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On Communication of Analog Data from a Bounded Source Space

By AARON D. WYNER and JACOB ZIV

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We consider the problem of the transmission of discrete-time analog data with a variety of fidelity criteria. The outputs of the analog source are assumed to belong to a bounded set. Bounds on the minimum achievable average distortion for memoryless sources are derived both for the case where the coding delay is infinite (an extension of the Shannon Theory) and also for some cases where the coding delay is finite. Several examples are given, for which the upper and lower bounds coincide.

Further, we discuss the case where the assumption of the existence of a probabilistic model for the source is dropped. We adopt as our fidelity criterion the supremum over all possible source-output n-sequences **x**, *of the conditional expectation of the distortion given* **x** *("guaranteed distortion"). The Shannon Theory is not directly applicable in determining the minimum guaranteed distortion. We do obtain results for two important cases. Some generalizations and applications are also discussed.*

I. INTRODUCTION

In this paper we are concerned with communication of discrete-time analog data over a communication channel with a variety of fidelity criteria. The central assumption about the analog source is that its outputs belong to a bounded set, typically the interval $[-A/2, A/2]$. We begin with a rough outline of our results, leaving the precise formulation and statement to Section II. Proofs are found in Section III.

Suppose that we have a data source which emits a sequence of symbols $x_1, x_2, \cdots \in \mathfrak{X}$ (an arbitrary set) at a rate of ρ_S per second. This sequence is fed into an "encoder" which assigns to each successive block of n source symbols, say $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, a channel input of duration $n/\rho_S = T$ seconds. At the receiving end of the channel, the T-second output is transformed by a "decoder" into an n-sequence, say $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$, which is delivered to the destination. The "distortion" between the source output sequence $\mathbf{x}$ and the received sequence $\hat{\mathbf{x}}$ is defined as $d^{(n)}(\mathbf{x}, \hat{\mathbf{x}}) = n^{-1} \sum_{k=1}^{n} d(x_k, \hat{x}_k)$, where $d(x, \hat{x}) \geqq 0$ is an arbitrary function.

The classical problem is that of a "memoryless" source, where successive source outputs are statistically independent with identical probability distribution. In this case it is meaningful to let the system performance criterion (fidelity criterion) be the statistical expectation of the distortion $d^{(n)}(\mathbf{x}, \hat{\mathbf{x}})$. A quantity of interest is $\bar{d}^*(T)$, the smallest attainable value of the fidelity criterion when the coding delay is T seconds. The Shannon Theory gives the asymptotic behavior of $\bar{d}^*(T)$ as $T \to \infty$. In many cases this limit is difficult to evaluate analytically. Theorem 1 (in Section 2.2) considers the case where the source output set $\mathfrak{X} = [-A/2, A/2]$, and the function $d(x, \hat{x})$ depends only on the difference $\hat{x} - x$. This theorem gives a lower bound on $\text{limit}_{T\to\infty} \bar{d}^*(T)$. The examples which follow this theorem illustrate the applicability and utility of the bound.

There are two cases in which we are particularly interested. In the first, the source set $\mathfrak{X} = \{0, 1, \cdots, K-1\}$ with a uniform distribution, and $d(x, \hat{x}) = 0$ or 1 according as $x = \hat{x}$ or $x \neq \hat{x}$. Thus the fidelity criterion is the error-rate. For this case let $\bar{d}^*(T) = P_e(K, T)$. In the second case, $\mathfrak{X} = [-A/2, A/2]$ with a uniform distribution, and $d(x, \hat{x}) =$ 0 or 1 according as $|x - \hat{x}| < \delta$ or $|x - \hat{x}| \geqq \delta$ (where $\delta > 0$). In this case let $\bar{d}^*(T) = Q(T, A, \delta)$. It turns out that P_e and Q are intimately related. In fact it is a consequence of Theorem 2 (Section 2.2) that if $A/(2\delta) = K_0$, an integer, then $Q(T, A, \delta) = P_e(T, K_0)$. This result is valid for all values of the delay parameter T. From this result it can be deduced that the optimal encoder for the analog source $\mathfrak{X} = [-A/2, A/2]$ is a "uniform" quantizer followed by an optimal "digital" encoder. This is the only known case for which analog-to-digital conversion is known to be optimal for finite T for the transmission of analog data from a memoryless source.

We now drop the assumption of a memoryless source. In fact we do not even assume that there is a probabilistic model for the source. Instead of the expectation of the distortion, we adopt as our fidelity criterion, the supremum, over all possible source output n-sequences $\mathbf{x}$, of the conditional expectation of the distortion given $\mathbf{x}$. We call this criterion the "guaranteed distortion". Let $\hat{d}^*(T)$ be the minimum attainable guaranteed distortion for a system with delay parameter T. The Shannon Theory is not directly applicable in determining $\hat{d}^*(T)$. We do obtain results for the two interesting cases discussed below.

In the first, $\mathfrak{X} = \{0, 1, \cdots, K - 1\}$ and $d(x, \hat{x}) = 0$ or 1, respectively, when $x = \hat{x}$ or $x \neq \hat{x}$. For this case let $\hat{d}^*(T) = \hat{P}_e(T, K)$. It is a consequence of Theorem 3 (Section 2.3) that $\text{limit}_{T\to\infty} \hat{P}_e(T, K) = \text{limit}_{T\to\infty} P_e(T, K)$, which is known from the Shannon Theory.

In the second case, $\mathfrak{X} = [-A/2, A/2]$ and $d(x, \hat{x}) = 0$ or 1, respectively, when $| x - \hat{x} | < \delta$ or $| x - \hat{x} | \geqq \delta$. For this case, let $\hat{d}^*(T) = \hat{Q}(T, A, \delta)$. Theorem 4 (Section 2.3) relates $\hat{P}_e$ and $\hat{Q}$ by

$$\hat{Q}(T, A, \delta) = \hat{P}_e(T, M),$$

where M is the unique integer satisfying $(M - 1) \leqq A/(2\delta) < M$. Here too, we can deduce the optimality of analog-to-digital conversion. Theorem 4 is generalized by Theorem 5 (Section 2.4) to apply to an arbitrary set $\mathfrak{X}$ with a distance-like measure defined on it (replacing $| x - \hat{x} |$).

In Section 2.5, we give some applications of the above results. In particular we obtain some results for the distortion $d(x, \hat{x}) = | x - \hat{x} |^e$.

In order to state our results completely and precisely, it is unfortunately necessary to give a rather large collection of definitions and to introduce a large number of symbols. In order to ease the reader's burden somewhat, we have included a glossary of symbols in the appendix.

II. STATEMENT OF THE PROBLEM AND PRINCIPAL RESULTS

In Section 2.1 we define a "channel" (and its "capacity") in a very general and abstract way. We do this because the nature of the channel does not figure explicitly in our results (except for the channel capacity), and we want our results to apply as broadly as possible. In Section 2.2 we describe the communication system which we shall consider, and state our results for the case of a "memoryless" information source. The remainder of the results follows in Sections 2.3–2.5.

2.1 *Channel and Channel Capacity*

A *channel* is defined as follows. For every $T > 0$ we have a set $\mathfrak{W}_T$ of "allowable" inputs and a set $\mathfrak{Z}_T$ of possible outputs. Every T

seconds some $w \in \mathfrak{W}_T$ is transmitted through the channel, and the channel output z is a member of $\mathfrak{Z}_T$. The output is related to the input $w \in \mathfrak{W}_T$ by a probability measure μ_w on the set $\mathfrak{Z}_T$. Thus given that $w \in \mathfrak{W}_T$ is transmitted, the probability that $z \in B$ [where B is a (measurable) subset of $\mathfrak{Z}_T$] is $\mu_w(B)$. For example $\mathfrak{W}_T$ and $\mathfrak{Z}_T$ may be the set of binary sequences of length $[T]^{-}$†. The measure μ_w is then a discrete conditional probability distribution. Another example is the case where $\mathfrak{W}_T$ and $\mathfrak{Z}_T$ are sets of real valued functions defined on the interval $[0, T]$, and the members of $\mathfrak{W}_T$ have "energy" not exceeding PT.

With T specified, a *block code* with parameter N is a set of N pairs $\{(w_i, B_i)\}_{i=1}^N$, where $w_i \in \mathfrak{W}_T$ are called *code words* and the collection of B_i is a set of disjoint (measurable) subsets of $\mathfrak{Z}_T$ called *decoding sets*. If code word $w_i (1 \leqq i \leqq N)$ is transmitted, the resulting error probability is

$$\lambda_i = \Pr \{z \notin B_i \mid w_i \text{ is transmitted}\} = 1 - \mu_{w_i}(B_i). \tag{1}$$

The *word error probability* for the code is

$$\lambda = \max_{1 \leqq i \leqq N} \lambda_i . \tag{2}$$

Let $\lambda^*(T, N)$ be the smallest attainable word error probability for a code with parameters T and N. The *channel capacity* C is defined as the supremum of those numbers $R \geqq 0$, for which

$$\lambda^*(T, [e^{RT}]^-) \to 0, \quad \text{as} \quad T \to \infty .$$

Let us define the *average word error probability* by

$$\bar{\lambda} = \frac{1}{N} \sum_{i=1}^{N} \lambda_i . \tag{3}$$

Thus $\bar{\lambda}$ is the resulting average error probability which results when each of the N code words are equally likely to be transmitted. Let us define $\bar{\lambda}^*(T, N)$ as the smallest attainable value of $\bar{\lambda}$ for a code with parameters T and N. Since $\bar{\lambda} \leqq \lambda$ for any code, it follows from the above definition of channel capacity that for any $R < C$,

$$\bar{\lambda}^*(T, [e^{RT}]^-) \to 0, \quad \text{as} \quad T \to \infty .$$

Further it is known that for a large class of channels including the memoryless gaussian channel and discrete memoryless channels,[1]

$$\bar{\lambda}^*(T, [e^{CT}]^-) \to \tfrac{1}{2}, \quad \text{as} \quad T \to \infty . \tag{4}$$

[It is also true that for many of these same channels if $R > C$,

† Throughout this paper we denote by $[x]^-$ and $[x]^+$ the largest integer $\leq x$ and the smallest integer $\geq x$ respectively $(0 \leq x < \infty)$.

$\bar{\lambda}^*(T, [e^{RT}]^-)$ tends to 1 as $T \to \infty$, but we do not need this fact here.]

Let us remark here that for a large class of channels (including "memoryless" channels and "finite state channels"), the capacity C is known to be the supremum of a quantity called the "information". In fact this equivalence is the essence of the Fundamental Theorem of Information Theory. It will not be necessary, however, to explore this equivalence further.

2.2 *Memoryless Source and Communication With a Fidelity Criterion*

Consider the communication system of Figure 1. The output of the *source* is a sequence of random variables $X_1, X_2, \cdots$ from an arbitrary subset $\mathfrak{X}$ of Euclidean p-space. Assume that these random variables are statistically independent and identically distributed with probability density function $P_S(x)$, $x \in \mathfrak{X}$. If we allow impulses in the density function, then the X_k can be discrete random variables. Say that the source outputs appear at a rate of ρ_S per second. The *encoder* waits T seconds (called the "delay") during which time $n = \rho_S T$ symbols, say $X_1, X_2, \cdots, X_n \in \mathfrak{X}$, have appeared at its input. (Assume that $\rho_S T$ is an integer.) Denote the T-second output of the source by the random n-vector $\mathbf{X} = (X_1, X_2, \cdots, X_n) \in \mathfrak{X}^n$.

The *channel* is defined as above (Section 2.1), so that during the T seconds which it takes for the n-vector $\mathbf{X}$ to appear, the channel can process an input belonging to the channel input set $\mathfrak{W}_T$. It is the task of the encoder to assign to each possible source output n-vector $\mathbf{X} = \mathbf{x}$, a channel input $f_E(\mathbf{x}) \in \mathfrak{W}_T$. The channel output is a member Z of the channel output set $\mathfrak{Z}_T$, and it is the task of the *decoder* to assign to each possible $Z = z$ an n-vector $\hat{\mathbf{X}} = f_D(z) \in \mathfrak{X}^n$. Note that the source and channel statistics define a joint probability density on the random n-vectors $\mathbf{X}$ and $\hat{\mathbf{X}}$.

