end{cases} \\ n_g^2 - 1 \qquad d < x < f(z) \quad \text{if} \quad d < f(z) \\ -(n_g^2 - 1) \qquad f(z) < x < d \quad \text{if} \quad d > f(z) \\ 0 \begin{cases} x > f(z) & \text{if} \quad d < f(z) \\ x > d & \text{if} \quad d > f(z) \end{cases} \end{cases}. \tag{31}$$

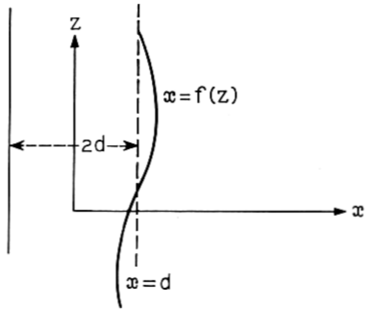

Fig. 3 — Illustration of the wall distortion function $f(z)$.

The field distribution E_y of this waveguide is a solution of

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + (n_0^2 + \Delta n^2)k^2 E_y = 0 \tag{32}$$

with H_x and H_z given by equations (4) and (5). The modes of the perfect waveguide form a complete orthogonal set for all TE modes with no variation in the y-direction. It is, therefore, possible to express any field distribution on the waveguide with imperfect walls by the expansion

$$E_y = \sum_n C_n(z)\, \mathcal{E}_n + \sum \int_0^\infty g(\rho, z)\, \mathcal{E}(\rho)\, d\rho. \tag{33}$$

The first summation extends over all even and odd modes of the discrete spectrum of guided modes. The integral extends over all modes of the continuum, and the summation sign in front of the integral indicates summation over even and odd modes. The expansion coefficients C_n and $g(\rho)$ are unknown functions of z.

To obtain a coupled system of differential equations for the expansion coefficients we substitute equation (33) into equation (32). Multiplying the resulting equation by

$$\frac{\beta_m}{2\omega\mu}\, \mathcal{E}_m^* ,$$

integrating over x from $-\infty$ to $+\infty$, and using the orthogonality relations and the fact that $\mathcal{E}_n$ and $\mathcal{E}(\rho)$ are the (discrete and continuous) modes of the perfect guide leads to

$$\frac{\partial^2 C_m}{\partial z^2} - 2i\beta_m \frac{\partial C_m}{\partial z} = F_m(z) \tag{34}$$

with

$$F_m(z) = -\frac{\beta_m k^2}{2\omega\mu P}\left[\sum_n C_n(z)\int_{-\infty}^{\infty} \mathcal{E}_m^* \, \Delta n^2 \, \mathcal{E}_n \, dx + \sum \int_0^{\infty} d\rho g(\rho, z) \int_{-\infty}^{\infty} \mathcal{E}_m^* \, \Delta n^2 \, \mathcal{E}(\rho) \, dx\right]. \tag{35}$$

Similarly multiplying by

$$\frac{\beta'}{2\omega\mu} \, \mathcal{E}^*(\rho')$$

leads to

$$\frac{\partial^2 g(\rho')}{\partial z^2} - 2i\beta' \frac{\partial g(\rho')}{\partial z} = G(\rho' z) \tag{36}$$

with

$$G(\rho', z) = -\frac{\beta' k^2}{2\omega\mu P}\left[\sum_n C_n(z)\int_{-\infty}^{\infty} \mathcal{E}^*(\rho') \, \Delta n^2 \, \mathcal{E}_n \, dx + \sum \int_0^{\infty} d\rho g(\rho, z) \int_{-\infty}^{\infty} \mathcal{E}^*(\rho') \, \Delta n^2 \, \mathcal{E}(\rho) \, dx\right]. \tag{37}$$

No n-label on the power term P is necessary since we assume that all the normal modes are normalized to the same amount of power. The actual power carried by each mode relative to the power of the other modes is given by the C_n coefficients. Solutions of equations (34) and (35) with appropriate initial conditions provide us with exact solutions of the imperfect waveguide. It is interesting to note that this method of solution does not require the consideration of boundary conditions.

The normal modes $\mathcal{E}_n$ and $\mathcal{E}(\rho)$ were assumed to have the time and z-dependence of equation (2); this means they represent waves traveling in the positive z-direction. However, the solutions of equations (34) and (36) introduce waves traveling in positive as well as negative z-direction. To see this, let us assume that $\Delta n^2 = 0$ so that $F_n(z) = 0$. The equation

$$\frac{\partial^2 C_n}{\partial z^2} - 2i\beta_n \frac{\partial C_n}{\partial z} = 0 \tag{38}$$

has the solution

$$C_n(z) = A + Be^{2i\beta_n z} \tag{39}$$

with constant A and B. The product of A with $\mathcal{E}_n$ results in a wave traveling in the positive z-direction but the product of B exp $(2i\beta_n z)$

with $\mathcal{E}_n$ results in a wave traveling in the negative z-direction. So, even though we started out with waves traveling in the positive z-direction the expansion (33) contains partial waves traveling in positive as well as negative z-direction.

For the purpose of obtaining perturbation solutions of equations (34) and (36), an integral form of these equations is more useful. Treating equations (34) and (36) as inhomogeneous differential equations, we can immediately write the following integral equations

$$C_m = A_m + B_m e^{2i\beta_m z} + \frac{1}{2i\beta_m} \int_0^z [e^{2i\beta_m(z-\zeta)} - 1] F_m(\zeta)\, d\zeta, \tag{40}$$

$$g(\rho', z) = C(\rho') + D(\rho') e^{2i\beta' z} + \frac{1}{2i\beta'} \int_0^z [e^{2i\beta'(z-\zeta)} - 1] G(\rho', \zeta)\, d\zeta. \tag{41}$$

It is important to know which part of equations (40) and (41) is associated with waves traveling in the positive or negative z-direction. Therefore, we introduce the notation.

$$C_m = C_m^{(+)} + C_m^{(-)} \tag{42}$$

with

$$C_m^{(+)}(z) = A_m - \frac{1}{2i\beta_m} \int_0^z F_m(\zeta)\, d\zeta, \tag{43}$$

$$C_m^{(-)}(z) = \left\{ B_m + \frac{1}{2i\beta_m} \int_0^z e^{-2i\beta_m \zeta} F_m(\zeta)\, d\zeta \right\} e^{2i\beta_m z}. \tag{44}$$

The superscript $(+)$ indicates the coefficient which after substitution into equation (33) produces waves traveling in positive z-direction, while $(-)$ indicates the part which produces waves traveling in negative z-direction. A similar notation and resulting equations is used for $g(\rho', z)$; however, the corresponding equations are obvious and are therefore omitted.

