Second and Third Order Modulation Terms in the Distortion Produced when Noise Modulated FM Waves are Filtered

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This paper is concerned with the distortion produced in a frequency modulation wave when it passes through a filter. The phase or frequency modulation representing the signal is assumed to be a band of gaussian noise. The main result is an expression for the power spectrum $W_\theta(f)$ of the output phase angle $\theta(t)$. This expression holds for any filter, contains all of the distortion terms due to second and third order modulation, and is suited to computer evaluation. It is useful in many cases, but it has the shortcoming of not containing any modulation terms higher than the third order.

A second result is an approximation to $W_\theta(f)$, based on log $(1 + x) \approx x$ (that is, a "first-order" approximation), which is encountered in the derivation of the main result. Although it does not contain all of the second and third order modulation terms, it does contain higher order modulation terms which may give most of the distortion in some cases. The results given here are compared with those obtained earlier.

I. PREFACE

This work is a sequel to "Distortion and Crosstalk of Linearly Filtered Angle-Modulated Signals" by E. Bedrosian of the Rand Corporation and myself.[1] One of the principal results of that paper is an expression for the distortion produced when a frequency modulation wave, modulated by gaussian noise, passes through a filter assumed to be symmetrical about the carrier frequency.

The assumption of symmetry simplified the analysis, but led to zero second order modulation; and consequently the results do not apply to many cases of practical interest.

The main result of this paper is an expression for the distortion which contains all of the second and third order modulation terms produced by a general filter. It includes the earlier result as a special case.

As before, the input phase angle is assumed to be a gaussian noise.

Some time ago Mr. Bedrosian and I worked out, independently, the second order modulation terms. His analysis is somewhat different from mine and throws a different light on the problem. Since each approach is of interest in its own right and since we were unable to combine the two without losing useful results, we decided to publish our work separately. An early version of Bedrosian's analysis is given in a RAND memorandum.[2]

II. INTRODUCTION

When an angle-modulated wave (FM or PM) passes through a filter, the signal becomes distorted. For a multichannel system this distortion may produce crosstalk. In many practical cases the second and third order modulation terms give a good measure of the distortion. These terms have been studied by a number of investigators. In this paper we obtain some general expressions for them for the case in which the modulation is gaussian.

Our main results includes all of the second and third order modulation products. In this respect, it is more general than some of the earlier expressions for the distortion (see Medhurst,[3] Magnusson,[4] and Liou[5]). However, it does not give higher order modulation terms, some of which appear in earlier "first order" approximations. A first order approximation (similar to the earlier ones) occurs in our derivation of the main result. It is stated, along with the main result, in Section III.

As in Ref. 1, the complex form of the filter input is

$$s(t) = \exp [i\omega_0 t + i\varphi(t)] \tag{1}$$

where the carrier frequency is $\omega_0 = 2\pi f_0$ and the signal is carried by the real input phase angle $\varphi(t)$. Let the filter have the transfer function $G(f)$ and the impulse response $g(t)$:

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} G(f) \exp (i\omega t)\, df, \\ G(f) &= \int_{-\infty}^{\infty} g(t) \exp (-i\omega t)\, dt, \qquad \omega = 2\pi f, \end{aligned} \tag{2}$$

where the response $g(t)$ may be complex and may be different from zero when $t < 0$.

The filter is regarded as a bandpass filter for which $G(f)$ is large only near $\pm f_0$. Let normalized functions be defined by

$$\Gamma(f) = \frac{G(f_0 + f)}{G(f_0)}, \qquad \gamma(t) = \frac{g(t) \exp (-i\omega_0 t)}{G(f_0)}. \tag{3}$$

From the definitions and the Fourier relations (2) it follows that

$$\gamma(t) = \int_{-\infty}^{\infty} \Gamma(f) \exp(i\omega t)\, df, \qquad \Gamma(f) = \int_{-\infty}^{\infty} \gamma(t) \exp(-i\omega t)\, dt,$$
$$\int_{-\infty}^{\infty} \gamma(t)\, dt = 1, \qquad \Gamma^*(-f) = \int_{-\infty}^{\infty} \gamma^*(t) \exp(-i\omega t)\, dt, \tag{4}$$

where the asterisk denotes "conjugate complex."

The filter output corresponding to the input $s(t)$ is

$$\begin{aligned} s_0(t) &= \int_{-\infty}^{\infty} g(u)s(t-u)\, du \\ &= \int_{-\infty}^{\infty} g(u) \exp[i\omega_0(t-u) + i\varphi(t-u)]\, du \\ &= \left[G(f_0) \int_{-\infty}^{\infty} \gamma(u) \exp[i\varphi(t-u)]\, du \right] \exp(i\omega_0 t) \\ &= \exp(-\alpha_0 - i\beta_0)\{R(t) \exp[i\theta(t)]\} \exp(i\omega_0 t) \\ &= \exp(-\alpha_0 - i\beta_0)\{\exp[i\Theta(t)]\} \exp(i\omega_0 t). \end{aligned} \tag{5}$$

The definition of $\gamma(u)$ is used in going from the second to the third line. In going from the third to the fourth line, the attenuation and phase shift, α_0 , β_0 at the carrier frequency have been introduced by writing $G(f_0)$ as $\exp(-\alpha_0 - i\beta_0)$. The complex phase angle $\Theta(t)$ is related to the envelope $R(t)\exp(-\alpha_0)$ and phase angle $-\beta_0 + \theta(t)$ of the output [α_0 and β_0 are constants which do not depend on $\varphi(t)$] by

$$\exp[i\Theta(t)] = R(t) \exp[i\theta(t)], \qquad i\Theta(t) = \ln R(t) + i\theta(t),$$
$$\theta(t) = \operatorname{Re} \Theta(t), \qquad \ln R(t) = -\operatorname{Im} \Theta(t). \tag{6}$$

Comparing the third and fifth lines of equation (5) leads to

$$\Theta(t) = -i \ln \left[\int_{-\infty}^{\infty} \gamma(u) \exp[i\varphi(t-u)]\, du \right]. \tag{7}$$