Now ideally we would like $\mathbf{X} \equiv \hat{\mathbf{X}}$. But this is most often not possible due to imperfections (for example, noise) in the channel. Thus we define a *fidelity criterion* which we use as a measure of the reliability of the system. Suppose we are given a non-negative *distortion* function $d(x, \hat{x})$ defined on $\mathfrak{X} \times \mathfrak{X}$. Typical choices of the distortion function are $d(x, \hat{x}) = |x - \hat{x}|^s (s > 0)$ when $\mathfrak{X}$ is a subset of the reals (that is, the dimensionality $p = 1$), or the "Hamming" distance

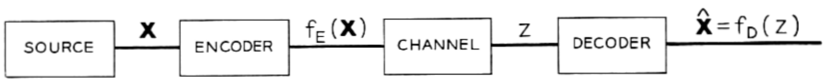

Fig. 1 — Communication system.

$$d(x, \hat{x}) = d_H(x, \hat{x}) = \begin{cases} 0, & x = \hat{x}, \\ 1, & x \neq \hat{x}, \end{cases} \tag{5}$$

where $\mathfrak{X}$ is a discrete (that is, countable) set.

The distortion between the n-vectors, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$ is

$$d^{(n)}(\mathbf{x}, \hat{\mathbf{x}}) = n^{-1} \sum_1^n d(x_k, \hat{x}_k).$$

Our system performance (fidelity) criterion, which we seek to minimize, is

$$\bar{d} = Ed^{(n)}(\mathbf{X}, \hat{\mathbf{X}}),$$

where E denotes expectation (with respect to the joint probability distribution of $\mathbf{X}$ and $\hat{\mathbf{X}}$). For a given delay T, which corresponds to $n = \rho_S T$, let $\bar{d}^*(T)$ denote the infimum (with respect to all encoder-decoder pairs) of the attainable values of $\bar{d}$ (for given ρ_S and source-channel statistics). Although we usually do not know $\bar{d}^*(T)$ exactly, we do know its asymptotic behavior as $T \to \infty$. We proceed as follows.

For $0 \leqq \beta \leqq \infty$, define $\mathfrak{M}(\beta)$ as the set of probability density functions $p(x, \hat{x})$ defined on $\mathfrak{X} \times \mathfrak{X}$ which satisfy

(*i*) $\int_{\mathfrak{X}} p(x, \hat{x})\, d\hat{x} = P_S(x)$, the source output probability density function,
(*ii*) $\int_{\mathfrak{X}} \int_{\mathfrak{X}} d(x, \hat{x}) p(x, \hat{x})\, dx\, d\hat{x} \leqq \beta$.

The *information* corresponding to the density $p(x, \hat{x}) \in \mathfrak{M}(\beta)$ is defined as

$$I\{p(x, \hat{x})\} = \int_{\mathfrak{X}} \int_{\mathfrak{X}} p(x, \hat{x}) \log \frac{p(x, \hat{x})}{P_S(x) p_2(\hat{x})}\, dx\, d\hat{x}, \tag{6}$$

where $p_2(\hat{x}) = \int_{\mathfrak{X}} p(x, \hat{x})\, dx$. It is easy to show that $I \geqq 0$ with equality if and only if $p(x, \hat{x}) = P_s(x) p_2(\hat{x})$. Finally define the *equivalent rate* of the source

$$R_{eq}(\beta) = \inf_{p(x, \hat{x}) \in \mathfrak{M}(\beta)} I\{p(x, \hat{x})\}. \tag{7}$$

$R_{eq}(\beta)$ is usually called the "rate-distortion function". Note that $R_{eq}(\beta)$ depends only on β and $P_s(x)$.

Let us now return to the quantity $\bar{d}^*(T)$. Shannon's well known theorems tell us the following.[2] For a given communication system (as in Fig. 1),

$$\begin{aligned} &(i)\ \bar{d}^*(T) \geqq \bar{d}_0\,, \qquad \text{for all } T, \\ &(ii)\ \bar{d}^*(T) \to \bar{d}_0\,, \qquad \text{as} \quad T \to \infty, \end{aligned} \tag{8}$$

where $\bar{d}_0$ is the smallest solution of

$$\rho_s R_{\text{eq}}(\bar{d}_0) \leqq C,$$

and C is the capacity of the channel.

Some intuitive insight into the meaning of Shannon's theorem can be gained by thinking of $\rho_s R_{\text{eq}}(\beta)$ as the equivalent rate in nats per second of the source (when reproduced with distortion β). It is reasonable then to suppose that the minimum attainable distortion $\bar{d}_0$ is that distortion for which the source rate is just equal to the channel capacity C.

There are two well-known cases for which $R_{\text{eq}}(\beta)$ is known explicitly. The first is the case where $\mathfrak{X}$ = the reals, $P_S(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2\sigma^2)$, and $d(x, \hat{x}) = (x - \hat{x})^2$. In this case, $R_{\text{eq}}(\beta) = \frac{1}{2} \log \sigma^2/\beta^2$, so that $\bar{d}_0 = \sigma^2 \exp(-2C/\rho_S)$.

The second case (which is important in the sequel) is $\mathfrak{X} = \{0, 1, 2, \cdots, K-1\}$ $(K = 2, 3, \cdots)$, $P_S(x) = \sum_{k=0}^{K-1} (1/K)\delta(x - k)$[$\delta(x)$ is the unit impulse], and $d(x, \hat{x})$ is given by equation (5). In other words, the source output is a sequence of independent random variables, each equally distributed on the K-ary alphabet $\{0, 1, \cdots, K-1\}$. The quantity $\bar{d}$ is the average fraction of symbols received in error, and is often called the "error-rate". In this case, we write $\bar{d}^*(T) = P_e(T, K)$, where the dependence of P_e on K as well as T is indicated explicitly. For this case it is known that[2]

$$R_{\text{eq}}(\beta) = \begin{cases} \log K - h(\beta) - \beta \log (K-1), & \beta \leqq \dfrac{K-1}{K}, \\[2ex] 0, & \beta \geqq \dfrac{K-1}{K}, \end{cases} \tag{9a}$$

where

$$h(\beta) = -\beta \log \beta - (1-\beta) \log (1-\beta), \quad (0 \leqq \beta \leqq 1). \tag{9b}$$

Shannon's theorem, equation (8), tells us that

$$P_e(T, K) \to \gamma(K, \rho_S, C), \qquad T \to \infty, \tag{10a}$$

where $\gamma(K, \rho_S, C)$ is the smallest solution of

$$\rho_s R_{\text{eq}}(\gamma) \leqq C, \tag{10b}$$

and C is the channel capacity. A graph of $\gamma(K, \rho_S, C)$ versus C/S_S for

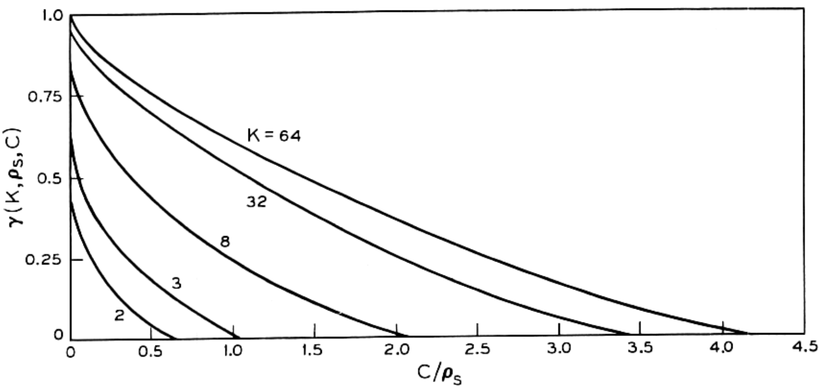

Fig. 2 — $\gamma(K, \rho_s, C)$ versus C/ρ_s (K-a parameter).

various values of K is given in Fig. 2. Notice that $\gamma(K, \rho_S, C)$ decreases from $(K - 1)/K$ to zero as C/ρ_S increases from zero to log K.

Let us also remark that the quantity $P_e(T, K)$ is related to $\bar{\lambda}^*(T, M)$ (the smallest attainable average *word* error probability). In fact it is easy to show that

$$\frac{1}{n} \bar{\lambda}^*(T, K^n) \leqq P_e(T, K) \leqq \bar{\lambda}^*(T, K^n) \tag{11}$$

where $n = \rho_S T$ (assumed to be an integer).

Now, in the general case [arbitrary $P_S(x)$ and $d(x, \hat{x})$], it is usually not possible to obtain a closed form expression for $R_{\mathrm{eq}}(\beta)$. Theorem 1, which is stated below, gives a useful bound on $R_{\mathrm{eq}}(\beta)$ for the case where $P_S(x)$ is a density and $\mathfrak{X}$ is a bounded set. This theorem is an extension of a result of Shannon.[2] The proof is given in Section 3.1.

Let $\mathfrak{X}$ be the interval $[-A/2, A/2]$, where $A(0 < A < \infty)$ is arbitrary. Let the source outputs X have density $P_S(x)$, and let $d(x, \hat{x}) = r(x - \hat{x})$, where $r(u)$ satisfies

$$\begin{aligned} &(i)\ r(u) = r(-u), \\ &(ii)\ r(u) \geqq 0, \text{ with equality at } u = 0, \\ &(iii)\ r(u) \text{ is continuous at } u = 0. \end{aligned} \tag{12}$$

Then it can be shown (see Ref. 3, Appendix A) that for $0 < \beta \leqq 1/A \int_{-A/2}^{A/2} r(u)\, du$, there exists a unique $\lambda_0(\beta)$ which satisfies

$$\int_{-A/2}^{A/2} r(u) e^{-\lambda_0(\beta) r(u)}\, du = \beta \int_{-A/2}^{A/2} e^{-\lambda_0(\beta) r(u)}\, du. \tag{13}$$

Define the probability density $g_\beta(x)$ on $\mathfrak{X}$ by

$$g_\beta(x) = \left[\int_{-A/2}^{A/2} e^{-\lambda_0(\beta) r(x)}\, dx\right]^{-1} e^{-\lambda_0(\beta) r(x)}, \tag{14a}$$

[note that $\int r(x) g_\beta(x)\, dx = \beta$], and let

$$H_1(\beta) = -\int_{-A/2}^{A/2} g_\beta(x) \log g_\beta(x)\, dx, \tag{14b}$$

be the corresponding "entropy".

For $A = \infty$, equation (13) has a solution in many cases. In particular, when $r(u) = |u|^s$ $(s > 0)$, equation (10) has a solution for $0 < \beta < \infty$. Thus $g_\beta(x)$ and $H_1(\beta)$ are meaningful for $A = \infty$ also.

We now state the lower bound on $R_{eq}(\beta)$ as Theorem 1.

Theorem 1: For the source defined above, for $0 < \beta \leqq A^{-1} \int_{-A/2}^{A/2} r(u)\, du$,

$$R_{eq}(\beta) \geqq H_S - H_1(\beta), \tag{15a}$$

where

$$H_S = -\int_{-A/2}^{A/2} P_S(x) \log P_S(x)\, dx \tag{15b}$$

is the entropy of the source density $P_S(x)$, *and* $H_1(\beta)$ *is defined in equations* (13) *and* (14). *Inequality* (15a) *also holds for* $A = \infty$, *when* $r(u) = |u|^s$ $(s > 0)$.

Examples:

(*i*) Say $\mathfrak{X}$ = the reals, and $d(x, \hat{x}) = r(x - \hat{x}) = |x - \hat{x}|^s$, where $s > 0$ is arbitrary. Theorem 1 is applicable with $A = \infty$. Solving equation (13), yields $\lambda_0(\beta) = (s\beta)^{-1}$ and

$$g_\beta(x) = \frac{s^{(s-1)/s}}{2\beta^{1/s}\Gamma\left(\frac{1}{s}\right)} \exp\left[-|x|^s/(s\beta)\right],$$

so that

$$R_{eq}(\beta) \geqq H_S - H_1(\beta), \tag{16a}$$

where

$$H_1(\beta) = \frac{1}{s} \log\left[\frac{2^s e \Gamma^s\left(\frac{1}{s}\right)\beta}{s^{s-1}}\right], \tag{16b}$$

and H_S is given by equation (15b).