The constants A_m, B_m, and so on, occurrng in equations (43), (44), and the corresponding equations for $g(\rho', z)$ must be determined from initial conditions. We always assume that the lowest order guided mode is incident on the imperfect waveguide at $z = 0$. Using the subscript 0 for this incident mode we get immediately from equation (43)

$$C_m^{(+)} = 0 \quad \text{for} \quad m \neq 0 \qquad \text{at} \quad z = 0$$

or

$$A_m = 0 \quad \text{for} \quad m \neq 0, \tag{45}$$

but

$$A_0 = 1. \tag{46}$$

We imagine that at $z = L$ the waveguide is connected to a perfect guide so that at that point there are no waves traveling in negative z-direction. This leads to the condition

$$B_m = -\frac{1}{2i\beta_m} \int_0^L e^{-2i\beta_m \zeta} F_m(\zeta) \, d\zeta \tag{47}$$

for all values of m. The power loss ΔP of the incident mode due to mode conversion is given by

$$\frac{\Delta P}{P} = \sum_{n=1}^{\infty} [| C_n^{(+)}(L) |^2 + | C_n^{(-)}(0) |^2] + \sum \int_0^{\infty} [| g^{(+)}(\rho, L) |^2 + | g^{(-)}(\rho, 0) |^2] \, d\rho. \tag{48}$$

Equation (48) states that the total power lost by mode conversion from the incident mode escapes at $z = L$ in spurious modes traveling in position z-direction and at $z = 0$ in spurious modes traveling in negative z-direction. The factor P is the normalized power factor of equations (12) and (19); it is the power incident in mode 0. Notice that because of equations (45) and (47) only the integral terms of equations (43) and (44) (taken from $z = 0$ to $z = L$) enter into equation (48).

The integral equations (43) and (44) can only be solved approximately. We perform first order perturbation theory by using $C_m(0)$ instead of $C_m(z)$ and $g(\rho, 0)$ instead of $g(\rho, z)$ in equations (35) and (37). Furthermore, we realize that $C_m^{(-)}(0)$ for all m is a quantity of first order and will therefore be neglected in equations (35) and (37). The same is true for $C_m^{(+)}(0)$ with $m \neq 0$. In the spirit of first order perturbation theory we use therefore

$$C_m = \delta_{0m} \tag{49}$$

and

$$g(\rho) = 0 \tag{50}$$

in equations (35) and (37).

The perturbation theory is feasible not only when $n_g^2 - 1 \ll 1$ but also when $n_g^2 - 1$ is arbitrarily large but the geometrical deviation of the guide walls from perfect straightness is slight. In either case we obtain from equations (35) and (37) the simple approximations

$$F_m(z) = -\frac{\beta_m k^2}{2\omega\mu P}(n_g^2 - 1)\{[f(z) - d]\,\mathcal{E}_0(d, z)\,\mathcal{E}_m^*(d, z)$$
$$- [h(z) + d]\,\mathcal{E}_0(-d, z)\,\mathcal{E}_m^*(-d, z)\}, \tag{51}$$

$$G(\rho, z) = -\frac{\beta k^2}{2\omega\mu P}(n_g^2 - 1)\{[f(z) - d]\,\mathcal{E}^*(\rho, d, z)\,\mathcal{E}_0(d, z)$$
$$- [h(z) + d]\,\mathcal{E}^*(\rho, -d, z)\,\mathcal{E}_0(-d, z)\}. \tag{52}$$

The function $f(z)$ describes the dielectric-air interface in the vicinity of $x = d$, while $h(z)$ describes it near $z = -d$. We assumed that $f(z)$ and $h(z)$ depart so little from $x = d$ and $x = -d$ that the functions $\mathcal{E}(x, z)$ could be replaced by $\mathcal{E}(\pm d, z)$.

IV. EVALUATION OF THE SPURIOUS MODE AMPLITUDES

We begin the discussion of the consequences of our scattering theory by calculating the coefficients $C_m^{(+)}$ and $g^{(+)}$. We obtain [from equations (43) and (51) with the help of equations (3a) and (13) for the even modes] the following

$$C_{me}^{(+)}(L) = \frac{Lk^2}{2i}(n_g^2 - 1)\frac{\cos \kappa_0 d \, \cos \kappa_m d}{\left[\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)\left(\beta_m d + \frac{\beta_m}{\gamma_m}\right)\right]^{\frac{1}{2}}}(\varphi_m - \psi_m). \tag{53}$$

The coefficients φ_m and ψ_m are defined by

$$\varphi_m = \frac{1}{L}\int_0^L [f(z) - d]e^{-i(\beta_0 - \beta_m)z}\, dz \tag{54}$$

and

$$\psi_m = \frac{1}{L}\int_0^L [h(z) + d]e^{-i(\beta_0 - \beta_m)z}\, dz. \tag{55}$$

These are the Fourier coefficients of the functions $f(z) - d$ and $h(z) + d$ which are expanded in a domain

$$0 \leqq z \leqq L.$$

The amplitude of the mth even mode depends on the Fourier components of the wall function whose "spatial frequency" Γ is

$$\Gamma_m = \frac{2\pi}{\Lambda_m} = \beta_0 - \beta_m\,. \tag{56}$$

The corresponding expression for the even modes of the continuous spectrum is:

$$g_e^{(+)}(\rho, L) = \frac{Lk^2}{2i(\pi)^{\frac{1}{2}}}(n_g^2 - 1)\frac{\cos \kappa_0 d \cos \sigma d[\varphi(\beta) - \psi(\beta)]}{\left[\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)\beta\left(\cos^2 \sigma d + \frac{\sigma^2}{\rho^2}\sin^2 \sigma d\right)\right]^{\frac{1}{2}}} \quad (57)$$

with [$\beta = \beta(\rho)$ see equation (16)]

$$\varphi(\beta) = \frac{1}{L}\int_0^L [f(z) - d]e^{-i(\beta_0 - \beta)z}\, dz, \quad (58)$$

$$\psi(\beta) = \frac{1}{L}\int_0^L [h(z) + d]e^{-i(\beta_0 - \beta)z}\, dz. \quad (59)$$

The corresponding expressions for the odd modes are

$$C_{n0}^{(+)}(L) = \frac{Lk^2}{2i}(n_g^2 - 1)\frac{\cos \kappa_0 d \sin \kappa_n d}{\left[\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)\left(\beta_n d + \frac{\beta_n}{\gamma_n}\right)\right]^{\frac{1}{2}}}(\varphi_n + \psi_n), \quad (60)$$

$$g_0^{(+)}(\rho, L) = \frac{Lk^2}{2i(\pi)^{\frac{1}{2}}}(n_g^2 - 1)\frac{\cos \kappa_0 d \sin \sigma d[\varphi(\beta) + \psi(\beta)]}{\left[\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)\beta\left(\sin^2 \sigma d + \frac{\sigma^2}{\rho^2}\cos^2 \sigma d\right)\right]^{\frac{1}{2}}}. \quad (61)$$

The Fourier coefficients φ and ψ are given by equations (54), (55), (58), and (59) except that β_n and β are now the propagation constants of the odd modes.