The analysis may be simplified by introducing the "linear portion" $\Phi(t)$ of $\Theta(t)$. Working with the case in which the input $\varphi(t)$ is small gives

$$\begin{aligned} \Theta(t) &= -i \ln \left[1 + i \int_{-\infty}^{\infty} \gamma(u)\varphi(t-u)\, du + \cdots \right] \\ &\approx (-i)i \int_{-\infty}^{\infty} \gamma(u)\varphi(t-u)\, du, \end{aligned}$$

and this leads us to define the linear portion of $\Theta(t)$ as

$$\Phi(t) = \int_{-\infty}^{\infty} \gamma(u)\varphi(t - u)\, du. \tag{8}$$

The complex phase angle $\Theta(t)$ may be separated into its linear and nonlinear portions by adding and subtracting $i\Phi(t)$ in the exponent in equation (7):

$$\Theta(t) = \Phi(t) - i \ln \left[\int_{-\infty}^{\infty} \gamma(u) \exp\,[i\varphi(t - u) - i\Phi(t)]\, du \right]. \tag{9}$$

Adding and subtracting 1 in the integrand and using the fact that the integral of $\gamma(u)$ is unity gives the fundamental relation

$$\Theta(t) = \Phi(t) - i \ln\,[1 + K(t)] \tag{10}$$

where

$$K(t) = \int_{-\infty}^{\infty} du\, \gamma(u) \{\exp\,[i\varphi(t - u) - i\Phi(t)] - 1\}. \tag{11}$$

In most cases of practical interest, $K(t)$ tends to be small. When $| K(t) | < 1$, expansion of the logarithm gives the series

$$\Theta(t) = \Phi(t) - iK(t) + \frac{i}{2} K^2(t) - \frac{i}{3} K^3(t) + \cdots \tag{12}$$

upon which our analysis is based. When $\varphi(t)$ is gaussian, $| K(t) |$ will occasionally exceed unity. It appears that results obtained from the series of equation (12) represent the first few terms of an asymptotic series. This is further discussed in Appendix F.

If $\varphi(t)$ is small for all values of t, expansion of the exponential in the definition of $K(t)$ [equation (11)] shows that $K(t)$ is $0(\varphi^2)$. Our main expression for the distortion, given in Section III, neglects terms of order φ^8. For this accuracy, equation (12) can be written as

$$\Theta(t) = \Phi(t) - iK(t) + \frac{i}{2} K^2(t) + O(\varphi^6). \tag{13}$$

Since the variable portion $\theta(t)$ of the output phase angle is the real part of $\Theta(t)$, the dc portion, θ_{dc}, of $\theta(t)$ is the average value of Re $\Theta(t)$. When the input $\varphi(t)$ is a stationary gaussian process with zero mean,

$$\begin{aligned} \theta_{dc} &= \mathrm{Re}\, \langle \Theta(t) \rangle_{\mathrm{av}} \\ &= \mathrm{Re} \left\langle -iK(t) + \frac{i}{2} K^2(t) \right\rangle_{\mathrm{av}} + O(\varphi^6) \\ &= \mathrm{Im}\, \langle K(t) - 2^{-1}K^2(t) \rangle_{\mathrm{av}} + O(\varphi^6) \end{aligned} \tag{14}$$

where $\langle\ \rangle_{av}$ denotes "ensemble average" and $\langle\Phi(t)\rangle_{av}$ is zero because $\Phi(t)$ depends linearly on $\varphi(t)$. Notice that from equation (5), the total output phase angle is $\hat{\theta}(t) = -\beta_0 + \theta(t)$ and that the dc part of $\hat{\theta}(t)$ is

$$\hat{\theta}_{dc} = -\beta_0 + \theta_{dc} .$$

The two-sided power spectra $W_{\hat{\theta}}(f)$, $W_\theta(f)$ of $\hat{\theta}(t)$, $\theta(t)$ contain the dc spikes $(-\beta_0 + \theta_{dc})^2\ \delta(f)$, $\theta_{dc}^2\ \delta(f)$, respectively. Furthermore,

$$W_{\hat{\theta}}(f) = (\beta_0^2 - 2\beta_0\theta_{dc})\ \delta(f) + W_\theta(f).$$

Here $\delta(f)$ is the unit impulse function.

In the following work it is convenient to ignore β_0, and we shall call $\theta(t)$ itself the output phase angle.

III. STATEMENT OF PRINCIPAL RESULTS

In all of the results stated here, the input phase angle $\varphi(t)$ is gaussian with zero mean. The two-sided power spectrum of $\varphi(t)$ is $W_\varphi(f)$. The two-sided power spectrum of the output phase angle $\theta(t)$ is $W_\theta(f)$.

3.1 *Second and Third Order Modulation Terms in $W_\theta(f)$*

The principal result given in this paper† is an expression for $W_\theta(f)$ which contains all of the second and third order modulation terms:

$$\begin{aligned} W_\theta(f) = {} & \theta_{dc}^2\ \delta(f) + \tfrac{1}{4}W_\varphi(f)\ \lvert U(f) + U^*(-f) \rvert^2 \\ & + \frac{1}{8}\int_{-\infty}^{\infty} d\rho\ W_\varphi(\rho)W_\varphi(f-\rho)\ \lvert T(\rho, f-\rho) - T^*(-\rho, -f+\rho) \rvert^2 \\ & + \frac{1}{24}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\ W_\varphi(\rho)W_\varphi(\sigma)W_\varphi(f-\rho-\sigma) \\ & \cdot \lvert S(\rho, \sigma, f-\rho-\sigma) + S^*(-\rho, -\sigma, -f+\rho+\sigma) \rvert^2 \\ & + O(\varphi^6 W_\varphi). \end{aligned} \tag{15}$$

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† Note added at press time: Equation (15) gives essentially the first few terms of a general expansion due to A. Mircea, Rev. Roum. Sci. Tech.—Electrotechn et Energ., 1967, *t. 12*, No. 3, pp. 359–371, and Proc. IEEE (Correspondence), October 1966, *54*, pp. 1463–1466. I regret the oversight of Mircea's excellent work. Use of his results would have substantially improved this article.