(*ii*) Quadratic Distortion: Let $\mathfrak{X}$ = the reals, and $d(x, \hat{x}) = (x - \hat{x})^2$. Then from example (*i*), with $s = 2$,

$$R_{\text{eq}}(\beta) \geqq H_S - \tfrac{1}{2} \log 2\pi e\beta. \tag{17a}$$

Further Shannon[4] has given the following upper bound to $R_{\text{eq}}(\beta)$:

$$R_{\text{eq}}(\beta) \leqq \tfrac{1}{2} \log \frac{\sigma^2}{\beta}, \qquad \beta \leqq \sigma^2, \tag{17b}$$

where $\sigma^2 = \int x^2 P_S(x)\, dx$. Note that when $P_S(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-x^2/2)$, the upper and lower bounds of inequalities (17) coincide for $\beta \leqq \sigma^2$. [Since $R_{\text{eq}}(\beta)$ is non-increasing, $R_{\text{eq}}(\beta) = 0$ for $\beta > \sigma^2$.]

Another case of interest is $\mathfrak{X} = [-A/2, A/2](A < \infty)$, $P_S(x) = A^{-1}$, and $d(x, \hat{x}) = (x - \hat{x})^2$. In this case Theorem 1 (applied for finite A) provides a lower bound on $R_{\text{eq}}(\beta)$ which is tighter than that of inequality (17a) and can be evaluated numerically. An upper bound can be found by computing $I[p_0(x, \hat{x})]$, where $p_0(x, \hat{x})$, a joint probability density for X and $\hat{X}$, is defined by the following: The variate X has density $P_S(x)$. The variate $\hat{X} = \alpha(X + Y)$, where the Y is a Gaussian variate, independent of X, with

$$EY = 0 \quad \text{and} \quad EY^2 = \beta A^2/(A^2 - 12\beta),$$

and

$$\alpha = (A^2 - 12\beta)/A^2.$$

Note that $E(X - \hat{X})^2 = \beta$. The information $I[p_0(x, \hat{x})]$ corresponding to $p_0(x, \hat{\mathbf{x}})$ can also be evaluated numerically and is an upper bound to $R_{\text{eq}}(\beta)$. Figure 3 is a graph of these bounds on $R_{\text{eq}}(\beta)$, and also of $\bar{d}_0$, the solution of $\rho_S R_{\text{eq}}(\bar{d}_0) = C$.

(*iii*) Say $\mathfrak{X} = [-A/2, A/2]$. Let $P_S(x) = A^{-1}$ and $d(x, \hat{x}) = r(x - \hat{x})$ where, in addition to satisfying conditions (12), $r(u)$ satisfies

$$r(u) = r(v) \quad \text{if} \quad u \equiv v \pmod{A}. \tag{18}$$

[If, for example, $A = 2\pi$ and $\mathfrak{X}$ represents an angle, then equation (18) must hold.] For $r(u)$ satisfying condition (18), the bound (15a) on $R_{\text{eq}}(\beta)$ of Theorem 1 holds with equality, namely, $R_{\text{eq}} = H_S - H_1(\beta)$. (Section 3.1)

(*iv*) Threshold Distortion: Let $\mathfrak{X} = [-A/2, A/2]$ and let $d(x, \hat{x})$ be the "threshold" distortion defined by

$$d(x, \hat{x}) = d_\delta(x, \hat{x}) = r_\delta(x - \hat{x}), \tag{19a}$$

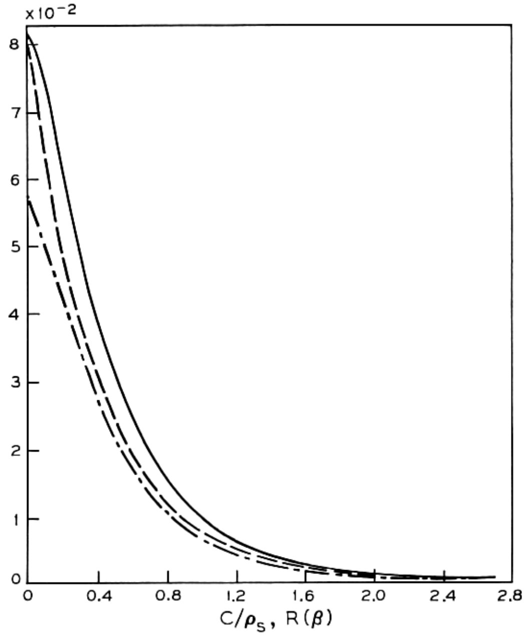

Fig. 3 — Bounds on β/A^2 versus $R_{eq}(\beta)$ or d_0/A^2 versus C/ρ_S. (*i*) ——— upper bound; (*ii*) — — — lower bound (Theorem 1); (*iii*) — - — - — lower bound (17a).

where

$$r_\delta(u) = \begin{cases} 1, & |u| \geqq \delta, \\ 0, & |u| < \delta. \end{cases} \tag{19b}$$

In this case, the bound (15) of Theorem 1 is

$$R_{eq}(\beta) \geqq H_S - h(\beta) - \log 2\delta - \beta \log (A/2\delta - 1), \tag{20}$$

where $h(\beta)$ is defined in equation (9b). There is a case where inequality (20) is satisfied with equality, namely $P_S(x) = A^{-1}$ and $A/(2\delta) = K_0 = 1, 2, \cdots$. For this case, we show in Section 3.1 that

$$R_{eq}(\beta) = \begin{cases} \log K_0 - h(\beta) - \beta \log (K_0 - 1), & 0 \leqq \beta \leqq \dfrac{K_0 - 1}{K_0}, \\ 0, & \beta \geqq \dfrac{K_0 - 1}{K_0}. \end{cases} \tag{21}$$

Notice the striking similarity of equations (21) and (9) for the discrete K-ary source. We will have more to say about this later.

When $A/(2\delta)$ is not an integer, we show (in Section 3.1) the right member of (21) is an upper bound to $R_{eq}(\beta)$ with K_0 replaced by $[A/2\delta]^+ = K_+$. Thus with inequality (20),

$$\log \left[\frac{A}{2\delta}\right] - h(\beta) - \beta \log \left[\frac{A}{2\delta} - 1\right] \leqq R_{eq}(\beta)$$

$$\leqq \log K_+ - h(\beta) - \beta \log (K_+ - 1). \qquad (22)$$

A Result for Finite T for the Threshold Distortion is as follows. Let $\mathfrak{X} = [-A/2, A/2]$, $P_S(x) = A^{-1}$, and $d(x, \hat{x}) = d_\delta(x, \hat{x})$, the threshold distortion given in equations (19), as in example (*iv*) above. In the system of Fig. 1, let $\bar{d}^*(T) = Q(T, A, \delta)$, where the dependence on A and δ is indicated explicitly. The results in example (*iv*) [equation (21)] and equation (10) imply that for $A/2\delta = K_0$, $\lim_{T\to\infty} Q(T, A, \delta) = \lim_{T\to\infty} P_e(T, K_0) = \gamma(K_0, \rho_S, C)$. This correspondence between Q and P_e is extended to finite T in the Theorem 2 (proved in Section 3.2.).

Theorem 2: *Let* $K_+ = [A/2\delta]^+$, $K = [A/2\delta]^-$. *For all* T,

$$P_e(T, K_-) \leqq Q(T, A, \delta) \leqq P_e(T, K_+). \qquad (23)$$

The quantities P_e *and* Q *are defined, of course, for the same channel and source output rate* ρ_S.

A case of particular interest is $A/2\delta = K_0$, an integer, so that $K_+ = K_- = K_0$ and Theorem 2 yields

$$P_e(T, K_0) = Q(T, A, \delta), \quad \text{all} \quad T. \qquad (24)$$

For this case we deduce from equation (24) that (for all T) the optimal encoder for the analog source is a K_0-level "uniform" quantizer with quantization levels $[(2i - K_0 - 1)\delta]_{i=1}^{K_0}$ followed by an optimal "digital" encoder. This is the only known case for which analog-to-digital conversion is known to be optimal for $T < \infty$ for the transmission of analog data.

2.3 *Case Where The Source Has No Statistics*

Suppose that the source output is, as in Section 2.2, a sequence of symbols from the source alphabet $\mathfrak{X}$, which appear at a rate of ρ_S per second. However, in this case, as distinct from above, we assume that there is no known statistical model for the source. Say that, as in Section 2.2, the encoder waits T seconds during which time $n = \rho_S T$ source symbols $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathfrak{X}^n$ have appeared. Again, as above, the encoder output is $f_E(\mathbf{x}) \in \mathcal{W}_T$, the channel output is $Z \in \mathfrak{Z}_T$ and the decoder output is $\hat{\mathbf{X}} = f_D(Z) \in \mathfrak{X}^n$. The encoder-decoder pair and the channel statistics induce a probability density for $\hat{\mathbf{X}}$ on $\mathfrak{X}^n$ which depends on $\mathbf{x}$ (the source output). Denote this density by $f(\hat{\mathbf{x}} \mid \mathbf{x})$. Assuming, as in Section 2.3, that a non-negative distortion function $d(x, \hat{x})$ on $\mathfrak{X} \times \mathfrak{X}$ is given, then the average distortion when the source output is $\mathbf{x}$ is

$$\bar{d}(\mathbf{x}) = \int_{\mathfrak{X}^n} \left[\frac{1}{n} \sum_{k=1}^{n} d(x_k, \hat{x}_k) \right] f(\hat{\mathbf{x}} \mid \mathbf{x}) \, d\hat{\mathbf{x}}, \tag{25}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$. Since we cannot take a meaningful statistical average over $\mathbf{x}$, we adopt as our fidelity criterion, the "guaranteed" distortion

$$\hat{d} = \sup_{\mathbf{x} \in \mathfrak{X}^n} \bar{d}(\mathbf{x}). \tag{26}$$

Let $\hat{d}^*(T)$ be the smallest attainable value of $\hat{d}$ for a given delay T (which corresponds to $n = \rho_S T$).

For the special case where $\mathfrak{X} = \{0, 1, \cdots, K - 1\}$ and $d(x, \hat{x}) = d_H(x, \hat{x})$ [given by equations (5)] let $\hat{d}^*(T) = \hat{P}_e(T, K)$ where the dependence on K is made explicit. Consider $P_e(T, K)$ (the average error-rate in Section 2.2). Clearly,

$$\hat{P}_e(T, K) \geqq P_e(T, K).$$

The following theorem [taken together with equations (10)] shows that as $T \to \infty$, P_e and $\hat{P}_e$ are asymptotically equal. The proof is in Section 3.4.

Theorem 3: *For the communication system described above with K-ary source alphabet, source output rate* ρ_S, *and channel capacity* C,

$$\lim_{T \to \infty} \hat{P}_e(T, K) = \gamma(K, \rho_S, C), \tag{27}$$

where $\gamma(K, \rho_S, C)$ *is given by inequality* (10b).

A second important special case is $\mathfrak{X} = [-A/2, A/2]$, and $d(x, \hat{x}) = d_\delta(x, \hat{x})$, the threshold distortion given by equations (19). In this case let $\hat{d}^*(T) = \hat{Q}(T, \delta, A)$. The quantity $\hat{Q}$ can be related to $\hat{P}_e$, and Theorem 4 (proved in Section 3.3) is analogous to Theorem 2, though somewhat sharper.

Theorem 4: *For* $0 < \delta \leqq A$, *let* $M(\delta)$ *be the integer satisfying*

$$M - 1 \leqq \frac{A}{(2\delta)} < M. \tag{28a}$$

Then for all T,

$$\hat{Q}(T, A, \delta) = \hat{P}_e[T, M(\delta)]. \tag{28b}$$

The quantities $\hat{P}_e$ and $\hat{Q}$ are defined, of course, for the same channel and source output rate ρ_S.

In constrast to Theorem 2, this theorem asserts the equality of corresponding values of $\hat{Q}$ and $\hat{P}_e$ for *all* values of $A/(2\delta)$. Also as in Theorem 2, this theorem implies that the optimal encoder for the source $\mathfrak{X} = [-A/2, A/2]$, with $d = d_\delta$ [with a fidelity criterion as in equation (26)] is a uniform quantizer [with $M(\delta)$ levels] followed by an optimal digital encoder (see part (*i*) of the proof of Theorem 4).

Theorems 3 and 4 can be combined to obtain the following.