The corresponding expressions for $C^{(-)}$ and $g^{(-)}$ are obtained by replacing β_m with $-\beta_m$ and β with $-\beta$ in equations (54), (55), (56), (58), and (59).

V. SINUSOIDAL WALL DEFLECTIONS

As a specific example, let us assume that the wall imperfections have sinusoidal shape. Then

$$f(z) - d = a \sin \theta z \quad (62)$$

and

$$h(z) + d = -a \sin (\theta z + \alpha). \quad (63)$$

The phase factor α allows us to consider either a waveguide whose width varies sinusoidally

$$\alpha = 0, \quad (64)$$

or one whose direction changes sinusoidally

$$\alpha = \pi. \tag{65}$$

We obtain from equation (54) with

$$\theta = \beta_0 - \beta_m \tag{66}$$

the Fourier component

$$\varphi_m = \frac{a}{2i} \tag{67}$$

and from equation (55)

$$\psi_m = -\frac{a}{2i} e^{i\alpha}. \tag{68}$$

A term of the order $a/L \ll 1$ was omitted in equations (67) and (68). It is apparent that only one spurious mode is excited by the sinusoidal wall deflection since condition (66) can be satisfied for only one value of β_m. If condition (66) is not satisfied, φ_m and ψ_m are of the order of $a/L \ll 1$. The fractional power scattered into one spurious guided mode due to a sinusoidal wall irregularity is [from equations (48), (53), (67) and (68)]

$$\left(\frac{\Delta P}{P}\right)_{eg} \quad \frac{L^2a^2k^4}{4}(n_g^2 - 1)^2 \frac{\cos^2 \kappa_0 d \cos^2 \kappa_m d}{\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)\left(\beta_m d + \frac{\beta_m}{\gamma_m}\right)} \cos^2 \frac{\alpha}{2} \tag{69}$$

for even modes or [from equations (48) and (60)]

$$\left(\frac{\Delta P}{P}\right)_{og} \quad \frac{L^2a^2k^2}{4}(n_g^2 - 1)^2 \frac{\cos^2 \kappa_0 d \sin^2 \kappa_m d}{\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)\left(\beta_m d + \frac{\beta_m}{\gamma_m}\right)} \sin^2 \frac{\alpha}{2} \tag{70}$$

for odd modes. However only one even or one odd mode can be excited by one particular sinusoidal wall deviation since it is impossible to satisfy the "resonance" condition (66) for more than one mode simultaneously.

If $\alpha = 0$, that is if the width of the guide changes sinusoidally, only even modes can be excited while sinusoidal deviations from straightness ($\alpha = \pi$) couple the even fundamental mode only to odd spurious modes. It must also be noticed that for a long period length

$$\Lambda = \frac{2\pi}{\theta} \tag{71}$$

equation (66) can be satisfied only for forward scattering modes. To couple to backward scattering modes, the period length D must be approximately equal to half the wavelength of the guided modes. The fact that only one spurious mode is coupled to the incident mode by sinusoidal wall imperfections (it can be shown that the coupling to the continuous mode spectrum is also weak if one guided mode can couple strongly) allows us to give a much better description of the coupling process.

Since the mode amplitudes C_m can change only slowly in the distance of one wavelength we can neglect the second derivative of C_m in equation (34). Labeling the incident mode 0 and the one coupled spurious mode 1 we can write the equation system (34) in the following form

$$\frac{\partial C_0}{\partial z} = -\kappa_{01} C_1 , \tag{72}$$

$$\frac{\partial C_1}{\partial z} = \kappa_{01}^* C_0 , \tag{73}$$

with

$$\kappa_{01} = \frac{k^2 a}{2} (n_g^2 - 1) \frac{\cos \kappa_0 d \cos \kappa_1 d}{\left[\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)\left(\beta_1 d + \frac{\beta_1}{\gamma_1}\right)\right]^{\frac{1}{2}}} \exp\left(i\frac{\alpha}{2}\right) \cos\frac{\alpha}{2}. \tag{74}$$

The coupling coefficient κ_{01} of equations (74) holds for coupling from an even mode 0 to an even mode 1. The case of coupling from an even mode 0 to an odd mode 1 can be treated similarly. In fact, except for an unimportant phase factor, we get it from $(1/L)[(\Delta P/P)_{og}]^{\frac{1}{2}}$ of equarion (70). In equation (72) we omitted a term with C_0 on the right-hand side, and similarly a term with C_1 was omitted in equation (73). These terms would be multiplied by sinusoidally varying functions and would describe the local change of phase velocity as the guide dimensions vary. These terms give no contribution if we use an average over C_0 and C_1 over the mechanical period length of equation (71).

Assuming $C_0 = 1$, $C_1 = 0$ at $z = 0$ the equation system (72) and (73) has the solution

$$C_0 = \cos |\kappa_{01}| z, \tag{75}$$

$$C_1 = \left(\frac{\kappa_{01}^*}{\kappa_{01}}\right)^{\frac{1}{2}} \sin |\kappa_{01}| z. \tag{76}$$

Total exchange of energy is possible between the two coupled modes.