Here the dc part, θ_{dc}, of $\theta(t)$ is the imaginary part of

$$\begin{aligned} D_c = &-\frac{1}{2}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho)S(\rho, -\rho) \\ &-\frac{1}{2}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho)W_\varphi(\sigma)S(-\sigma, -\rho)[\tfrac{3}{2}S(\sigma, \rho) - \Gamma(\sigma + \rho)] \\ &+ O(\varphi^6) \end{aligned} \tag{16}$$

and

$$\begin{aligned} T(\rho, f - \rho) = &\ S(\rho, f - \rho) \\ &+ \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\sigma)[2S(\sigma, \rho)S(-\sigma, f - \rho) - S(\sigma, f - \sigma) \\ &- \Gamma(\sigma)\Gamma(-\sigma)S(\rho, f - \rho) + S(\rho + \sigma, f - \rho - \sigma)] \end{aligned} \tag{17}$$

$$\begin{aligned} U(f) = &\ \Gamma(f) + \int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho)\Gamma(\rho)S(-\rho, f) \\ &+ \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho)W_\varphi(\sigma)\{-\tfrac{1}{2}\Gamma(\rho+\sigma)S(-\rho-\sigma, f) + \Gamma(\sigma) \\ &\cdot[3S(-\sigma, \rho)S(-\rho, f) - S(\rho, f - \rho - \sigma) + S(\rho - \sigma, f - \rho)]\}. \end{aligned} \tag{18}$$

The $\Gamma(f)$ is the normalized filter transfer function defined by equation (3) and the functions S are discussed in Appendix B. They depend only on the filter. That is, they are independent of $W_\varphi(f)$, and are defined symbolically by

$$S(x_1, \cdots, x_n) = \prod_{k=1}^{n} [y^{x_k} - \Gamma(x_k)] \tag{19}$$

where the power y^z of y is to be replaced by $\Gamma(z)$ after multiplying out the product. The S's are symmetric functions of their arguments. For $n = 2$ and 3,

$$\begin{aligned} S(\rho, \sigma) &= y^{\rho+\sigma} - y^\rho\Gamma(\sigma) - y^\sigma\Gamma(\rho) + \Gamma(\rho)\Gamma(\sigma) \\ &= \Gamma(\rho + \sigma) - \Gamma(\rho)\Gamma(\sigma), \\ S(\rho, \sigma, \nu) &= \Gamma(\rho + \sigma + \nu) - \Gamma(\rho + \sigma)\Gamma(\nu) \\ &\quad - \Gamma(\rho + \nu)\Gamma(\sigma) - \Gamma(\sigma + \nu)\Gamma(\rho) + 2\Gamma(\rho)\Gamma(\sigma)\Gamma(\nu). \end{aligned} \tag{20}$$

The $S(\rho, \sigma, \nu)$ of Ref. 1 is the negative of the one used here.

In the "order of" symbol appearing in equation (15) the W_φ enters for dimensional reasons. This is in line with

$$\int_{-\infty}^{\infty} df\, W_\varphi(f) = \langle \varphi^2(t) \rangle_{av} \tag{21}$$

as can be seen by integrating both sides of equation (15) from $f = -\infty$ to $f = \infty$.

In some instances the expression for $W_\theta(f)$ is useful when rms $\varphi(t)$ is large but rms $d\varphi(t)/dt$ is small.

Bedrosian has computed curves showing the second order distortion for the case of quadratic phase shift, that is, for $\Gamma(f)$ of the form $\exp[iAf^2]$.[2]

As equation (15) stands, it is oriented towards phase modulation. For frequency modulation the time derivatives $\varphi'(t) = d\varphi(t)/dt$, $\theta'(t) = d\theta(t)/dt$ replace the phase angles $\varphi(t)$, $\theta(t)$ as the items of interest. Since the power spectrum $W_{\theta'}(f)$ of $\theta'(t)$ is equal to $(2\pi f)^2 W_\theta(f)$, multiplying (15) by $(2\pi f)^2$ converts it into an expression for $W_{\theta'}(f)$. On the right side of (15), the factor $W_\varphi(f)$ in the first line is replaced by $W_{\varphi'}(f)$ and the W_φ's appearing in the integrands may be transformed into $W_{\varphi'}$'s without introducing infinities. The last statement is seen to be true for the second order modulation integral when we write

$$W_\varphi(\rho) W_\varphi(f - \rho) = W_{\varphi'}(\rho) W_{\varphi'}(f - \rho)\ (2\pi)^{-4} \rho^{-2} (f - \rho)^{-2}$$

and observe that the product $\rho^{-1}(f - \rho)^{-1} T(\rho, f - \rho)$ remains finite even when ρ and $(f - \rho)$ approach zero. The third order term may be treated in a similar way.

In many applications $W_{\varphi'}(f)$ is proportional to D^2 where

$$D^2 = \left\langle \left[\frac{\varphi'(t)}{2\pi} \right]^2 \right\rangle_{av}$$

and D is the rms frequency deviation in cycles per second. Then the second and third order modulation integrals in (15) are proportional to D^4 and D^6, respectively, as D tends to zero. This suggests that the remainder term, $0(\varphi^6 W_\varphi)$, is proportional to D^8. For this reason we shall sometimes refer to (15) as the "small deviation" approximation. When the FM signal to crosstalk ratio in dB is plotted as a function of log D, the behavior of the resulting curve as $D \to 0$ can be computed from (15). Indeed, if the second order modulation predominates, (15) furnishes an asymptote to the curve with a slope of 6 dB per octave. If, because of symmetry in the filter, the second order modulation term in (15) is zero, the third order term gives an asymptote with a slope of 12 dB per octave.

Some idea of how equation (15) begins to fail as D increases from 0 may be obtained by considering the case $\varphi(t) = A \sin \omega_a t$. Then the rms frequency deviation is $D = f_a A/2^{\frac{1}{2}}$ and the filter output is

$$s_0(t) = \sum_{-\infty}^{\infty} J_n(A) G(f_0 + nf_a) \exp [i(\omega_0 + n\omega_a)t]$$

where $J_n(A)$ is a Bessel function and $\omega_a = 2\pi fa$. Consider only the second harmonic. It is proportional to $J_2(A)$, and the approximation underlying (15) is roughly equivalent to replacing $J_2(A)$ by $A^2/8$, the leading term in its power series. The value of A which makes $A^2/8$ exceed $J_2(A)$ by 3 dB is $A \approx 2.0$ and the corresponding D is 1.4 f_a . If the baseband of a gaussian FM wave were flat and extended from 0 to B, the expected number of zeros per second would be 1.16 B. This is the same as the number of zeros of $A \sin \omega_a t$ with $f_a = 0.58\ B$. This representative value of f_a leads to the estimate that (15) will be in error by 3 dB when $D \approx (1.4)\ (0.58)B \approx 0.8\ B$. Comparison of (15) with experimental values indicates that the 3 dB error point typically occurs when D lies between $B/2$ and B.

3.2 *Power Spectrum of $a\theta(t) + b \ln R(t)$.*

Equation (15) for $W_\theta(f)$ may be modified to give information regarding the fluctuation of the envelope $R(t)$. This information may be of interest, say, in determining the distortion produced by "AM to PM conversion."[5] More generally, suppose that one is interested in the power spectrum $W_x(f)$ of

$$x(t) = a\theta(t) + b \ln R(t) = \text{Re}\ [(a + ib)\Theta(t)] \tag{22}$$

where a and b are arbitrary real constants. Then $W_x(f)$ is given by an expression obtained from equation (15) upon replacing $U(f)$, $T(\rho, f-\rho)$ and $S(\rho, \sigma, f-\rho-\sigma)$ by $(a+ib)U(f)$, $(a+ib)T(\rho, f-\rho)$, and $(a+ib)S(\rho, \sigma, f-\rho-\sigma)$, respectively, so that $U^*\ (-f)$ is replaced by $(a-ib)U^*\ (-f)$, and so on. (See Appendix E.)