Corollary: *For* $0 < \delta \leqq A$, *let* $M(\delta)$ *be as in Theorem* 4. *Then*

$$\lim_{T \to \infty} \hat{Q}(T, A, \delta) = \gamma[M(\delta), \rho_S, C], \tag{29}$$

where γ *is given by inequality* (10b).

2.4 *Generalization to Arbitrary Source Alphabets*

In this section we consider the case where the source alphabet $\mathfrak{X}$ is an arbitrary space with an arbitrary metric or metric-like function defined on it. We then give a generalization of Theorem 4. First we give some preliminary definitions.

Let $\mathfrak{X}$ be a set and let $\rho_0(x, \hat{x})$ be real-valued function defined on $\mathfrak{X} \times \mathfrak{X}$ with the properties

$$(i) \quad \rho_0(x, \hat{x}) = \rho_0(\hat{x}, x) \tag{30a}$$

$$(ii) \quad \rho_0(x, \hat{x}) \geqq 0 \text{ with equality when } x = \hat{x}. \tag{30b}$$

If in addition $\rho_0(x, \hat{x})$ satisfies

$$(iii) \quad \rho_0(x, \hat{x}) \leqq \rho_0(x, y) + \rho_0(y, \hat{x}), \tag{30c}$$

then $\rho_0(x, \hat{x})$ is a metric; but we will not require inequality (30c) to hold. For $x \in \mathfrak{X}$ and $\Delta > 0$, let $S_x(\Delta) = \{\hat{x} \in \mathfrak{X} : \rho_0(x, \hat{x}) < \Delta\}$ be the (open) sphere of radius Δ about x.

A set $A \subseteq \mathfrak{X}$ is called a "Δ-covering" (of $\mathfrak{X}$) if $\bigcup_{x \in A} S_x(\Delta)$ contains $\mathfrak{X}$, and A is called a "Δ-packing" (of $\mathfrak{X}$) if $S_x(\Delta) \cap S_{\hat{x}}(\Delta)$ is empty for all $x, \hat{x} \in A$, $x \neq \hat{x}$. Let $M_C(\Delta)$ be the minimum number of points which can constitute a Δ-covering of $\mathfrak{X}$, and let $M_P(\Delta)$ be the maximum number of points which can constitute a Δ-packing. These quantities are related by the following lemma (proved in Section 3.4).

Lemma 1: Let $\eta = sup_{x,y,z\epsilon\mathfrak{X}}\, \rho_0(x, y)/[\rho_0(x, z) + \rho_0(z, y)]$. *Then for* $\Delta > 0$,

$$M_C(2\eta\Delta) \leqq M_P(\Delta). \tag{31}$$

In particular, if ρ_0 is a metric, $\eta \leqq 1$. Inequality (31) is of course meaningful only if $\eta < \infty$.

Now consider the communication system discussed in Section 2.3 with an arbitrary source space $\mathfrak{X}$†. Let ρ_0 satisfy expressions (30a) and (30b), and define the "threshold" distortion $d_\delta(x, \hat{x})$ by

$$d_\delta(x, \hat{x}) = \begin{cases} 1, & \rho_0(x, \hat{x}) \geqq \delta, \\ 0, & \rho_0(x, \hat{x}) < \delta. \end{cases} \tag{32}$$

Let $\hat{d}$ be the guaranteed distortion defined by equation (26) with the distortion $d(x, \hat{x}) = d_\delta(x, \hat{x})$ [given by equation (32)]. Finally, let $\hat{G}(T, \delta)$ be the smallest attainable value of $\hat{d}$ for a system with delay T. (The dependence of $\hat{G}$ on δ is made explicit.) Of course $\hat{G}(T, \delta)$ also depends on ρ_S as well as the channel characteristics. The special case treated in Section 2.3 is $\mathfrak{X} = [-A/2, A/2]$, $\rho_0(x, \hat{x}) = | x - \hat{x} |$. In this case $\hat{G}(T, \delta) = \hat{Q}(T, A, \delta)$.

The following is a generalization of Theorem 4 and is proved in Section 3.3.

Theorem 5: Let $M_C(\Delta)$ *and* $M_P(\Delta)$ *be as defined above for the source alphabet* $\mathfrak{X}$ *[with a* $\rho_0(x, \hat{x})$*]. Then* G(T, δ) *satisfies*

$$\hat{P}_e[T, M_P(\delta)] \leqq \hat{G}(T, \delta) \leqq \hat{P}_e[T, M_C(\delta)], \tag{33}$$

where $\hat{P}_e$ *is defined in Section* 2.3. *Note that* $\hat{P}_e$ *and* $\hat{G}$ *are defined for the same channel and source output rate* ρ_S.

Theorem 5 reduces to Theorem 4 on noting that for $\mathfrak{X} = [-A/2, A/2]$ and $\rho_0(x, \hat{x}) = | x - \hat{x} |$,

$$M_P(\delta) = M_C(\delta) = M(\delta), \tag{34}$$

where $M(\delta)$ is defined by inequality (28a). Let us remark that although $M_P \equiv M_C$, the maximum δ-packing is not in general identical to their minimum δ-covering. For example, when $\delta = A/4$, $M(\delta) = 3$, and the maximum δ-packing is unique, namely

$$\left\{-\frac{A}{2}, 0, \frac{A}{2}\right\},$$

† To be precise, we must assume that the space $\mathfrak{X}$ and the encoder and decoder functions are measurable.

which is not a δ-covering. There are many δ-coverings, for example

$$\left\{-\frac{A}{3}, 0, \frac{A}{3}\right\}.$$

2.5 *Some Applications*

2.5.1 *Rate at Which $Q(T, A, \delta)$ Approaches its Limit*

Consider again the source with $\mathfrak{X} = [-A/2, A/2]$, $P_S(x) = A^{-1}$, and distortion $d(x, \hat{x}) = d_\delta(x, \hat{x})$ [defined by expressions (19)]. Suppose further that $A/(2\delta) = K_0$, an integer and that the channel capacity $C = \rho_S \log K_0$. In this case $\gamma(K_0, \rho_S, C) = 0$ [see expressions (9) and (10)], so that from expressions (24) and (10a)

$$\lim_{T\to\infty} Q(T, A, \delta) = 0.$$

We will now obtain a lower bound on the rate at which this limit is approached. From the first inequality in inequality (11), using $n = \rho_S T$,

$$P_e(T, K) \geqq \frac{1}{\rho_S T} \bar{\lambda}^*(T, e^{CT}). \tag{35}$$

For those channels for which expression (4) holds, the right member of inequality of (35) $\sim (2\rho_S T)^{-1}$. Combining expressions (24) and (35) we have that

$$Q(T, A, \delta) \geqq \frac{1}{2\rho_S T}[1 + \xi(T)], \tag{36}$$

where $\xi(T) \to 0$ as $T \to \infty$. Thus for the class of channels for which expression (4) holds and these parameter values, $Q(T, A, \delta)$, approaches its limit no faster than T^{-1}. Determination of the similar bounds on the rate of approach of Q to its limit for other parameters is an open question.

2.5.2 *The* sth-*Mean Distortion*

Consider the case where $\mathfrak{X} = [-A/2, A/2]$, and the distortion $d(x, x) = |x - \hat{x}|^s$ $(s > 0)$. When $P_S(x) = A^{-1}$, let the smallest attainable average distortion $\bar{d}^*(T) \triangleq \bar{\epsilon}^s(T)$. For the case of no source statistics (as in Section 2.3), let the smallest attainable guaranteed distortion $\hat{d}^*(T) \triangleq \hat{\epsilon}^s(T)$. We establish some properties of $\bar{\epsilon}^s$ and $\hat{\epsilon}^s$ below.

For any random variable Y (such that $|Y| \leqq A$), and any δ_1, $\delta_2 (0 \leqq \delta_1, \delta_2 \leqq A)$,

$$\delta_1^s \Pr\{|Y| \geqq \delta_1\} \leqq E|Y|^s \leqq \delta_2^s \Pr\{|Y| < \delta_2\} + A^s \Pr\{|Y| \geqq \delta_2\}. \tag{37}$$

It follows from inequality (37) that for arbitrary δ_1, δ_2 ($0 \leqq \delta_1, \delta_2 \leqq A$),

$$\delta_1^s Q(T, A, \delta_1) \leqq \bar{\epsilon}^s(T) \leqq \delta_2^s[1 - Q(T, A, \delta_2)] + A^s Q(T, A, \delta_2), \tag{38a}$$

and

$$\delta_1^s \hat{Q}(T, A, \delta_1) \leqq \hat{\epsilon}^s(T) \leqq \delta_2^s[1 - \hat{Q}(T, A, \delta_2)] + A^s \hat{Q}(T, A, \delta_2), \tag{38b}$$

where Q and $\hat{Q}$ are defined in Sections 2.2 and 2.3 respectively. Applications of Theorems 2 and 4 (and $Q, \hat{Q} \geqq 0$) yields

$$\delta_1^s P_e(T, K_-) \leqq \bar{\epsilon}^s(T) \leqq \delta_2^s + A^s P_e(T, K_+), \tag{39a}$$

and

$$\delta_1^s \hat{P}_e[T, M(\delta_1)] \leqq \hat{\epsilon}^s(T) \leqq \delta_2^s + A^s \hat{P}_e[T, M(\delta_2)], \tag{39b}$$

where $K_+ = [A/2\delta_2]^+$, $K_- = [A/2\delta_1]^-$, and $M(\delta)$ is defined by inequality (28a). Thus $\bar{\epsilon}^s$ and $\hat{\epsilon}^s$ too are related to the digital error rates P_e and $\hat{P}_e$. Of course, δ_1 and δ_2 may be chosen to yield the tightest bounds.

Examples

(*i*) Since we know the asymptotic value of P_e and $\hat{P}_e$ as $T \to \infty$, we can apply inequalities (39) to obtain estimates of the limiting values $\bar{\epsilon}_0^s = \text{limit}_{T\to\infty}\, \bar{\epsilon}^s(T)$ and $\hat{\epsilon}_0^s = \text{limit}_{T\to\infty}\hat{\epsilon}^s(T)$. For example, when the channel capacity C is large, setting $A/2\delta_1 = \exp[(C/\rho_S)(1 + \Delta_1)]$ and $A/2\delta_2 = \exp[(C/\rho_S)(1 - \Delta_1)](\Delta_1, \Delta_2 > 0)$, yields, after some computation,

$$\bar{\epsilon}_0^s = \exp\left\{-\frac{sC}{\rho_S}[1 + \xi_1(C)]\right\}, \tag{40a}$$

$$\hat{\epsilon}_0^s = \exp\left\{-\frac{sC}{\rho_S}[1 + \xi_2(C)]\right\}, \tag{40b}$$

where $\xi_1, \xi_2 \to 0$ as $C \to \infty$. Thus for large C, $\bar{\epsilon}_0^s$ and $\bar{\epsilon}_1^s$ decay roughly exponentially in C.

Let us remark that parts of inequalities (40) are obtainable by other means. Specifically, $\bar{\epsilon}_0^s \geqq K_1(s)\exp[-sC/\rho_S]$ follows from inequality (16). Further, $\epsilon_0^s \leqq \exp[-(sC/\rho_S)(1 + \xi_1)]$ and $\hat{\epsilon}_0^s \leqq \exp[-(sC/\rho_S) \cdot (1 + \xi_2)]$ can be deduced from the work of Panter and Dite on quantization.[5] Finally the bound $\hat{\epsilon}_0^s \geqq \exp[-(sC/\rho_S)(1 + \xi_2)]$ is new.

(*ii*) In this example, we apply the first inequality of (39a) to show the possible gains (with the sth mean criterion) obtainable by using coding in a particular (though quite typical) case.

Suppose that the channel is the additive white Gaussian noise channel with average power P_0, one-sided spectral density N_0, with no bandwidth constraint.[6] To begin with, suppose $T = 1/\rho_S$, so that $n = 1$ and there is no "coding", that is, each T-second channel input depends on exactly one source output. When the source is the K-ary digital source (with equi-distributed symbols), it is known that the minimum attainable error rate is lower bounded by†

$$P_e(T, K) \geqq \tfrac{1}{2}\Phi\left\{\left[\frac{KP_0T}{(K-2)(2N_0)}\right]^{\frac{1}{2}}\right\}, \tag{41}$$

where

$$\Phi(\alpha) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\alpha} e^{-u^2/2}\, du,$$

is the cumulative error function.