The distance D over which all the energy is exchanged is given by

$$D = \frac{\pi}{2 \mid \kappa_{01} \mid}. \tag{77}$$

Finally, we need the power loss to the modes of the continuous spectrum. From equations (48), (57), (61), (62), and (63) we obtain

$$\left(\frac{\Delta P}{P}\right)_c = \frac{a^2 k^4}{\pi} (n_g^2 - 1)^2 \frac{\cos^2 \kappa_0 d}{\beta_0 d + \frac{\beta_0}{\gamma_0}}$$

$$\cdot \int_0^\infty \left[\frac{\cos^2 \sigma d \cos^2 \frac{\alpha}{2}}{\beta\left(\cos^2 \sigma d + \frac{\sigma^2}{\rho^2} \sin^2 \sigma d\right)} + \frac{\sin^2 \sigma d \sin^2 \frac{\alpha}{2}}{\beta\left(\sin^2 \sigma d + \frac{\sigma^2}{\rho^2} \cos^2 \sigma d\right)} \right]$$

$$\cdot \frac{\sin^2 [\theta - (\beta_0 - \beta)] \frac{L}{2}}{[\theta - (\beta_0 - \beta)]^2} \, d\rho. \tag{78}$$

The integration can be performed easily if one realizes that for large values of L only a very narrow region in the β range near $\beta - \beta_0 - \theta$ contributes to the integral. We consider all functions in the integrand as constant in this very narrow range and take them out of the integral with the exception of

$$\left\{ \frac{\sin [\theta - (\beta_0 - \beta)] \frac{L}{2}}{\theta - (\beta_0 - \beta)} \right\}^2 .$$

This remaining integral can easily be performed if we use equation (16) to obtain

$$d\rho = -\frac{\beta}{\rho} \, d\beta.$$

Following this procedure yields

$$\left(\frac{\Delta P}{P}\right)_c = \frac{L a^2 k^4}{2} (n_g^2 - 1)^2 \frac{\cos^2 \kappa_0 d}{\beta_0 d + \frac{\beta_0}{\gamma_0}}$$

$$\cdot \left\{ \frac{\rho \cos^2 \sigma d \cos^2 \frac{\alpha}{2}}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} + \frac{\rho \sin^2 \sigma d \sin^2 \frac{\alpha}{2}}{\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d} \right\}. \tag{79}$$

The parameters σ and ρ follow from equations (15) and (16) with

$$\beta = \beta_0 - \theta. \tag{80}$$

Equation (79) holds only for $\beta < k$; we get $\Delta P/P = 0$ for $\beta > k$. The most interesting aspect of equation (79) is its linear dependence on L. The scattering loss due to the modes of the continuous spectrum acts like a true loss process. By contrast, the corresponding equation (69) for the loss to guided modes is proportional to L^2 because coupling to a guided mode does not result in loss of energy but results in energy exchange between the two coupled modes. Energy loss to one of the guided modes is followed by energy gain when the energy exchange has reversed itself.

VI. NUMERICAL EXAMPLES FOR SINUSOIDAL IMPERFECTIONS

A few numerical examples resulting from equations (74) and (77) are listed in Table I. Two different values of the index of refraction n_g have been assumed, and for each value of the index three different values of $kd = 2\pi(d/\lambda_0)$ have been chosen so that one, two, or three guided modes can exist simultaneously. The mode with β_0 is the lowest

TABLE I—NUMERICAL EXAMPLES FOR SINUSOIDAL IMPERFECTIONS

n_g	kd	$\beta_0 d$	$\beta_1 d$	$\beta_2 d$	$\frac{aD}{d^2}$	Remarks
1.5	1.3	1.729	—	—	—	Single mode operation
	1.8	2.495	1.916	—	6.98	$0 - 1$ coupling $\alpha = \pi$
	3.0	4.336	3.831	3.051	6.17	$0 - 1$ coupling $\alpha = \pi$
					5.52	$0 - 2$ coupling $\alpha = 0$
1.01	8.0	8.041	—	—	—	Single mode operation
	15.0	15.113	15.022	—	42.54	$0 - 1$ coupling $\alpha = \pi$
	23.0	23.199	23.112	23.002	36.28	$0 - 1$ coupling $\alpha = \pi$
					43.69	$0 - 2$ coupling $\alpha = 0$

order even guided mode which is assumed to be incident on the waveguide with sinusoidal wall imperfections. This mode couples to the first odd mode with β_1 or the next even mode with β_2. The values for the normalized, dimensionless quantitity $(aD)/d^2$ [a = amplitude of the sinusoidal wall deviation according to equation (62) and (63), d = half width of the guide, and D = energy exchange length] have been obtained with the assumption that equation (66) is satisfied for the two modes which are coupled together. Coupling from mode 0 to mode 1 is considered only for the case of sinusoidal straightness deviations of the waveguide ($\alpha = \pi$) while coupling between even modes 0 to 2 is considered only for sinusoidal changes of the thickness of the waveguide ($\alpha = 0$). It is immediately apparent from Table I that the energy exchange length D is shorter for a guide with larger values of the refractive index.

To obtain a feeling for the numbers involved in this mode coupling phenomenon, let us assume that $n_g = 1.5$ and that the free space wavelength is $\lambda_0 = 1\mu$. The value of $kd = 1.8$ corresponds to $d = 0.286\mu$. To achieve total exchange of energy between modes 0 and 1 in $D = 1$ cm requires the extremely small amplitude $a = 5.72\ 10^{-5}\mu$ or $a = 0.572$ Å!† The length of the mechanical period in this example is $\Lambda = 3.1\mu$.

Next, let us assume that the index of refraction is $n_g = 1.01$. Using again, $\lambda_0 = 1\mu$, we obtain from $kd = 15.0$ the value $d = 2.39\mu$ for the half width of the waveguide. Requiring again, $D = 1$ cm, we find $a =$ 243 Å.

We can look at this problem in a different way. It is unlikely that any optical waveguide has a strictly sinusoidal deviation from perfect straightness. In fact, the numbers just presented show that it would be impossible to produce such a waveguide intentionally. However, we have seen [equation (53)] that the mode conversion between two guided modes is produced by a Fourier component of the actual deviation function. It is therefore not necessary to have a strictly sinusoidal straightness deviation. Any arbitrary deviation from straightness can be decomposed into a Fourier series and the Fourier component at the mechanical frequency which satisfies equation (66) is responsible for the coupling. In the more general case of arbitrary straightness deviations, there can be no complete exchange of energy between any two modes since power loss to other guided modes and the continuous

† A mechanical period of a fraction of an Angstrom is somewhat unphysical due to the granular nature of matter. However, this result can be restated to say that complete power conversion occurs in 0.1 mm if the amplitude is $a = 57.2$ Å.

spectrum of modes compete with each other since all of them are coupled simultaneously.