3.3 *Second and Third Order Modulation Terms for "Small and Slow" Frequency Deviations*

The expression (15) for $W_\theta(f)$ simplifies when

(*i*) $\Gamma(f)$ can be expanded as a power series

$$\Gamma(f) = \frac{G(f_0 + f)}{G(f_0)} = 1 + \sum_{n=1}^{\infty} \frac{\alpha_n f^n}{n!} \tag{23}$$

and

(*ii*) the effective spread of $W_\varphi(f)$ is so small that the Γ's used in equations (15) to (20) can be replaced by the first few terms of their power series expansion. Roughly, this means that the top baseband frequency is small compared with the filter bandwidth. The instantaneous frequency changes slowly in comparison with the envelope of the impulse response of the filter, and the quasistatic case is approached. With these assumptions the resulting simplified form of $W_\theta(f)$ is given by equation (126). A more complete form of the small and slow deviation approximation is given in equation (133) which brings out the asymptotic nature of the results.

The sum of the second and third order modulation terms given by the integrals in equation (126) [which are the simplified versions of the corresponding integrals in equation (15)] is

$$W_\theta^I(f) = 2^{-1}(\lambda_{2i} + 2^{-1}D^2\lambda_{4i})^2 \int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho)W_\varphi(f-\rho)\rho^2(f-\rho)^2$$
$$+ 6^{-1}(\lambda_{3i})^2 \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho)W_\varphi(\sigma)W_\varphi(f-\rho-\sigma)\rho^2\sigma^2(f-\rho-\sigma)^2. \tag{24}$$

Here $W_\theta^I(f)$ is the portion of $W_\theta(f)$ which gives the interchannel interference, that is, the noise a listener would hear in an idle channel in a multichannel frequency division multiplex angle-modulation system. D is the rms frequency deviation in cps,

$$D^2 = \left\langle \left[\frac{\varphi'(t)}{2\pi}\right]^2 \right\rangle_{av} \tag{25}$$

where $\langle\ \rangle_{av}$ denotes ensemble average. The quantities λ_{ni} are the imaginary parts of the semi-invariants λ_n defined by the expansion

$$\ln \Gamma(f) = \sum_{n=1}^{\infty} \frac{\lambda_n f^n}{n!}. \tag{26}$$

Equations (125) express the first five λ_n's in terms of the first five α_n's, the coefficients in the expansion of $\Gamma(f)$.

The corresponding approximation for the power spectrum $W_x^I(f)$ of $x = a\theta(t) + b \ln R(t)$ [see equation (22)] is obtained by replacing λ_{ni} in equation (24) by $(a\lambda_{ni} + b\lambda_{nr})$ where λ_{nr} denotes the real part of λ_n.

In the important FM case in which the baseband signal $\varphi'(t)$ has a flat power spectrum, the two-sided power spectrum of $\varphi'(t)$ may be taken to be

$$W_{\varphi'}(f) = \begin{cases} 0, & |f| > B \\ (2\pi D)^2/(2B), & |f| \leqq B \end{cases} \tag{27}$$

where the baseband extends from $f = 0$ to $f = B$ (that is, from $-B$ to $+B$ for two-sided spectra), and D is the rms frequency deviation. Then

$$W_{\varphi}(f) = W_{\varphi'}(f)/(2\pi f)^2 = \begin{cases} 0, & |f| > B \\ D^2/(2Bf^2), & |f| \leqq B. \end{cases} \tag{28}$$

When this expression is used, the integrals in equation (24) may be evaluated and it is found that, for $0 \leqq |f| \leqq B$,

$$W_{\theta}^{I}(f) = \frac{D^4}{8B^2}(2B - |f|)(\lambda_{2i} + 2^{-1}D^2\lambda_{4i})^2 + \frac{D^6}{48B^3}(3B^2 - f^2)(\lambda_{3i})^2, \qquad 0 \leqq |f| \leqq B. \tag{29}$$

The average signal power in an elementary frequency band extending from f to $f + \Delta f$ in the input base band is $W_{\varphi'}(f)\,\Delta f$ (radians per second)^2. The ratio of the interference power to the output power in the same elementary band is

$$\frac{W_{\theta'}^{I}(f)\,\Delta f}{W_{\theta'}(f)\,\Delta f} = \frac{W_{\theta}^{I}(f)}{W_{\theta}(f)}. \tag{30}$$

For the flat baseband FM case we may approximate $W_{\theta}(f)$ by $W_{\varphi}(f) = D^2/(2Bf^2)$ and use equation (29). This leads to the approximation

$$\frac{W_{\theta'}^{I}(f)\,\Delta f}{W_{\theta'}(f)\,\Delta f} = \frac{f^2D^2}{4}\left[\left(2 - \frac{|f|}{B}\right)(\lambda_{2i} + 2^{-1}D^2\lambda_{4i})^2 + \frac{D^2}{6}\left(3 - \frac{f^2}{B^2}\right)(\lambda_{3i})^2\right], \tag{31}$$

for the ratio of the interchannel interference power to the signal power in the elementary band $(f, f + \Delta f)$. This ratio has meaning only if $|f| \leqq B$.

Liou has given an approximation which is equivalent to equation (24) for $W_{\theta}^{I}(f)$ with several more terms included.[5] This approximation is discussed in Section XII.

The small and slow deviation approximation described above gives results which agree well with Monte Carlo computations made by C. L. Ruthroff.[6] For illustration we take the simplest of Ruthroff's cases, the one in which the transfer admittance of the filter is

$$G(f) = \frac{1}{1 + 2x + 2x^2 + x^3}, \qquad x = i(f - f_o)/f_c . \tag{32}$$

Here f_o is the carrier frequency and f_c is the filter semibandwidth.