We now apply the lower bound of inequality (39a) together with inequality (41) to obtain a lower bound on $\bar{\epsilon}^s(T)$ when the channel signal power P_0 made large, while $T = 1/\rho_S$ is held fixed. Setting $\delta_1 = P_0^{-1}$, we obtain from inequalities (39a) and (41) and $\Phi(\alpha) \sim (2\pi\alpha^2)^{-1} \exp(-\alpha^2/2)$ (as $\alpha \to \infty$), that (with $T = \rho_S^{-1}$ held fixed)

$$\bar{\epsilon}^s(T) = \bar{\epsilon}^s\left(\frac{1}{\rho_S}\right) \geqq \exp\left\{-\frac{P_0}{2N_0\rho_S}\left[1 + \xi_3(P_0)\right]\right\} \tag{42}$$

where $\xi_3(P_0) \to 0$ as $P_0 \to \infty$.

Now suppose that for a given channel (and a given P_0) we allow T to become large. In other words, we permit "source coding" in blocks of length $n = \rho_S T$. Since the channel capacity $C = P_0/N_0$, we have from equation (40a) that

$$\lim_{T\to\infty} \bar{\epsilon}^s(T) = \bar{\epsilon}_0^s = \exp\left\{-\frac{sP_0}{2N_0\rho_S}\left[1 + \xi_4(P_0)\right]\right\}, \tag{43}$$

where $\xi_4(P_0) \to 0$ as $P_0 \to \infty$.

Now let $\theta > 0$ be arbitrary, and let P_1 be sufficiently large so that for $P_0 \geqq P_1$,

$$|\xi_3(P_0)|, \qquad |\xi_4(P_0)| < \theta.$$

Then from inequality (42), with $P_0 \geqq P_1$, the best attainable mean sth

† This bound follows from Ref. 1 [equation (82)] when the signal energy nP in that reference is replaced by P_0T our signal energy, and M is replaced by K.

error with no coding is bounded by

$$\bar{\epsilon}^{s}\left(\frac{1}{\rho_S}\right) \geqq \exp\left[-\frac{P_0}{2N_0\rho_S}(1+\theta)\right]. \tag{44}$$

The best attainable sth error with infinite delay T is from equation (43) with $P_0 \geqq P_1$, bounded by

$$\bar{\epsilon}_0^{s} \leqq \exp\left[-\frac{sP_0}{N_0\rho_S}(1-\theta)\right]. \tag{45}$$

We conclude that coding with large delay offers a saving of at least a factor of $(2s)$ in power P_0 or rate ρ_S (when $P_0 \geqq P_1$). This of course is interesting when $s > \frac{1}{2}$. Similar results for $s = 2$ have been derived by Ziv and Zakai.[7] This result can be generalized to arbitrary n (here we studied $n = 1$) and arbitrary channels simply by using appropriate bounds on $P_e(T, K)$.

III. PROOFS OF THEOREMS

3.1 *Proof of Theorem* 1 *and Related Examples*

3.1.1 *Proof of Theorem* 1

Shannon [Ref., 2, pp. 155–156] has shown that for a difference distortion measure $d(x, \hat{x}) = r(x - \hat{x})$, that

$$R_{\text{eq}}(\beta) \geqq H_S - \Phi(\beta), \tag{46}$$

where H_S is given by equation (15b) and $\Phi(\beta)$ is the maximum attainable entropy $H\{f(x)\}$ for a probability density $f(x)$ which satisfies

$$\int_{-\infty}^{\infty} r(x)f(x)\,dx \leqq \beta. \tag{47}$$

The entropy $H\{f(x)\}$ is defined by

$$H\{f(x)\} = -\int_{-\infty}^{\infty} f(x)\log f(x)\,dx. \tag{48}$$

A trivial modification of Shannon's argument shows that when $\mathfrak{X} = [-A/2, A/2]$, inequality (46) remains valid if $f(x)$ is further restricted to satisfy

$$f(x) = 0, \qquad |x| > \frac{A}{2}. \tag{49}$$

Now the density $g_\beta(x)$ [defined by expressions (13) and (14a)] satisfies conditions (47) and (49) and has entropy $H_1(\beta)$ [defined by equation (14)]. We prove Theorem 1 by showing that if the density $f(x)$ satisfies conditions (47) and (49), then $H\{f(x)\} \leqq H_1(\beta)$.

Let us write $g_\beta(x) = Be^{-\lambda r(x)}$ where $\lambda = \lambda_0(\beta)$ and where

$$B = \left[\int_{-A/2}^{A/2} e^{-\lambda_0(\beta) r(x)}\, dx\right]^{-1}.$$

Then

$$H_1(\beta) = -\int_{-A/2}^{A/2} g_\beta(x) \log g_\beta(x)\, dx$$

$$= -\log B + \lambda \int_{-A/2}^{A/2} r(x) g_\beta(x)\, dx = -\log B + \lambda\beta.$$

Since $f(x)$ satisfies condition (47),

$$H_1(\beta) \geqq -\log B + \lambda \int_{-A/2}^{A/2} r(x) f(x)\, dx$$

$$= -\int f(x) \log Be^{-\lambda r(x)}\, dx = -\int f(x) \log g_\beta(x)\, dx.$$

Thus

$$H\{f(x)\} - H_1(\beta) \leqq -\int_{-A/2}^{A/2} f(x) \log f(x)\, dx + \int f(x) \log g_\beta(x)\, dx$$

$$= \int_{-A/2}^{A/2} f(x) \log \frac{g_\beta(x)}{f(x)}\, dx \leqq \int_{-A/2}^{A/2} f(x)\left[\frac{g_\beta(x)}{f(x)} - 1\right] dx = 1 - 1 = 0,$$

where the second inequality follows from $\log u \leqq u - 1$. Theorem 1 follows.

Note that Theorem 1 will hold for $A = \infty$ as long as we can find $g_\beta(x)$. Examination of the derivation which establishes the existence of $g_\beta(x)$ (Ref. 3, Appendix A) shows that Theorem 1 is valid in particular for $A = \infty$ and $r(x) = |x|^s$, $s > 0$.

3.1.2 *Determination of* R_{eq} (β) *in Example (iii)*

For $\mathfrak{X} = [-A/2, A/2]$, $P_S(x) = A^{-1}$, and $d(x, \hat{x}) = r(x - \hat{x})$, where $r(u)$ satisfies conditions (12); $H_S = \log A$. Theorem 1 implies

$$R_{eq}(B) \geqq \log A - H_1(\beta). \tag{50}$$

We now show that if, in addition, $r(u)$ satisfies equation (18), then inequality (50) is satisfied with equality. Let X and $\hat{X}$ be random variables such that the density for X is $P_S(x) = A^{-1}(|x| \leqq A/2)$, and $\hat{X} = X + Y$ where the random variable Y is independent of X and has density $g_\beta(y) = Be^{-\lambda_0 r y}$ [defined by equations (13) and (14a)]. The information of $p(x, \hat{x})$, the joint density for X, $\hat{X}$, is

$$I\{p(x, \hat{x})\} = H\{p_2(\hat{x})\} - \int_{-A/2}^{A/2} P_S(x) H\{p(\hat{x} \mid x)\} \, dx,$$

where $p_2(\hat{x})$ is the density for $\hat{X}$, $p(\hat{x} \mid x)$ is the conditional density for $\hat{X}$ given that $X = x$, and $H\{\ \}$ is the entropy defined in equation (48).

Now $p(x, \hat{x}) = P_S(x) p(\hat{x} \mid x) = A^{-1} B e^{-\lambda_0 r(\hat{x} - x)}$, so that

$$p_2(\hat{x}) = A^{-1} B \int_{-A/2}^{A/2} e^{-\lambda_0 r(\hat{x} - x)} \, dx = A^{-1} B \int_{\hat{x} - A/2}^{\hat{x} + A/2} e^{-\lambda_0 r(u)} \, du.$$

when $\hat{x} \geqq 0$ this becomes, letting $v = u - A$ and using equation (18)

$$\begin{aligned} p_2(\hat{x}) &= A^{-1} B \int_{\hat{x} - A/2}^{A/2} e^{-\lambda_0 r(u)} \, du + A^{-1} B \int_{A/2}^{\hat{x} + A/2} e^{-\lambda_0 r(u)} \, du \\ &= A^{-1} B \int_{\hat{x} - A/2}^{A/2} e^{-\lambda_0 r(u)} \, du + A^{-1} B \int_{-A/2}^{\hat{x} - A/2} e^{-\lambda_0 r(v)} \, dv. \end{aligned}$$

Hence, since $\int g_\beta(x) = 1$,

$$p_2(\hat{x}) = A^{-1} B \int_{-A/2}^{A/2} e^{-\lambda_0 r(u)} \, du = A^{-1}.$$

For $\hat{x} < 0$, a similar proof yields $p_2(\hat{x}) = A^{-1}$. Thus $H\{p_2(\hat{x})\} = \log A$. Further $p(\hat{x} \mid x) = g_\beta(\hat{x} - x)$, and a similar use of equation (18) yields $H\{p(\hat{x} \mid x)\} = H_1(\beta)$, independent of x. Thus we conclude that $I\{p(x, \hat{x})\} = \log A - H_1(\beta)$. Since $p(x, \hat{x}) \in \mathfrak{M}(\beta)$, this and inequality (50) imply $R_{eq}(\beta) = \log A - H_1(\beta)$.

3.1.3 *Proof for Example (iv)*

We first verify equation (21) for the case $A/(2\delta) = K_0$, an integer. That $R_{eq}(\beta)$ is greater than or equal to the right member of equation (21) follows from inequality (20) (since $H_S = \log A$) and from $R_{eq}(\beta) \geqq 0$. To show that $R_{eq}(\beta)$ is less than or equal to the right member of expression (21) we produce a density $p_0(x, \hat{x})$ for which $I\{p_0(x, \hat{x})\}$ equals the right member of equation (21). But first we digress to define "entropy" for a discrete random variable.

Consider a discrete probability density $f(x) = \sum_i a_i \, \delta(x - x_i)$. Then the "discrete entropy of $f(x)$ is defined by

$$H_D\{f(x)\} = -\sum_i a_i \log a_i \, . \tag{51}$$

Now, say that $p(x, \hat{x})$ is the probability density for two random variables X, $\hat{X}$, such that $\hat{X}$ takes values at a countable number of points. Then the marginal density for $\hat{X}$, denoted $p_2(\hat{x})$ and the conditional density for $\hat{X}$ given $X = x$ [denoted $p(\hat{x} \mid x)$] are discrete densities. It is easy to show that the information can be written

$$I\{p(x, \hat{x})\} = H_D\{p_2(\hat{x})\} - \int p_1(x) H_D\{p_2(\hat{x} \mid x)\} \, dx, \tag{52}$$

where $p_1(x)$ is the marginal density for X.

Return now to Example (*iv*). Let $0 \leqq \beta \leqq (K_0 - 1)/K_0$, and let $p_0(x, \hat{x})$ be the density for X, $\hat{X}$, where X has density $P_S(x) = A^{-1}$ and $\hat{X}$ has conditional density $p_0(\hat{x} \mid x)$ given as follows. Partition the interval $[-A/2, A/2]$ into K_0 subintervals $\{I_i\}_0^{K_0-1}$ of width 2δ. Let x_i be the midpoint of I_i $(i = 0, 1, 2, \cdots, K_0 - 1)$. Then for $x \in I_i$

$$p_0(\hat{x} \mid x) = (1 - \beta) \, \delta(\hat{x} - x_i) + \frac{\beta}{(K_0 - 1)} \sum_{j \neq i} \delta(\hat{x} - x_j).$$

In other words, $\hat{X}$ is an imperfectly quantized version of X. With probability $(1 - \beta)$, $\hat{X}$ is the midpoint of the subinterval in which X lies, and with probability β, $\hat{X}$ is uniformly distributed among the remaining $(K_0 - 1)$ midpoints. Note that $P_S(x)$ and $p_0(\hat{x} \mid x)$ together determine $p_0(x, \hat{x})$, and that $p_0(x, \hat{x}) \in \mathfrak{M}(\beta)$.