We can now ask the question: What amplitude of the mechanical straightness deviation is required to transfer 10 percent of power from mode 0 to mode 1 in a distance of $L = 1$ cm? Again, we use the previous examples. From equation (76) [or directly from equations (53) and (77)] we obtain

$$\frac{\Delta P}{P} = |\kappa_{01}|^2 L^2 = \frac{\pi^2}{4} \frac{L^2}{D^2}.$$

For the first example we obtain with $n_g = 1.5$, $\Delta P/P = 0.1$, $d = 0.286\mu$, and $aD/d^2 = 6.98$ the value $a = 0.115$ Å.† This result shows that if the Fourier component of the mechanical straightness deviation with a period length of 3.1μ is $a = 0.12$ Å (measured over a distance of 1 cm) the power loss caused by mode conversion to the first odd mode is 10 percent.

For the second example, we use again $n_g = 1.01$, $\Delta P/P = 0.1$, $d = 2.39\mu$, and $aD/d^2 = 42.54$ and obtain $a = 48.8$ Å. The important Fourier component in this case has a period of $\Lambda = 135\mu$. The power loss to the modes of the continuous spectrum caused by a sinusoidal change in thickness of the waveguide (which is very similar to its effect as a straightness deviation) can be calculated from equation (79) with $\alpha = 0$.

Let us consider only one case, $n_g = 1.01$, $kd = 15$, $\Lambda/d = 25$. For these values we obtain from equation (79)

$$\frac{d^3}{a^2 L} \frac{\Delta P}{P} = 4.6 \times 10^{-2}.$$

Assuming again $\Delta P/P = 0.1$ for a guide length $L = 1$ cm, we obtain with $d = 2.39\mu$

$$a = 5.46 \times 10^{-2}\mu = 546 \text{ Å}.$$

This number can be compared to the value $a = 48.8$ Å which gave 10 percent loss by conversion to one guided mode. However, for a meaningful comparison, we must remember that all the Fourier components of a Fourier expansion of the guide imperfections scatter power into the modes of the continuous spectrum. The total loss would have to be obtained by integrating the scattering loss over the spectral dis-

† Again it is more reasonable to restate this example to say that 10 percent loss occurs over a distance of $L = 0.1$ mm if $a = 12$ Å.

tribution of the Fourier components of the mechanical Fourier spectrum. Instead of doing this integration we use a different approach in Section VII.

VII. STATISTICAL TREATMENT OF WALL IMPERFECTIONS†

Equation (48) gives the relative loss of a guided mode caused by a definite (deterministic) distortion of the boundary of a dielectric waveguide. A quantity that may be even more interesting is the average of equation (48) taken over an ensemble of statistically identical systems.

For simplicity, let us assume that one wall of the waveguide is perfect while the other is randomly distorted. If both walls are randomly distorted, with no correlation between the distortions on opposite walls the loss value doubles compared to the case of only one wall being distorted. If the distortions on opposite sides of the waveguide are perfectly correlated the amount of loss is at most increased four times. So to simplify the discussion we assume

$$h(z) + d = 0. \tag{81}$$

In order to be able to calculate $\langle \Delta P/P \rangle_{av}$, we must evaluate

$$\langle | \varphi_m |^2 \rangle_{av} = \frac{1}{L^2} \int_0^L dz \int_0^L dz' R(z - z') e^{-i(\beta_0 - \beta_m)(z - z')} \tag{82}$$

We assumed that the correlation function

$$R(z - z') = \langle [f(z) - d][f(z') - d] \rangle_{av} \tag{83}$$

depends only on the difference between the coordinates z and z' but not on their individual values.

A change of integration variables allows us to write

$$\langle | \varphi_m |^2 \rangle_{av} = \frac{2}{L^2} \int_0^L (L - u) R(u) \cos (\beta_0 - \beta_m) u \, du. \tag{84}$$

To obtain equation (84) we made use of the fact that $R(u)$ is an even function.

The particular form of $R(u)$ depends on the statistics of the wall imperfections. However, all correlation functions have two features in common. They all have their maximum value at $u = 0$ and decrease to zero as $u \to \infty$. If $R(u)$ would not become 0 as $u \to \infty$ there would be a

† An excellent statistical treatment of random coupling effects in metallic waveguides can be found in Ref. 7.

systematic distortion of the waveguide boundary instead of the assumed random behavior. To get an idea of what one might expect, we assume the following form for the correlation function

$$R(u) = A^2 \exp\left(-\frac{|u|}{B}\right). \tag{85}$$

A is the rms deviation of the wall from perfect straightness and B is the correlation length. Using equation (85) we obtain from equation (84)

$$\langle | \varphi_m |^2 \rangle_{av} = \frac{2A^2}{L} \frac{1}{(\beta_0 - \beta_m)^2 + \frac{1}{B^2}} \left\{ \frac{1}{B} + \frac{(\beta_0 - \beta_m)^2 - \frac{1}{B^2}}{L\left((\beta_0 - \beta_m)^2 + \frac{1}{B^2}\right)} \right\} \tag{86}$$

where we neglected terms with exp $(-L/B)$ assuming that L/B is sufficiently large. In fact if

$$L \gg B, \tag{87}$$

equation (86) can be simplified further:

$$\langle | \varphi_m |^2 \rangle_{av} = \frac{2A^2}{BL} \frac{1}{(\beta_0 - \beta_m)^2 + \frac{1}{B^2}}. \tag{88}$$

Using equation (88) we obtain, from equation (53) for the ensemble average of the square magnitudes of the even guided modes,

$$\langle | C_{me} |^2 \rangle_{av} = \frac{A^2 k^4 L}{2B} (n_g^2 - 1)^2 \cdot \frac{\cos^2 \kappa_0 d \cos^2 \kappa_m d}{\left((\beta_0 - \beta_m)^2 + \frac{1}{B^2}\right)\left(\beta_0 d + \frac{\beta_0}{\gamma_0}\right)\left(\beta_m d + \frac{\beta_m}{\gamma_m}\right)}. \tag{89}$$

The corresponding expression for the odd modes is very similar except that $\cos^2 \kappa_m d$ is replaced by $\sin^2 \kappa_m d$ and β_m, κ_m, and γ_m are the parameters of the odd modes.