Putting $f = f_o$ gives $G(f_o) = 1$. Putting $f = f' + f_o$ gives $x = if'/f_c$ and

$$\begin{aligned} \ln \Gamma(f') &= \ln \frac{G(f' + f_o)}{G(f_o)} \\ &= -\ln(1 + 2x + 2x^2 + x^3) \\ &= -2\frac{x}{1!} + 2\frac{x^3}{3!} - 48\frac{x^5}{5!} + \cdots \\ &= \sum_{n=1}^{\infty} \lambda_n \frac{f'^n}{n!}. \end{aligned} \tag{33}$$

The last line follows from the series of equation (26) defining the λ_n's as the coefficients in the expansion of $\ln \Gamma(f')$. In going from line 2 to line 3, the logarithm is expanded by setting $\alpha_1 = 2$, $\alpha_2 = 4$, $\alpha_3 = 6$, $\alpha_n = 0$ for $n > 3$ in

$$\ln\left(1 + \sum_1^{\infty} \alpha_n x^n/n!\right) = \sum_1^{\infty} \lambda_n x^n/n!$$

and by using the expressions (125) for the λ_n's.

Substituting $x = if'/f_c$ in line 3 of equation (33) and comparing the result with the last line gives

$$\lambda_1 = -2(if_c^{-1}), \qquad \lambda_2 = 0, \qquad \lambda_3 = 2(if_c^{-1})^3, \qquad \lambda_4 = 0.$$

Hence $\lambda_{2i} = 0$, $\lambda_{3i} = -2f_c^{-3}$, $\lambda_{4i} = 0$, and the approximation of equation (31) for the ratio of the interchannel interference power to the signal power leads to

$$-10 \log_{10} \frac{W_\theta^I(f)\,\Delta f}{W_\theta(f)\,\Delta f} \approx -10 \log_{10} \left[\frac{f^2 D^4}{6 f_c^6}\left(3 - \frac{f^2}{B^2}\right)\right]. \tag{34}$$

In his Fig. 15 Ruthroff has plotted values of

$$-10 \log_{10} [W_\theta^I(f)\,\Delta f / W_\theta(f)\,\Delta f]$$

for several different values of D/B and f/B with $B = 7$ MHz and

$f_c = 119$ MHz.[6] The agreement with our equation (34) is good at $f/B = 1.0$. Our D/B is the same as Ruthroff's σ/W and our $f/B = 1.0$ corresponds to Ruthroff's slot 10. At $f/B = 0.4$, equation (34) gives values which are about 3 or 4 decibels less than the Monte Carlo values, but this may still be regarded as good agreement.

Similar agreement is found when the small and slow deviation approximation is applied to a number of the other cases examined by Ruthroff.

3.4 *A "First Order" Approximation*

Although equation (15) is useful in some FM distortion problems, in some cases it is of no help. One example concerns the distortion produced by an ideal filter centered on the carrier frequency and having a semibandwidth exceeding $3B$, where B is the baseband. That is, $W_\varphi(f)$ is 0 for $|f| > B$. In this case the distortion is produced by modulation terms of order higher than the third, and these are neglected in equation (15).

For such problems "first order" approximations can sometimes be used. The term "first order" refers to the approximation $\ln(1 + x) \approx x$ where x is of the nature of $K(t)$ in equation (10); it does not refer to the order of the modulation products in x. Different choices of x lead to different first order approximations. The first order approximation given by the first two terms in the series of equation (12) for $\Theta(t)$ is

$$\theta(t) \approx \text{Re}\,\Phi(t) + \text{Im}\,K(t). \tag{35}$$

The output phase angle $\theta(t)$ may be written as the sum

$$\theta(t) = \theta_{dc} + \theta_\ell(t) + \theta_{n\ell}(t)$$

where $\theta_\ell(t) = \text{Re}\,\Phi(t)$ is the "linear portion" of $\theta(t)$, and $\theta_{n\ell}(t)$ given by

$$\begin{aligned}\theta_{n\ell}(t) &= \theta(t) - \theta_\ell(t) - \theta_{dc} \\ &= \theta(t) - \text{Re}\,\Phi(t) - \theta_{dc}\end{aligned} \tag{36}$$

is the time varying part of the "nonlinear distortion" in $\theta(t)$. The first order approximation for $\theta_{n\ell}(t)$ corresponding to the first order approximation of equation (35) for $\theta(t)$ is

$$\theta_{n\ell}(t) \approx \text{Im}\,K(t) - \theta_{dc} \approx \text{Im}\,K(t) - \text{Im}\,\langle K(t)\rangle_{\text{av}} = y(t) \tag{37}$$

where

$$y(t) = \text{Im}\,[K(t) - \langle K(t)\rangle_{\text{av}}]. \tag{38}$$

The work of Section VI, which is part of the derivation of equation

(15), shows that the power spectrum of $y(t)$ is

$$W_y(f) = 2^{-1} \operatorname{Re} [P(f) + Q(f) + P^*(-f) + Q^*(-f)] \tag{39}$$

where

$$
\begin{aligned}
P(f) &= -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \exp(-i\omega\tau) \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma(u)\gamma(v) \\
&\qquad \cdot \exp[a(u) + a(v)]\{\exp[2c(u, v, \tau)] - 1\} \\
Q(f) &= \frac{1}{2} \int_{-\infty}^{\infty} d\tau \exp(-i\omega\tau) \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma^*(u)\gamma(v) \\
&\qquad \cdot \exp[a^*(u) + a(v)]\{\exp[2\hat{c}(u, v, \tau)] - 1\} \\
a(u) &= -\frac{1}{2} \int_{-\infty}^{\infty} df\, W_\varphi(f) H_u(f) H_u(-f) \\
c(u, v, \tau) &= -\frac{1}{2} \int_{-\infty}^{\infty} df\, W_\varphi(f) H_u(-f) H_v(f) \exp(i\omega\tau) \\
\hat{c}(u, v, \tau) &= \frac{1}{2} \int_{-\infty}^{\infty} df\, W_\varphi(f) H_u^*(f) H_v(f) \exp(i\omega\tau) \\
H_u(f) &= \exp(-i\omega u) - \Gamma(f), \qquad \omega = 2\pi f.
\end{aligned} \tag{40}
$$

The function $Q(f)$ is real when f is real, and $P(f)$ is an even function of f.