Further, by symmetry, $\hat{X}$ is uniformly distributed on the K_0 midpoints, so that

$$H_D\{p_{02}(\hat{x})\} = \log K_0 ,$$

where $p_{02}(\hat{x})$ is the marginal density for $\hat{X}$ [corresponding to $p_0(x, \hat{x})$]. Also

$$H_D\{p_0(\hat{x} \mid x)\} = h(\beta) + \beta \log (K_0 - 1),$$

independent of x. Thus equation (52) yields

$$I\{p_0(x, \hat{x})\} = \log K_0 - h(\beta) - \beta \log (K_0 - 1),$$

the right member of expression (21). This establishes equation (21) for $0 \leqq \beta \leqq (K_0 - 1)/K_0$. Since $R_{eq}[(K_0 - 1)/K_0] = 0$ and $R_{eq}(\beta)$ is nonincreasing, we have $R_{eq}(\beta) = 0$ for $\beta \geqq (K_0 - 1)/K_0$, establishing expression (21).

It remains to verify the upper bound of expressions (22). But this follows immediately on noting that for fixed A and β, $R_{eq}(\beta)$ is a decreasing function of δ. Thus decreasing δ to $\delta' = A/2[A/2\delta]^+$ results in an increase in $R_{eq}(\beta)$. Since $A/(2\delta')$ is an integer, we can apply expression (21) to obtain the upper bound of expression (22).

3.2 *Proof of Theorem* 2

Theorem 2 relates the attainable distortions for a digital source and an analog source when connected to a given channel. The proof is in two parts [corresponding to the two inequalities in expression (23)], the second of which uses a bounding technique introduced by Ziv and Zakai.[7]

In part (*i*) we are given an encoder and decoder for the digital source (with appropriate parameters), which when connected to the channel as in Fig. 1 results in an average Hamming distortion $\bar{d} = \bar{d}_H$. We show how to quantize the outputs of the analog source (with appropriate parameters) to essentially simulate the digital source. When this quantizer is connected to the digital encoder, we show that we attain an average distortion for the analog source $\bar{d}_\delta \leqq \bar{d}_H$. This leads us directly to the second inequality of expression (23).

In part (*ii*) we establish the first inequality of expression (23) in an essentially dual way. We begin by assuming the existence of an analog encoder and decoder. We then show how to modulate the outputs of the digital source to virtually simulate the analog source. Unfortunately, this is not as easy as the quantization in part (*i*), and we have to make use of an "averaging" argument in the course of the proof.

(*i*) Let us denote by $\mathcal{S}_a$, the analog source whose output is a sequence $X_1, X_2, \cdots$ of independent random variables, each uniformly distributed on the source space $\mathfrak{X}_a = [-A/2, A/2]$. The random variables appear at a rate of ρ_S per second. For this source we use the distortion $d(x, \hat{x}) = d_\delta(x, \hat{x})$ defined by equations (19). Assume first that $A/(2\delta) = K_0$ an integer, and consider the following (uniform) quantizer. Partition the interval $[-A/2, A/2]$ into K_0 subintervals $\{I_i\}_0^{K_0-1}$ of width (2δ) where

$$I_i = (e_i, e_{i+1}], \qquad i = 0, 1, \cdots, K_0 - 1, \tag{53a}$$

and

$$e_i = (2\delta)\left[\left(i - \frac{K_0}{2}\right)\right], \qquad i = 0, 1, \cdots, K_0 . \tag{53b}$$

To be precise, the first interval I_0 should be closed on the left. The quantizer q is defined by

$$q(x) = i, \quad \text{if} \quad x \in I_i \qquad \left(-\frac{A}{2} \leqq x \leqq \frac{A}{2}\right). \tag{54}$$

Let us now consider the digital source $\mathcal{S}_d$ whose output is a sequence $S_1, S_2, \cdots$ of independent discrete random variables, each uniformly distributed on the K_0-ary set $\mathfrak{X}_d = \{0, 1, \cdots, K_0 - 1\}$. These random variables also appear at ρ_S per second. (Note that we use S_k instead of X_k as in Section II to distinguish the outputs of $\mathcal{S}_d$ from those of $\mathcal{S}_a$.) Say that the distortion $d = d_H$ as defined in equation (5).

Suppose that $\mathcal{S}_d$ can be connected with delay T to a channel as in Fig. 1 with (digital) encoder $f_E^{(d)}$ and decoder $f_D^{(d)}$, and average distortion $\bar{d}_H$. We now show how to connect the "analog" source $\mathcal{S}_a$ to the channel [with the help of $f_E^{(d)}$ and $f_D^{(d)}$] to attain an average distortion $\bar{d}_\delta \leqq \bar{d}_H$. Consider the system in Fig. 4. In T seconds the output of the analog source is an n-vector $(n = \rho_S T)\mathbf{X} = (X_1, \cdots, X_n)$. The "quantizer" output is the n-vector $\mathbf{S} = (S_1, S_2, \cdots, S_n)$, where $S_k = q(X_k)$ $(k = 1, 2, \cdots, n)$. Note that the S_k are independent and uniformly distributed on $\{0, 1, \cdots, K_0 - 1\}$, as are the outputs of the digital source $\mathcal{S}_d$. The digital encoder and decoder $f_E^{(d)}$ and $f_D^{(d)}$ are as given above, and the output of the latter is the K_0-ary vector $\hat{\mathbf{S}} = (\hat{S}_1, \cdots, \hat{S}_n)$. Thus

$$Ed^{(n)}(\mathbf{S}, \hat{\mathbf{S}}) = \bar{d}_H .$$

The "converter" output is the n-vector $\hat{\mathbf{X}} = (\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_n)$ where

$$\hat{X}_k = (2\hat{S}_k - K_0 + 1)\delta.$$

In other words if $\hat{S}_k = i$, then $\hat{X}_k$ is the midpoint $(e_i + e_{i+1})/2$ of the ith subinterval. Disregarding the case when X_k is equal to one of the endpoints e_i of the subintervals, (an event with zero probability), it is clear that $| X_k - \hat{X}_k | \geqq \delta$ if and only if $S_k \neq \hat{S}_k$ $(k = 1, 2, \cdots, n)$. Thus

$$\bar{d}_\delta = Ed_\delta^{(n)}(\mathbf{X}, \hat{\mathbf{X}}) = Ed_H^{(n)}(\mathbf{S}, \hat{\mathbf{S}}) = \bar{d}_H .$$

It follows that

$$Q(T, A, \delta) \leqq P_e\left(T, \frac{A}{2\delta}\right), \tag{55}$$

when $A/(2\delta)$ is an integer. The second inequality of expression (23)

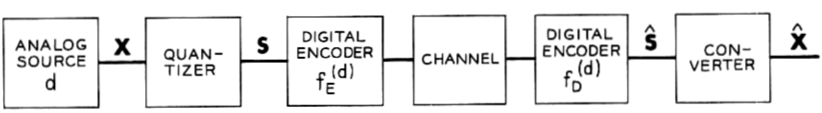

Fig. 4 — An analog communication scheme.

follows on noting that $Q(T, A, \delta)$ is a nonincreasing function of δ. Thus decreasing δ to $\delta' = A/2K_+$ does not result in a decrease in $Q(T, A, \delta)$. Since $A/(2\delta')$ is an integer, we can apply inequality (55) to obtain the second inequality of expression (23). This completes part (*i*).

(*ii*) Let us suppose that the analog source $\mathcal{S}_a$ defined in part (*i*) is connected with delay T to a channel as in Fig. 1. The T-second source output is the n-vector $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ and the decoder output is the n-vector $\hat{\mathbf{X}} = (X_1, X_2, \cdots, X_n)$. Say we attain an average distortion

$$\bar{d}_\delta = Ed_\delta^{(n)}(\mathbf{X}, \hat{\mathbf{X}}).$$

Letting $E[d_\delta^{(n)}(\mathbf{X}, \hat{\mathbf{X}}) \mid \mathbf{X} = \mathbf{x}]$ be the conditional expectation of $d_\delta^{(n)}(\mathbf{X}, \hat{\mathbf{X}})$ given $\mathbf{X} = \mathbf{x}$, we can write

$$\bar{d}_\delta = \int_{[-A/2, A/2]^n} E[d_\delta^{(n)} \mid \mathbf{X} = \mathbf{x}] \frac{1}{A^n} d\mathbf{x}. \tag{56}$$

Suppose that $(A/2\delta) = K_0$, an integer. Let us partition the interval $[-A/2, A/2]$ into K_0 subintervals of width 2δ as in equations (53). Let $\mathcal{E}$ be the set of left end-points of these subintervals, that is,

$$\mathcal{E} = \{e_i\}_{i=1}^{K_0-1}. \tag{57}$$

Now consider the n-cube $[-A/2, A/2]^n$. Note that the random n-vector $\mathbf{X}$ is uniformly distributed on this cube. The partition of the interval $[-A/2, A/2]$ defines a partition of the n-cube into K_0^n subcubes, each the product of n subintervals. Let the members of $\mathcal{E}^n$ be denoted by the n-vectors $\boldsymbol{\xi}_j$, $j = 1, \cdots, K_0^n$, and let C_j be the corresponding subcube. (That is, C_j is the product of the subintervals whose left end-points are the coordinates of $\boldsymbol{\xi}_j$.) Then clearly,

$$\left[-\frac{A}{2}, \frac{A}{2}\right]^n = \sum_{j=1}^{K^n_0} C_j,$$

where $\sum$ denotes disjoint union. Thus we can rewrite equation (56) as

$$\begin{aligned}\bar{d}_\delta &= \sum_{j=1}^{K^n_0} \int_{C_j} \frac{1}{A^n} E[d_\delta^{(n)} \mid \mathbf{X} = \mathbf{x}]\, d\mathbf{x} \\ &= \sum_{j=1}^{K^n_0} \frac{1}{K_0^n} \int_{[0,2\delta]^n} \frac{1}{(2\delta)^n} E[d_\delta^{(n)} \mid \mathbf{X} = \boldsymbol{\xi}_j + \boldsymbol{\alpha}]\, d\boldsymbol{\alpha},\end{aligned} \tag{58}$$

where the second equality follows from the change of variable of integration to $\boldsymbol{\alpha} = \mathbf{x} - \boldsymbol{\xi}_j$, and the fact that $A = 2\delta K_0$.

Some insight into what we have done may be gained by considering the special case where $K_0 = 2$ and $n = 2$. In this case the n-cube $[-A/2, A/2]^n$ is a square, and there are $K_0^n = 2^2 = 4$ members of $\mathcal{E}^n$ denoted ξ_1, ξ_2, ξ_3, and ξ_4. (See Fig. 5.) The subcubes are C_1, C_2, C_3, and C_4 as indicated.

Let us consider now the digital source $\mathcal{S}_d$ defined in part (*i*) whose output is the sequence $S_1, S_2, \cdots$. We would like to transmit the outputs of $\mathcal{S}_d$ through a channel (as in Fig. 1) with delay T, so that the source output must be an n-vector $(n = \rho_S T)\mathbf{S}$, and the decoder output an n-vector $\hat{\mathbf{S}}$. The fidelity criterion is

$$\bar{d}_H = E d_H^{(n)}(\mathbf{S}, \hat{\mathbf{S}}).$$

Now suppose that we are given an encoder-decoder, $f_E^{(a)}$, $f_D^{(a)}$, for the analog source $\mathcal{S}_a$ [for which $A/(2\delta) = K_0$], connected with delay T, to a given channel. Say this encoder-decoder attains an average distortion $\bar{d}_\delta$. We show that there exists an encoder-decoder for the K_0-ary digital source $\mathcal{S}_d$, connected with delay T, to the same channel such that the average distortion $\bar{d}_H \leqq \bar{d}_\delta$. From this we deduce immediately that for $A/(2\delta) = K_0$,

$$P_e(T, K_0) \leqq Q(T, A, \delta). \tag{59}$$

The digital encoder is given schematically in Fig. 6a. The analog encoder which we are given is $f_E^{(a)}(\mathbf{x})$, $\mathbf{x} \in [-A/2, A/2]^n$, and is realized in the right box of Fig. 6a. The function of the "modulator" is to assign to each n-vector $\hat{\mathbf{s}} \in \{0, 1, \cdots, K_0 - 1\}^n$, a member of $[-A/2, A/2]^n$. This is done as follows. Let $\mathcal{E}$ be the set defined by equation (57). For $s \in \{0, 1, 2, \cdots, K_0 - 1\}$, let

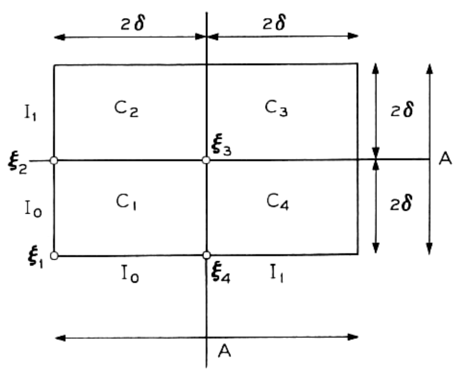

Fig. 5 — A digital encoder.