The total loss caused by coupling to all guided modes supported by the dielectric waveguide is the sum over all $\langle | C_m |^2 \rangle_{av}$ for even as well as odd modes traveling in positive ($\beta_m = +| \beta_m |$) as well as negative ($\beta_m = -| \beta_m |$) z-direction. It is noteworthy that equation (89) is proportional to L and not to L^2. The conversion to spurious guided modes by random imperfections appears as a true loss to the incident mode.

The losses due to the modes of the continuous spectrum are obtained from equations (48), (57), (61), (81) and (88) (with $\beta_m = \beta$):

$$\left\langle \frac{\Delta P}{P} \right\rangle_{av} = \frac{A^2 k^4 L}{2\pi B} (n_g^2 - 1)^2 \int_{-k}^{k} \left[\frac{\rho \cos^2 \kappa_0 d}{\left((\beta_0 - \beta)^2 + \frac{1}{B^2} \right) \left(\beta_0 d + \frac{\beta_0}{\gamma_0} \right)} \right.$$

$$\left. \cdot \left(\frac{\cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 d} + \frac{\sin^2 \sigma d}{\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d} \right) \right] d\beta. \quad (90)$$

The relation between β, σ, and ρ is given by equations (15) and (16) while β_0, κ_0, and γ_0 are related by equations (9) and (10) and their value is obtained by solving equation (11). The integral in equation (90) is extended over β from $-k$ to k, the range of real values of the propagation constant (in z-direction) of the modes of the continuous spectrum. Equation (90) thus includes the losses due to forward as well as backward scattered radiation. The radiation modes with imaginary values of β can carry power away from the waveguide only strictly perpendicular to its axis. This power loss, if any, is not included in equation (90).

VIII. NUMERICAL RESULTS FOR THE STATISTICAL CASE

Figures 4 through 9 show numerical evaluations of equations (89) and (90). These figures can be grouped into two classes. Figures 4 through 6 are drawn for a dielectric waveguide whose index of refraction is $n_g = 1.01$. Figures 7 through 9 apply to a waveguide with $n_g = 1.5$. Within each of these two classes, the kd value was chosen to allow for three different cases. Figures 4 and 7 apply to waveguides which can support only the lowest order guided mode. In this case there is power lost only to the modes of the continuous spectrum. Figures 5 and 8 apply to waveguides supporting two guided modes and Figs. 6 and 9 apply to waveguides supporting three guided modes. Each figure shows the normalized loss caused by scattering into modes of the continuous spectrum as solid lines and the loss to the possible guided modes as dotted lines. Also shown are the ratios of backward to forward scattered power as solid lines for the modes of the continuum and as dotted lined for the guided modes. The total power lost to the lowest order guided modes is the sum of the losses to the continuum and the spurious guided modes.

Several remarkable features of these loss curves are worthy of a comment. The losses caused by the modes of the continuum as well as by the guided modes peak at certain values of the correlation length B. The location of these peaks are different, however, for the continuum and guided modes.

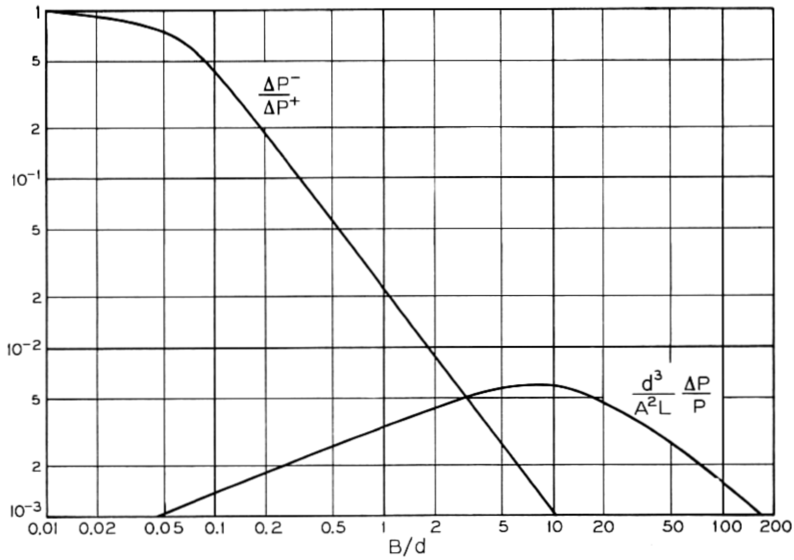

Fig. 4 — Normalized radiation loss (d^3/A^2L) $(\Delta P/P)$ and ratio of backward to forward scattered power $\Delta P^-/\Delta P^+$ as functions of the normalized correlation length B/d for $n_g = 1.01$ and $kd = 8.0$. Single guided mode operation (d = half width of waveguide, A = rms deviation of one waveguide wall, L = Length of waveguide section, n_g = index of refraction of waveguide, k = free space propagation constant).

The losses to the guided modes increase with increasing number of guided modes supported by the waveguide. However, the losses caused by the continuum of modes also increase as an increasing number of guided modes can be supported. This increase is less rapid, however, as one might expect because of the dependence of equation (90) on the fourth power of k. The fourth power dependence on frequency (or inverse wavelength) is typical for Rayleigh scattering by small particles, and it is not surprising that we encounter it here.

Finally, it is apparent from the curves showing the ratio of back-scattered to forward scattered power that forward scattering is predominant for large values of the correlation length. The ratio of $\Delta P^-/\Delta P^+$ levels off for large values of B. In some of the curves the leveling of the $\Delta P^-/\Delta P^+$ curves occurs out of the diagram but it is a common feature of all the curves. For small values of the correlation length there is as much scattering in the forward as in the backward direction.

For many practical applications, a waveguide supporting only one

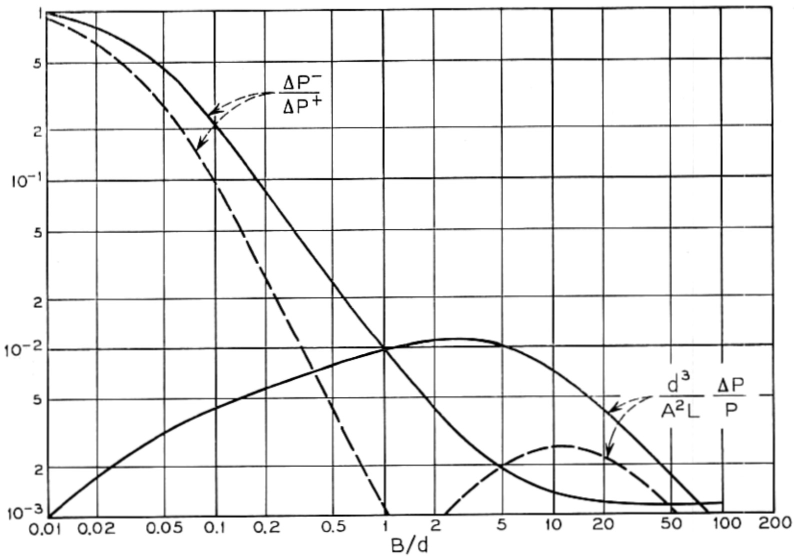

Fig. 5 — Normalized power loss and ratio of foreward to backward scattered power for radiation (solid curves) and spurious guided modes (dashed curves). Two guided modes ($n_g = 1.01$, $kd = 15$).