The first order approximation $W_y(f)$ for the power spectrum of $\theta_{nt}(t)$ contains some higher order modulation terms which are not contained in our main equation (15) for $W_\theta(f)$; conversely, equation (15) contains terms which are not in $W_y(f)$. In using the first order approximation of equation (35), which may be rewritten as

$$\theta(t) = \operatorname{Re} \Phi(t) + \operatorname{Im} K(t) + O(K^2),$$

one should guard against throwing away* [in the $O(K^2)$ terms] quantities which are of the same order as those being computed from $\operatorname{Re} \Phi(t) + \operatorname{Im} K(t)$ (the leading term). Although each case requires its own investigation, it is helpful to remember that $K(t)$ is $O(\varphi^2)$. Furthermore, when $\gamma(t)$ is real, $\operatorname{Im} K(t)$ is $O(\varphi^3)$. Also when $\Gamma(f) \equiv 1$, $K(t)$ becomes 0; and when

$$|\Gamma(f) - 1| < \epsilon \ll 1$$

for all real values of f (as in the case of small wave-guide echoes), it

* This type of error has been discussed by Enloe, Ruthroff, Gladwin, and Medhurst in Refs. 7 and 8.

may be conjectured that $K(t)$ itself is $O(\epsilon)$ irrespective of how large $\langle\varphi^2\rangle_{av}$ may be.

The approximation $\theta_{nl}(t) \approx y(t)$, is not quite the same as earlier first order approximations.[3,4,9-12] It is a closer, but more complicated, approximation because $\varphi(t - u) - \Phi(t)$ is used in the integral equation (11) for $K(t)$ instead of $\varphi(t - u) - \varphi(t)$ as in the earlier approximations. Appendix D gives some results obtained when the present analysis is repeated with $\varphi(t - u) - \varphi(t)$ in place of $\varphi(t - u) - \Phi(t)$.

Equation (39) for $W_y(f)$ gives a first order approximation for the power spectrum of $\theta_{nl}(t) = \theta(t) - \text{Re}\,\Phi(t) - \theta_{dc}$. The corresponding approximation for the power spectrum of

$$\ln R(t) + \text{Im}\,\Phi(t) - [\ln R(t)]_{dc} \approx \text{Re}\,[K(t) - \langle K(t)\rangle_{av}] = x(t)$$

is

$$W_x(f) = 2^{-1}\,\text{Re}\,[-P(f) + Q(f) - P^*(-f) + Q^*(-f)].$$

3.5 *Simplification When Filter has Symmetry* $\Gamma(-f) = \Gamma^*(f)$

When the filter has the symmetry

$$\Gamma(-f) = \Gamma^*(f) \tag{41}$$

about the carrier frequency, the even order modulation terms disappear, $S^*(-x_1, \cdots, -x_n)$ becomes equal to $S(x_1, \cdots, x_n)$, and equation (15) becomes

$$\begin{aligned} W_\theta(f) &= W_\varphi(f)\,|\,U(f)\,|^2 \\ &\quad + \frac{1}{6}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\; W_\varphi(\rho)W_\varphi(\sigma)W_\varphi(f - \rho - \sigma) \\ &\qquad \cdot |\,S(\rho, \sigma, f - \rho - \sigma)\,|^2 + O(\varphi^6 W_\varphi). \end{aligned} \tag{42}$$

Here $U(f)$ is still given by equation (18) and $S(\rho, \sigma, \nu)$ by equation (20). This expression for $W_\theta(f)$ agrees with one of the main results of Ref. 1 when the double integral in equation (18) for $U(f)$ is assumed to be so small that it may be neglected.

When $\Gamma(-f)$ is equal to $\Gamma^*(f)$, the coefficients α_n in the power series of equation (23) for $\Gamma(f)$ are real when n is even, and imaginary when n is odd. The same is true for the λ_n's of equation (26). Hence λ_{2i}, λ_{4i} are zero and the second order modulation terms disappear from the small and slow deviation approximations equations (24), (29), and (31) for $W_\theta^I(f)$.

The relation $\Gamma(-f) = \Gamma^*(f)$ implies that $\gamma(u)$ is real and that $H_u(-f)$ is equal to $H_u^*(f)$. Then $a(u)$, $c(u, v, \tau)$, and $\hat{c}(u, v, \tau)$ are real and $\hat{c}(u, v, \tau)$

is equal to $-c(u, v, \tau)$. Both $P(f)$ and $Q(f)$ become even and real. Equation (39) for the power spectrum of $y(t)$, that is, the first order approximation for the power spectrum of the nonlinear distortion $\theta_{nl}(t)$, becomes

$$W_y(f) = P(f) + Q(f). \tag{43}$$

Here the triple integral for $P(f)$ is the same as that given in equation (40); and the triple integral for $Q(f)$ may be obtained from the integral for $P(f)$ by changing the sign of $2c(u, v, \tau)$. Hence equation (43) becomes

$$W_y(f) = -\int_{-\infty}^{\infty} d\tau \exp(-i\omega\tau) \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma(u)\gamma(v) \cdot \exp[a(u) + a(v)] \sinh[2c(u, v, \tau)] \tag{44}$$

where $\omega = 2\pi f$, and $a(u)$, $c(u, v, \tau)$ are given by equation (40).

IV. INITIAL EXPRESSION FOR THE POWER SPECTRUM OF $\theta(t)$

When $\theta(t)$ is a stationary noise process its two-sided power spectrum $W_\theta(f)$ is the Fourier transform of its autocovariance:

$$W_\theta(f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau)\langle\theta(t)\theta(t+\tau)\rangle_{\text{av}}\, d\tau, \qquad \omega = 2\pi f. \tag{45}$$

Denoting functions with arguments t, $t + \tau$ by subscripts 1, 2 and using

$$\theta(t) = \operatorname{Re}\Theta(t) = 2^{-1}[\Theta(t) + \Theta^*(t)] \tag{46}$$

gives

$$\begin{aligned} \langle\theta(t)\theta(t+\tau)\rangle_{\text{av}} &= \langle\theta_1\theta_2\rangle_{\text{av}} \\ &= 2^{-1}\operatorname{Re}[\langle\Theta_1\Theta_2\rangle_{\text{av}} + \langle\Theta_1^*\Theta_2\rangle_{\text{av}}]. \end{aligned} \tag{47}$$

The procedure of Appendix A and equation (13) for the complex phase angle $\Theta(t)$ lead to

$$\begin{aligned} \langle\Theta_1\Theta_2\rangle_{\text{av}} &= A(\tau) + A(-\tau) + O(\varphi^8) \\ \langle\Theta_1^*\Theta_2\rangle_{\text{av}} &= B(\tau) + B^*(-\tau) + O(\varphi^8) \end{aligned} \tag{48}$$

where $A(\tau)$ and $B(\tau)$ are the ensemble averages

$$\begin{aligned} A(\tau) &= \langle\Phi_1(2^{-1}\Phi_2 - iK_2 + i2^{-1}K_2^2) - 2^{-1}K_1(K_2 - K_2^2)\rangle_{\text{av}} \\ B(\tau) &= \langle\Phi_1^*(2^{-1}\Phi_2 - iK_2 + i2^{-1}K_2^2) + 2^{-1}K_1^*(K_2 - K_2^2)\rangle_{\text{av}}. \end{aligned} \tag{49}$$

The remainder terms in equation (48) are $O(\varphi^8)$ instead of $O(\varphi^7)$ because the ensemble average of an odd order term is zero.