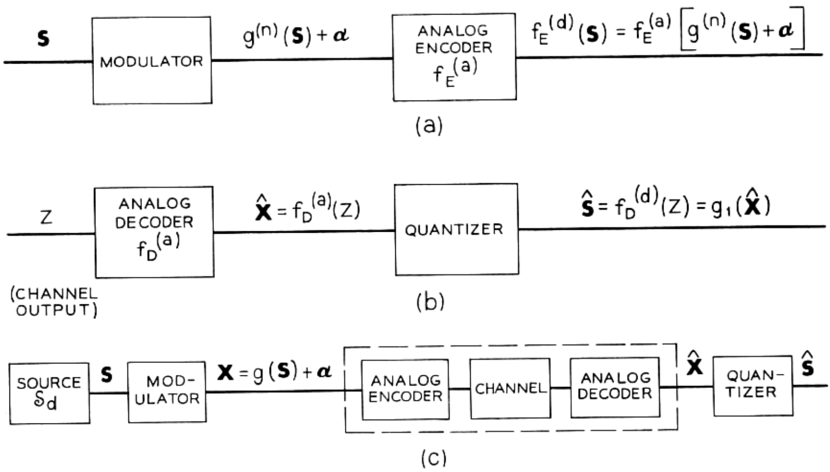

Fig. 6 — (a) Digital to analog encoder. (b) Analog to digital decoder. (c) Digital communication scheme.

$$g(s) = (2\delta)\left[s - \frac{K_0}{2}\right]$$

be the sth member of $\mathcal{E}$. For $\mathbf{s} = (s_1, s_2, \cdots, s_n) \varepsilon \{0, 1, \cdots, K_0 - 1\}^n$, let

$$g^{(n)}(\mathbf{s}) = [g(s_1), g(s_2), \cdots, g(s_n)].$$

When the input to the modulator is $\mathbf{s}$, its output is

$$\boldsymbol{\alpha} + g^{(n)}(\mathbf{s}),$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n) \varepsilon [0, 2\delta]^n$ is a fixed vector. Thus the digital encoder is

$$f_E^{(d)}(\mathbf{s}) = f_E^{(a)}[\boldsymbol{\alpha} + g^{(n)}(\mathbf{s})].$$

The digital decoder is given schematically in Fig. 6b. The left box is the analog decoder $f_D^{(a)}$ which we are given. Its output $\hat{\mathbf{x}}$ is a real n-vector. The right box is a quantizer. When its input is $\hat{\mathbf{x}} = (\hat{x}_1, \cdots, \hat{x}_n)$, its output is $g_1(\hat{\mathbf{x}}) = \hat{\mathbf{s}} = (\hat{s}_1, \cdots, \hat{s}_n)$, where $\hat{s}_k (k = 1, 2, \cdots, n)$ is a member of $\{0, 1, \cdots, K_0 - 1\}$ which minimizes $| g_k(\hat{s}_k) + \alpha_k - \hat{x}_k |$.

When the digital source $\mathcal{S}_d$ is connected to the channel with this encoder-decoder pair, the result is schematized in Fig. 6c. (Upper case X's and S's are used to signify random variables.) The portion of the system in the dotted lines is precisely the analog encoder-channel-decoder which would produce an average distortion $\bar{d}_\delta$ given by equation (56), *if* the analog input, $\mathbf{X}$, where uniformly distributed on the n-cube $[-A/2, A/2]^n$. But this is not the case here. In fact, $\mathbf{X}$ takes only one of K_0^n possible values. However, the quantity $E[d_\delta^{(n)}(\mathbf{X}, \hat{\mathbf{X}}) \mid \mathbf{X} = \mathbf{x}]$ is exactly the same in the system of Fig. 6c as in equation (56), for $\mathbf{x} = g^{(n)}(\mathbf{s}) + \boldsymbol{\alpha}(\mathbf{s} \in \{0, \cdots, K_0 - 1\}^n)$.

Let us write an expression for the average distortion $\bar{d}_H$ for the digital source. Note that $S_k \neq \hat{S}_k$, only if $|\hat{X}_k - [g(S_k) + \alpha_k]| \geqq \delta$. Thus

$$d_H^{(n)}(\mathbf{S}, \hat{\mathbf{S}}) \leqq d_\delta^{(n)}[g^{(n)}(\mathbf{S}) + \boldsymbol{\alpha}, \hat{\mathbf{X}}],$$

and

$$\begin{aligned}\bar{d}_H &= E d_H(\mathbf{S}, \hat{\mathbf{S}}) \\ &\leqq \sum_{\mathbf{s}} \frac{1}{K_0^n} E[d_\delta^{(n)}(\mathbf{X}, \hat{\mathbf{X}}) \mid \mathbf{X} = g^{(n)}(\mathbf{s}) + \boldsymbol{\alpha}],\end{aligned} \tag{60}$$

where $\sum_{\mathbf{s}}$ is the sum over the K_0^n equally-likely values of $\mathbf{s}$. Let us now average the right member of expression (60) over all $\boldsymbol{\alpha}$ in $[0, 2\delta]^n$, with $\boldsymbol{\alpha}$ assumed to be uniformly distributed. That average is

$$\int_{[0,2\delta]^n} \frac{d\mathbf{x}}{(2\delta)^n} \sum_{\mathbf{s}} \frac{1}{K_0^n} E[d_\delta(\mathbf{X}, \hat{\mathbf{X}}) \mid \mathbf{X} = g^{(n)}(\mathbf{s}) + \boldsymbol{\alpha}].$$

If we note that the set $\{g^{(n)}(\mathbf{s})\}$ are in one-to-one correspondence with the K_0^n members $\boldsymbol{\xi}_i$ of $\mathcal{E}^n$, this quantity may be written as

$$\sum_{i=1}^{K^n{}_0} \frac{1}{K_0^n} \int_{[0,2\delta]^n} \frac{1}{(2\delta)^n} E[d_\delta(\mathbf{X}, \hat{\mathbf{X}}) \mid \mathbf{X} = \boldsymbol{\xi}_i + \boldsymbol{\alpha}]\, d\boldsymbol{\alpha},$$

which equals $\bar{d}_\delta$ by equation (58). Since there must be at least one value of $\boldsymbol{\alpha}$ for which the right member of expression (60) is as small as the average, we have proved inequality (59).

The first inequality of expression (23) follows from inequality (59) on noting as in part (*i*) that $Q(T, A, \delta)$ is a decreasing function of δ.

3.3 *Proof of Theorems* 4 *and* 5

Since Theorem 5 includes Theorem 4 as a special case we need only give a proof of Theorem 5. Our task is further simplified since the basic idea of the proof of Theorem 5 is the same as in Theorem 2 (Section 3.2). Here too we break the proof into two parts. In part (*i*) we assume that we are given an encoder-decoder for the digital source and deduce the existence of an encoder-decoder for the general source (which plays the part of the analog source in Theorem 2). In part (*ii*) we do the opposite. However we do not have the complications here which necessitated an averaging argument in Section 3.2.

(*i*) We prove here that $\hat{G}(T, \delta) \leqq \hat{P}_e[T, M_C(\delta)]$, the second inequality of expression (33). The proof parallels that of part (*i*) in Section 3.2. Instead of the analog source space $\mathfrak{X}_a$ we have here a general space $\mathfrak{X}$. The distortion is $d_\delta(x, \hat{x})$ with $| x - \hat{x} |$ replaced by $\rho_0(x, \hat{x})$.

To transmit the source outputs which belong to $\mathfrak{X}$ we use the system in Fig. 4. The digital encoder-decoder is for a K_0-ary source where $K_0 = M_C(\delta)$. We assume that it attains a guaranteed distortion $\hat{d}_H$. The quantizer is defined as follows. Let $\{\beta_i\}_0^{K_0-1}$ be a minimum δ-covering of $\mathfrak{X}$. For $x \in \mathfrak{X}$, let $q(x)$ be the smallest $i (0 \leqq i \leqq K_0 - 1)$ such that $x \in S_{\beta_i}(\delta)$. Then if $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathfrak{X}^n$ is the source output, the quantizer output is $\mathbf{s} = q^n(\mathbf{x}) = [q(x_1), q(x_2), \cdots, q(x_n)]$. The output of the digital decoder is $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2, \cdots, \hat{S}_n)$ and the converter output is $\hat{\mathbf{X}} = (\hat{X}_1, \cdots, \hat{X}_n)$, where $\hat{X}_k = \beta_i$ when $\hat{S}_k = i$. Clearly, if $S_k = \hat{S}_i$, then $\rho_0(X_k, X_k) < \delta$. Thus for any source output $\mathbf{x}$,

$$\bar{d}_\delta(\mathbf{x}) \leqq \bar{d}_H[q^{(n)}(\mathbf{x})] \leqq \hat{d}_H ,$$

so that the overall guaranteed distortion $\hat{d}_\delta \leqq \hat{d}_H$, from which part (*i*) follows.

(*ii*) We prove here that $\hat{P}_e[T, M_P(\delta)] \leqq \hat{G}(T, \delta)$, the first inequality of expression (33). As in part (*i*), the proof of part (*ii*) parallels that in Section 3.2. Again $\mathfrak{X}_a$ is replaced by $\mathfrak{X}$ and $| x - \hat{x} |$ by $\rho_0(x, \hat{x})$.

As in Section 3.2, we assume that we are given an encoder-decoder for the general source with guaranteed distortion $\hat{d}_\delta$. We set $K_0 = M_P(\delta)$ and use the system of Fig. 6 to transmit the outputs of the K_0-ary digital source. The modulator is defined as follows. Let $\{\beta_i\}_{i=0}^{K_0-1}$ be a minimum δ-packing of $\mathfrak{X}$. If source output is $\mathbf{s} = (s_1, s_2, \cdots, s_n)$, then the modulator output is $g^{(n)}(\mathbf{s}) = (\beta_{s_1}, \beta_{s_2}, \cdots, \beta_{s_n})$. The output of the decoder is $\hat{\mathbf{X}} = (\hat{X}_1, \cdots, \hat{X}_n)$, and the quantizer output is $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2, \cdots, \hat{S}_n)$, where $\hat{S}_k = i$ when $\hat{X}_k \in S_{\beta_i}(\delta)$. If $X_k \notin S_{\beta_i}(\delta)$ for all i $(0 \leqq i \leqq K_0 - 1)$, then $\hat{S}_k = 0$.

Clearly, if $\rho_0(X_k, \hat{X}_k) < \delta$, then $S_k = \hat{S}_k$. Thus for any source output $\mathbf{s}$, the conditional expectation

$$\bar{d}_H(\mathbf{s}) \leqq \bar{d}_\delta[g^{(n)}(\mathbf{s})] \leqq \hat{d}_\delta .$$

Thus the overall guaranteed distortion is $\hat{d}_H \leqq \hat{d}_\delta$, completing the proof of part (*ii*) and the theorem.

3.4 *Proofs on Packing and Covering*

In this section we give a proof of Theorem 3, the main part of which is a lemma on covering of the K-ary n-cube. We also prove Lemma 1 relating packing and covering in Section 2.4.

3.4.1 *Proof of Theorem* 3

We first establish the following lemma.