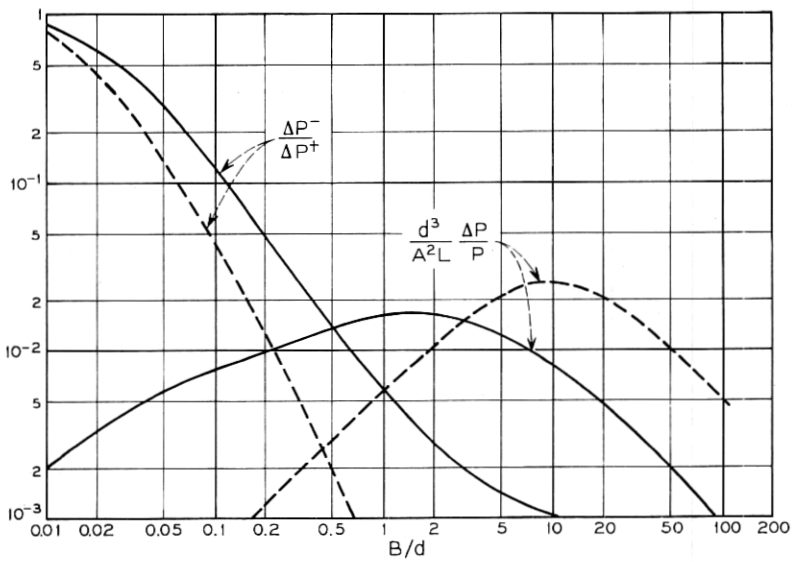

Fig. 6 — Similar to Fig. 5. Three guided modes ($n_g = 1.01$, $kd = 23$). — — — — two guided mode loss; ——— continuum loss.

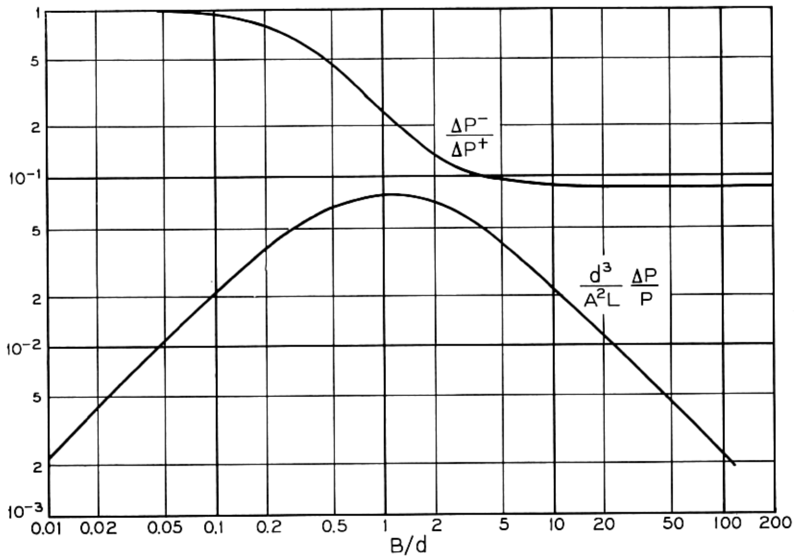

Fig. 7 — Similar to Fig. 4. One guided mode ($n_g = 1.5$, $kd = 1.3$).

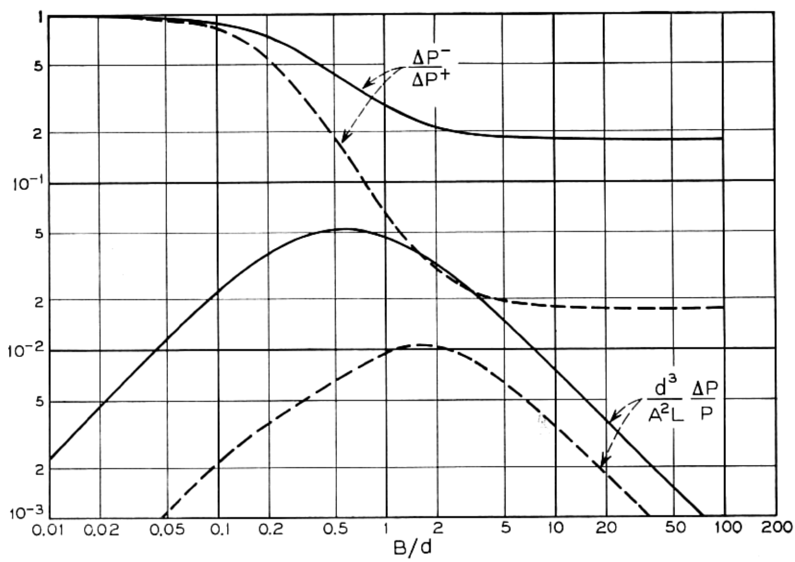

Fig. 8 — Similar to Fig. 5. Two guided modes ($n_g = 1.5$, $kd = 1.8$). — — — — one guided mode loss; ——— continuum loss.