It also follows from Appendix A that equation (45) for the power spectrum of $\theta(t)$ goes into

$$W_\theta(f) = 2^{-1} \operatorname{Re} [P(f) + Q(f) + P^*(-f) + Q^*(-f)] + O(\varphi^6 W_\varphi) \quad (50)$$

$$P(f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau) A(\tau)\, d\tau$$
$$Q(f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau) B(\tau)\, d\tau. \quad (51)$$

Although the functions $P(f)$, $Q(f)$ used here are not the same as those in Section III, they are of the same nature.

V. CALCULATION OF AVERAGES NEEDED FOR COVARIANCES

The equations of Section IV show that the value of $\langle \theta(t)\theta(t + \tau)\rangle_{\text{av}}$ depends upon various ensemble averages of products of $\Phi(t)$ and $K(t)$. When the input phase angle $\varphi(t)$ is gaussian, these averages may be computed by using a result proved in Ref. 1.

Let L be a linear operator which operates on functions of t, and let

$$L \exp(i\omega t) = \exp(i\omega t)\, \ell(f), \qquad \omega = 2\pi f. \quad (52)$$

Let $\varphi(t)$ be a stationary gaussian process with two-sided power spectrum $W_\varphi(f)$. Then

$$\langle \exp[iL\varphi(t)]\rangle_{\text{av}} = \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)\ell(f)\ell(-f)\right]. \quad (53)$$

Setting $xL\varphi(t)$ for $L\varphi(t)$ and comparing coefficients of x^2 in the power series expansions of the two sides of equation (53) shows that

$$\int_{-\infty}^{\infty} df\, W_\varphi(f)\ell(f)\ell(-f) = \langle [L\varphi(t)]^2\rangle_{\text{av}}\ .$$

That $\langle \exp[iL\varphi(t)]\rangle_{\text{av}}$ is equal to $\exp\{-2^{-1}\langle [L\varphi(t)]^2\rangle_{\text{av}}\}$ follows from the fact that the real and imaginary parts of $L\varphi(t)$ are correlated gaussian processes.

In dealing with $K(t)$ it is convenient to introduce the function $J(v, \tau)$ defined by

$$J(v, \tau) = \exp[i\varphi(t + \tau - v) - i\Phi(t + \tau)]. \quad (54)$$

The dependence of $J(v, \tau)$ on t is ignored because the right side of equation (54) is a stationary random process, and $J(v, \tau)$ will be used only to calculate ensemble averages. The examples, which follow from the definition of equation (11) of $K(t)$,

$$\begin{aligned}
\langle K_1\rangle_{av} &= \langle K(t)\rangle_{av} = \int_{-\infty}^{\infty} du\, \gamma(u)\langle J(u, o) - 1\rangle_{av} \\
\langle K_1K_2\rangle_{av} &= \langle K(t)K(t+\tau)\rangle_{av} \\
&= \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma(u)\gamma(v)\langle[J(u, o) - 1][J(v, \tau) - 1]\rangle_{av} \\
\langle K_1^*K_2\rangle_{av} &= \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma^*(u)\gamma(v)\langle[J^*(u, o) - 1][J(v, \tau) - 1]\rangle_{av}
\end{aligned} \tag{55}$$

show that averages of the type $\langle J(u, o)\rangle_{av}$, $\langle J(u, o)J(v, \tau)\rangle_{av}$, and $\langle J^*(u, o)J(v, \tau)\rangle_{av}$ are needed.

To calculate $\langle J(u, o)\rangle_{av}$ from the general result, equation (53), let L be the operator which carries $\varphi(t)$ into $\varphi(t - u) - \Phi(t)$. Replacing $\Phi(t)$ by the integral which defines it gives

$$L\varphi(t) = \varphi(t - u) - \int_{-\infty}^{\infty} ds\, \gamma(s)\varphi(t - s).$$

The function $\ell(f)$ associated with L is obtained by setting $\exp(i\omega t)$ in place of $\varphi(t)$:

$$\begin{aligned}
\exp(i\omega t)\ell(f) &= L[\exp(i\omega t)] \\
&= \exp[i\omega(t - u)] - \int_{-\infty}^{\infty} ds\, \gamma(s) \exp[i\omega(t - s)], \\
\ell(f) &= \exp(-i\omega u) - \Gamma(f) \equiv H_u(f).
\end{aligned}$$

Then equation (53) gives

$$\begin{aligned}
\langle \exp[iL\varphi(t)]\rangle_{av} &= \langle \exp[i\varphi(t - u) - i\Phi(t)]\rangle_{av} \\
&= \langle J(u, o)\rangle_{av} = \exp[a(u)]
\end{aligned} \tag{56}$$

where

$$a(u) = -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)H_u(f)H_u(-f). \tag{57}$$

The functions $a(u)$ and

$$H_u(f) = \exp(-i2\pi fu) - \Gamma(f) \tag{58}$$

play important parts in the analysis. The present $H_u(f)$ is the negative of the one used in Reference 1, a change made to simplify the analysis.

The calculation of $\langle J(u, o)J(v, \tau)\rangle_{av}$ proceeds in much the same way. Let

$$\begin{aligned} L\varphi(t) &= \varphi(t-u) - \Phi(t) + \varphi(t+\tau-v) - \Phi(t+\tau), \\ \ell(f) &= H_u(f) + \exp(i\omega\tau)H_v(f), \\ -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)\ell(f)\ell(-f) & \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)[H_u(f)H_u(-f) + H_v(f)H_v(-f) \\ &\qquad + 2\exp(i\omega\tau)H_u(-f)H_v(f)] \\ &= a(u) + a(v) + 2c(u, v, \tau) \end{aligned} \tag{59}$$

where $W_\varphi(f)$ is an even function of f and $c(u, v, \tau)$ is the integral

$$c(u, v, \tau) = -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)H_u(-f)H_v(f)\exp(i2\pi f\tau). \tag{60}$$

Consequently,

$$\langle J(u, o)J(v, \tau)\rangle_{av} = \exp[a(u) + a(v) + 2c(u, v, \tau)]. \tag{61}$$