Lemma 2: Let $\theta(0 < \theta < (K-1)/K)$ *be arbitrary, and let* r *satisfy*

$$R_{eq}(\theta) < r < \log K,$$

where $R_{eq}(\lambda)$ *is the equivalent rate for the* K*-ary source given by expressions* (9). *Using the terminology of Section 2.4, let* $\mathfrak{X} = \{0, 1, \cdots, K-1\}^n$ *(the K-ary n-cube) and* $\rho_0(\mathbf{x}, \hat{\mathbf{x}}) = d_H^{(n)}(x, \hat{x})$. *Then for* n *sufficiently large, there exists a* θ*-covering of* $\mathfrak{X}$ *with* $M = e^{rn}$ *points.*

Proof: Let $\{\mathbf{x}_i\}_1^M$ be a set of K-ary n-vectors. Let $F(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_M)$ be the number of members $\hat{\mathbf{x}}$ of $\mathfrak{X}$ such that $d_H^{(n)}(\mathbf{x}_i, \hat{\mathbf{x}}_i) \geqq \theta$ for all $i = 1, 2, \cdots, M$. If $F = 0$, then $\{\mathbf{x}_i\}_1^M$ is a θ-covering of $\mathfrak{X}$. We can write

$$F(\mathbf{x}_1, \cdots, \mathbf{x}_M) = \sum_{\hat{\mathbf{x}} \in \mathfrak{X}} \Phi(\hat{\mathbf{x}}, \mathbf{x}_1, \cdots, \mathbf{x}_M),$$

where

$$\Phi(\hat{\mathbf{x}}, \mathbf{x}_1, \cdots, \mathbf{x}_M) = \begin{cases} 1 & \text{if } d_H^{(n)}(\mathbf{x}_i, \hat{\mathbf{x}}) \geqq \theta, \text{ all } i = 1, 2, \cdots, M, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider an experiment in which $M = e^{rn}$ n-vectors $\{\mathbf{X}_i\}_1^M$ are chosen at random from $\mathfrak{X}$ independently with identical (uniform) distribution

$$\Pr\{\mathbf{X}_i = \mathbf{x}\} \equiv K^{-n}.$$

Then $F(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_M)$ is random variable with expectation

$$EF = \sum_{\hat{\mathbf{x}} \in \mathfrak{X}} E\Phi(\hat{\mathbf{x}}, \mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_M),$$

where, as indicated, $E\Phi$ is computed with $\hat{\mathbf{x}}$ held fixed. Now for a given $\hat{\mathbf{x}}$,

$$\begin{aligned} E\Phi(\hat{\mathbf{x}}, \mathbf{X}_1, \cdots, \mathbf{X}_M) &= \Pr\{\Phi = 1\} \\ &= \Pr \bigcap_{i=1}^{M} \{d_H^{(n)}(\mathbf{x}, \mathbf{X}_i) \geqq \theta_i\} \\ &= [\Pr\{d_H^{(n)}(\hat{\mathbf{x}}, \mathbf{X}_i) \geqq \theta\}]^M, \end{aligned}$$

where the last equality follows from the independence and identical distribution of the random vectors $\{\mathbf{X}_i\}$. Letting $a_n = \sum_{0 \leqq i < \theta n} \binom{n}{j} \cdot (K-1)^i K^{-n}$ be the probability that $d_H(\hat{\mathbf{x}}, \mathbf{X}_i) < \theta n$, we have

$$E\Phi(\hat{\mathbf{x}}, \mathbf{X}_1, \cdots, \mathbf{X}_M) = (1 - a_n)^M \leqq e^{-a_n M},$$

independent of $\mathbf{x}$. Thus

$$EF \leqq Me^{-a_n M}.$$

Now it is well known (see for example, Ref. 8, p. 173) that for $0 < \theta < (K-1)/K$, as $n \to \infty$,

$$a_n = e^{-nR\mathrm{eq}(\theta) + o(n)}.$$

Thus since $M = 2^{rn}$ and $r > R_{\mathrm{eq}}(\theta)$,

$$E(F) \leqq Me^{-a^n M} = e^{rn} \exp \{-e^{[r - R\mathrm{eq}(\theta)]n + o(n)}\} \to 0, \quad \text{as} \quad n \to \infty.$$

Now, there must be at least one particular set $\{\mathbf{x}_i\}_1^M$ such that

$$F(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_M) \leqq EF.$$

Thus if we choose n large enough so that $E(F) < 1$, $F(\mathbf{x}_1, \cdots \mathbf{x}_M) = 0$ (since F is an integer valued function). Thus $\{\mathbf{x}_i\}_1^M$ is the required covering.

The proof of Theorem 3 now follows the standard proof of a source-channel coding theorem, with Lemma 2 playing the role of the source coding theorem. (See Ref. 2.) Roughly speaking the proof is as follows. When $\gamma(K, \rho_S, C) = (K-1)/K$, the entire theorem is trivial, since we can attain a guaranteed distortion of $(K-1)/K$ without even using the channel by simply letting the decoder outputs take the value $i(0 \leqq i \leqq K-1)$ with probability $1/K$. Thus assume that $0 \leqq \gamma < (K-1)/K$.

The channel can transmit e^{RT} (where $R < C$, the channel capacity) in T seconds with arbitrarily high reliability (see Section 2.1). By the definition of $\gamma = \gamma(K, \rho_S, C)$ [expression (10b)],

$$R_{\mathrm{eq}}(\gamma) \leqq C/\rho_S . \tag{61}$$

Let $\epsilon > 0$ be arbitrary. In Lemma 2, let $r = C/\rho_S - \epsilon_1$, where $\epsilon_1 > 0$ will be chosen below. Then approximate the T-second source output (a K_0-ary n-vector, $n = \rho_S T$) by a (covering) set with $e^{rn} = e^{r\rho_S T}$ members. Since $r\rho_S < C$ we can transmit these n-vectors through the channel with arbitrarily high reliability. Further, with $\epsilon > 0$ arbitrary, and if

$$r > R_{\mathrm{eq}}(\gamma + \epsilon), \tag{62}$$

we have from Lemma 2 that the error in making the approximation will always be less than or equal to $(\gamma + \epsilon)$ for T sufficiently large. In fact, if we set

$$\epsilon_1 = R_{eq}(\gamma) - R_{eq}\left(\gamma + \frac{\epsilon}{2}\right) > 0$$

[since $R(\gamma)$ as defined in equation (9) is strictly decreasing for $\gamma < (K - 1)/K$], then [using inequality (61)]

$$r = \frac{C}{\rho_S} - \epsilon_1 = \frac{C}{\rho_S} - R_{eq}(\gamma) + R_{eq}\left(\gamma + \frac{\epsilon}{2}\right)$$

$$\geqq R_{eq}\left(\gamma + \frac{\epsilon}{2}\right) > R_{eq}(\gamma + \epsilon)$$

and condition (62) is satisfied.

We conclude that for T sufficiently large, we can make

$$\hat{d}_H \leqq \gamma + \epsilon$$

for arbitrary $\epsilon > 0$. Thus

$$\operatorname*{limit}_{T \to \infty} \hat{P}_e(T, K) \leqq \gamma + \epsilon \to \gamma, \quad \text{as} \quad \epsilon \to 0,$$

which is Theorem 3.

3.4.2 *Proof of Lemma* 1

We say that $A \subseteq \mathfrak{X}$ is a "maximal Δ-packing" if A is a Δ-packing, and if for all $v \not\in A$, the union $\{v\} \cup A$ is not a Δ-packing. We establish Lemma 1 by showing that every maximal Δ-packing is a $(2\eta\Delta)$-covering. Let A be a maximal Δ-packing. If A is not a $(2\eta\Delta)$-covering, then there exists a $v_0 \in \mathfrak{X}$ such that $\rho_0(v_0, u) > 2\eta\Delta$, for all $u \in A$. From condition (30b), $v_0 \not\in A$. We claim that $\{v_0\} \cup A$ is a Δ-packing, contradicting the maximality of A. If $w \in S_{v_0}(\Delta)$, then for all $u \in A$ (using the definition of η)

$$\rho_0(v_0, u) \leqq \eta[\rho_0(v_0, w) + \rho_0(w, u)],$$

so that

$$\rho_0(w, u) \geqq \frac{\rho_0(v_0, u)}{\eta} - \rho_0(v_0, w) > \frac{2\eta\Delta}{\eta} - \Delta = \Delta.$$

Thus $w \not\in S_u(\Delta)$ and $\{v_0\} \cup A$ is a Δ-packing, establishing the lemma.

APPENDIX

List of Symbols

$\mathcal{X}$		the source output space
$P_S(x)$		the source probability density function
ρ_S		source output rate (symbols per second)
x_i		$(x_i \in \mathcal{X})$ the ith output of the source
$\mathbf{x}$		$(x_1, x_2, \cdots x_n) \in \mathcal{X}^n$
$\mathcal{W}_T$		set of "allowable" channel inputs
$\mathfrak{z}_T$		the set of all channel outputs
T		the coding delay
n	$=$	$\rho_S T$
$f_E(\mathbf{x})$		the encoding function, $f_E(\mathbf{x}) \in \mathcal{W}_T$
$f_D(z)$		the decoding function, $f_D(z) \in \mathcal{X}^n$
$\hat{\mathbf{x}}$		the decoded n-vector, $\hat{\mathbf{x}} = f_D(z) \in \mathcal{X}^n$
N		number of code words in a code
R	$=$	$1/T \log N$, the rate of a code
λ		the word probability of error
$\lambda^*(T, N)$		smallest attainable word error probability for a code with parameters N and T
$\bar{\lambda}$		average probability of error
$d(x, \hat{x})$		the distortion function
$d_H(x, \hat{x})$	$=$	$\begin{cases} 0, & x = \hat{x} \\ 1, & x \neq \hat{x} \end{cases}$
$d^n(\mathbf{x}, \hat{\mathbf{x}})$	$=$	$1/n \sum_{K=1}^{n} d(x_K, \hat{x}_K)$
$\bar{d}$	$=$	$Ed^{(n)}(\mathbf{x}, \hat{\mathbf{x}})$
$\bar{d}^*(T)$		the smallest attainable $\bar{d}$ for a given delay T
$d_\delta(x, \hat{x})$	$=$	$\begin{cases} 1 & \lvert x - \hat{x} \rvert < \delta \\ 0 & \lvert x - \hat{x} \rvert \geqq \delta \end{cases}$
$\bar{d}(\mathbf{x})$		the expectation of $d^n(\mathbf{x}, \hat{\mathbf{x}})$ given $\mathbf{x}$
$\hat{d}$	$=$	$\sup_{\mathbf{x} \in \mathcal{X}^n} \bar{d}(\mathbf{x})$
$\hat{d}^*(T)$		the smallest attainable value of $\hat{d}$ for a given delay T
$Q(T, A, \delta)$		$\bar{d}^*(T)$ for $d_\delta(x, \hat{x})$ and $x \in [-A/2, A/2]$
$\hat{Q}(T, \delta, A)$		$\hat{d}^*(T)$ for $d_\delta(x, \hat{x})$ and $x \in [-A/2, A/2]$
$P_e(T, K)$		the minimum attainable per symbol error rate for an equiprobable K-ary memoryless source
$\gamma(K, \rho_S, C)$	$=$	$\lim_{T \to \infty} P_e(T, K)$

C	the channel capacity
$\hat{P}_e(T, K)$	the minimum attainable guaranteed per symbol error rate for a K-ary source
$G(T, \delta)$	generalization of $\hat{Q}$, defined in Section 2.4

REFERENCES

1. Shannon, C. E., "Probability of Error for Optimal Codes in a Gaussian Channel," B.S.T.J., *38,* No. 3 (May 1959), pp. 611–656.
2. Shannon, C. E., "Coding Theorems for a Discrete Source with a Fidelity Criterion," IRE Nat. Conv. Record, March 23–26, 1959, part 4, pp. 142–163.
3. Wyner, A. D., "Capabilities of Bounded Discrepancy Decoding," B.S.T.J., *44,* No. 6 (July-August 1965), pp. 1061–1022.
4. Shannon, "A Mathematical Theory of Communication," B.S.T.J., *27,* No. 3 (July 1948), pp. 379–423; No. 4 (October 1948), pp. 623–656.
5. Panter, P. F., and Dite, W., "Quantization Distortion in Pulse-Count Modulation with Non-Uniform Spacing of Levels," Proc. IRE, *39,* No. 1 (January 1951), pp. 44–48.
6. Wozencraft, J. M., and Jacobs, I. M., *Principles of Communication Engineering,* New York: Wiley, 1965.
7. Ziv, J., and Zakai, M. "Some Lower Bounds on Signal Parameter Estimation," IEEE Trans. Inform. Theory, *IT-15,* No. 3 (May 1969), pp. 386–391.
8. Ash, R. B., *Information Theory,* New York: Interscience Publishers, 1965.