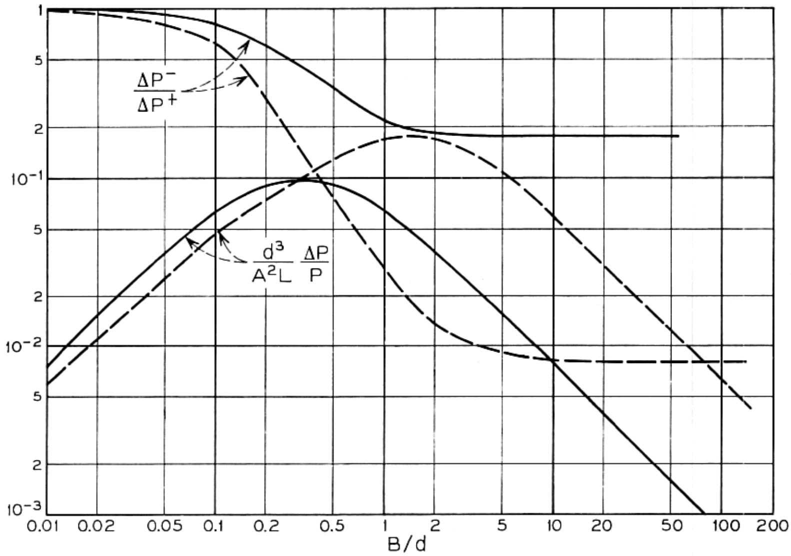

Fig. 9 — Similar to Fig. 5. Three guided modes ($n_g = 1.5$, $kd = 3$). — — — — two guided mode loss; continuum loss.

guided mode may be of most interest. Let us assume $\lambda_0 = 1\mu$. Figure 4, holding for $kd = 8.0$ and $n_g = 1.01$, applies to a waveguide whose half width is $d = 1.27\mu$. Taking the worst possible case of $B/d = 9$ or $B = 11.4\mu$, we find from Fig. 4

$$\frac{d^3}{A^2L}\frac{\Delta P}{P} = 6 \times 10^{-3}.$$

If we want to know how much rms deviation A of one wall of the guide would be required to cause a 10 percent loss ($\Delta P/P = 0.1$) in one centimeter of waveguide ($L = 1$ cm) we find $A = 5.85 \times 10^{-2}\mu = 585$ Å. The ratio of A over d gives an idea of the relative tolerance requirements:

$$\frac{A}{d} = 4.6 \times 10^{-2} = 4.6\%.$$

If the waveguide were to conform to the conditions of Fig. 6, we would have for $\lambda_0 = 1\mu$ a half width $d = 3.66\mu$. The losses caused by the two spurious modes are of the same order of magnitude as the radiation losses caused by the continuous spectrum. For $B/d = 10$ or $B = 36.6\mu$ we get a total loss of

$$\frac{d^3}{A^2L}\frac{\Delta P}{P} = 3.4 \times 10^{-2}.$$

To cause $\Delta P/P = 0.1$ for $L = 1$ cm requires that

$$A = 1.2 \times 10^{-1}\mu \quad \text{or} \quad \frac{A}{d} = 3.28\%.$$

The relative tolerance requirements are, therefore, approximately the same in both examples.

As a last example let us use Fig. 9 corresponding ($\lambda_0 = 1\mu$) to a waveguide with $n_g = 1.5$ and a half width $d = 0.477\mu$. For $B/d = 1.3$ or $B = 6.2\mu$ we find for the total loss

$$\frac{d^3}{A^2L}\frac{\Delta P}{P} = 2.3 \times 10^{-1}.$$

We get $\Delta P/P = 0.1$ with $L = 1$ cm for

$$A = 2.18 \times 10^{-3}\mu = 21.8\ \text{Å} \quad \text{or} \quad \frac{A}{d} = 0.457\%.$$

The perturbation theory, strictly speaking, holds only for small values of $\Delta P/P$. However, it is reasonable to expect that the power scattered into the radiation modes escapes sufficiently rapidly so that no appreciable amount of power reconversion from the radiation field to the guided mode occurs. The incremental power loss, $\Delta P/P = -\alpha L$, is therefore the same for any section of the guide so that we obtain the total scattering loss into the continuum of radiation modes $P = P_0e^{-\alpha L}$. We may now ask how much rms deviation is required to cause a radiation loss of 10 dB/km or $\alpha = 2.3$ km^{-1} $= 2.3 \times 10^{-5}$ cm^{-1}. Using $B/d = 10$, corresponding to the top of the loss curve of Fig. 4, we obtain the equation

$$\frac{d^3}{A^2} \times 2.3 \times 10^{-5} = 6 \times 10^{-3}$$

so that ($\lambda = 1\mu$, $n_g = 1.01$, $kd = 8.0$, $d = 1.27 \times 10^{-4}$ cm)

$$\frac{A}{d} = 6.98 \times 10^{-4} \quad \text{or} \quad A = 8.86 \times 10^{-8}\ \text{cm} = 8.86\ \text{Å}.$$

This figure dramatizes the stringent tolerance requirements of dielectric waveguides for long distance optical communications. In fact, such tolerances seem impossible to obtain. One can only hope that the correlation length can be kept far from the worst possible value of $B/d = 10$

(in this example) so that these extremely stringent tolerance requirements might be eased.

IX. CONCLUSION

We have analyzed the losses suffered by the lowest order symmetric mode propagating on a dielectric slab waveguide caused by imperfections of the waveguide boundaries. The analysis was simplified by assuming that there is no change in either the dielectric slab or the guided and unguided fields in one direction parallel to the slab. This assumption causes all our conclusions to be optimistic since variation of the slab in this direction can only cause additional losses. However, we expect that the results of this analysis give at least the correct order of magnitude of the actual scattering losses.

The statistical analysis was limited to a study of the effects which an exponential correlation function might have on the waveguide losses. The actual form of the correlation function may be quite different from this assumed exponential shape.† Conclusions regarding loss predictions are further hampered by a lack of knowledge of the expected correlation length.

However, our analysis does lead one to conclude that scattering losses suffered by optical fibers or other dielectric waveguide structures may be very serious. Deviations of the waveguide wall in the order of a few percent can cause a power loss of 10 percent or 0.46 dB/cm if the wall imperfection can be described by an exponential correlation function with a correlation length to guide half width ratio of approximately $B/d = 10$. An rms deviation of $A = 9$ Å causes a radiation loss of 10 dB/km if the free space wavelength is $\lambda_0 = 1\mu$ and the guide has an index of refraction of $n_g = 1.01$ (with vacuum on the outside). The width of the slab in this last example is $2d = 2.54\mu$.

The mode coupling and radiation loss theory has been experimentally confirmed at microwave frequencies. A report on these measurements is given in Ref. 8.

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† Several other correlation functions have been tried and it was found that the results are insensitive to the particular choice of the function for values of B/d less than the value corresponding to the loss peak. In particular, the maximum loss value and the position of this loss peak were the same for different correlation functions. However, the loss values for B/d larger than the value corresponding to the maximum of the curve are very strongly dependent on the choice of the correlation function.

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