Similarly, to calculate $\langle J^*(u, o)J(v, \tau)\rangle_{av}$ let

$$L\varphi(t) = -\varphi(t-u) + \Phi^*(t) + \varphi(t+\tau-v) - \Phi(t+\tau), \tag{62}$$

$$\begin{aligned} \ell(f) &= -\exp(-i\omega u) + \int_{-\infty}^{\infty} ds\, \gamma^*(s)\exp(-i\omega s) + \exp(i\omega\tau)H_v(f) \\ &= -H_u^*(-f) + \exp(i\omega\tau)H_v(f) \end{aligned}$$

where the Fourier transform of $\gamma^*(s)$ is $\Gamma^*(-f)$. The work of equations (59) and (60) goes through much as before with $-H_u^*(-f)$ in place of $H_u(f)$. The result is

$$\langle J^*(u, o)J(v, \tau)\rangle_{av} = \exp[a^*(u) + a(v) + 2\hat{c}(u, v, \tau)] \tag{63}$$

where

$$\hat{c}(u, v, \tau) = \frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)H_u^*(f)H_v(f)\exp(i2\pi f\tau). \tag{64}$$

All of the averages needed are given in Tables I and II. Items 1, 3, and 6 in Table I have just been computed, and the others may be ob-

TABLE I—ENSEMBLE AVERAGES OF PRODUCTS OF $J(v, \tau)$'S

No.	Average	$l(f)$	Value
1	$\langle J(u, o)\rangle_{av}$	$H_u(f)$	$\exp[a(u)]$
2	$\langle J(v, \tau)\rangle_{av}$	$\exp(i\omega\tau)H_v(f)$	$\exp[a(v)]$
3	$\langle J(u, o)J(v, \tau)\rangle_{av}$	$H_u(f) + \exp(i\omega\tau)H_v(f)$	$\exp[a(u) + a(v) + 2c(u, v, \tau)]$
4	$\langle J(u, o)J(v, \tau)J(w, \tau)\rangle_{av}$	$H_u(f) + \exp(i\omega\tau)[H_v(f) + H_w(f)]$	$\exp[a(u) + a(v) + a(w) + 2c(u, v, \tau) + 2c(u, w, \tau) + 2c(w, v, o)]$
5	$\langle J^*(u, o)\rangle_{av}$	$-H_u^*(-f)$	$\exp[a^*(u)]$
6	$\langle J^*(u, o)J(v, \tau)\rangle_{av}$	$-H_u^*(-f) + \exp(i\omega\tau)H_v(f)$	$\exp[a^*(u) + a(v) + 2\hat{c}(u, v, \tau)]$
7	$\langle J^*(u, o)J(v, \tau)J(w, \tau)\rangle_{av}$	$-H_u^*(-f) + \exp(i\omega\tau)[H_v(f) + H_w(f)]$	$\exp[a^*(u) + a(v) + a(w) + 2\hat{c}(u, v, \tau) + 2\hat{c}(u, w, \tau) + 2c(w, v, o)]$

TABLE II—ENSEMBLE AVERAGES OF PRODUCTS CONTAINING $\varphi(t - u)$

No.	Average	$l(f)$	Value
1	$\langle \varphi(t - u)\varphi(t + \tau - v)\rangle_{av}$		$\int_{-\infty}^{\infty} df W_\varphi(f) \exp[i\omega(\tau - v + u)]$
2	$\langle \varphi(t - u)J(v, \tau)\rangle_{av}$	$x \exp(-i\omega u) + \exp(i\omega\tau)H_v(f)$	$i\int_{-\infty}^{\infty} df W_\varphi(f) \exp[i\omega(\tau + u)]H_v(f) \exp[a(v)]$
3	$\langle \varphi(t - u)J(v, \tau)J(w, \tau)\rangle_{av}$	$x \exp(-i\omega u) + \exp(i\omega\tau)[H_v(f) + H_w(f)]$	$i\int_{-\infty}^{\infty} df W_\varphi(f) \exp[i\omega(\tau + u)][H_v(f) + H_w(f)] \exp[a(v) + a(w) + 2c(w, v, o)]$

tained in a similar manner. The entries in the last column of Table I may be verified by expressing the a's and c's as ensemble averages [see equation (67)] and using $\langle \exp [iL\varphi(t)] \rangle_{av} = \exp \{-2^{-1} \langle [L\varphi(t)]^2 \rangle_{av}\}$.

Table II gives averages of products in which one factor is $\varphi(t - u)$. The first average, $\langle \varphi(t - u)\varphi(t + \tau - v) \rangle_{av}$, is the Fourier transform of $W_\varphi(f)$. The second average, $\langle \varphi(t - u)J(v, \tau) \rangle_{av}$, is the coefficient of ix in the expansion of $\exp \langle [iL\varphi(t)] \rangle$ where

$$\begin{aligned} L\varphi(t) &= x\varphi(t - u) + \varphi(t + \tau - v) - \Phi(t + \tau) \\ \ell(f) &= x \exp(-i\omega u) + \exp(i\omega\tau)H_v(f). \end{aligned} \tag{65}$$

The third average may be computed in a similar way.

The following list brings together the integrals $a(v)$, $c(u, v, \tau)$, $\cdots$ which appear in Tables I and II:

$$\begin{aligned} a(u) &= -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)H_u(f)H_u(-f) \\ c(u, v, \tau) &= -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)H_u(-f)H_v(f) \exp(i2\pi f\tau) \\ a^*(u) &= -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)H_u^*(f)H_u^*(-f) \\ \hat{c}(u, v, \tau) &= \frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)H_u^*(f)H_v(f) \exp(i2\pi f\tau) \end{aligned} \tag{66}$$

where $H_u(f) = \exp(-i2\pi fu) - \Gamma(f)$, and replacing $H_u(f)$ by $-H_u^*(-f)$ in $a(u)$, $c(u, v, \tau)$ gives $a^*(u)$, $\hat{c}(u, v, \tau)$. Also

$$\begin{aligned} a(u) &= c(u, u, o), \qquad c(w, v, o) = c(v, w, o), \\ c(u, v, -\tau) &= c(v, u, \tau), \qquad \hat{c}(u, v, -\tau) = \hat{c}^*(v, u, \tau), \\ c(u, v, \tau) &= -\tfrac{1}{2}\langle [\varphi(t - u) - \Phi(t)][\varphi(t + \tau - v) - \Phi(t + \tau)] \rangle_{av} \\ \hat{c}(u, v, \tau) &= \tfrac{1}{2}\langle [\varphi(t - u) - \Phi^*(t)][\varphi(t + \tau - v) - \Phi(t + \tau)] \rangle_{av}\,. \end{aligned} \tag{67}$$

When $\Gamma(-f)$ is equal to $\Gamma^*(f)$, both $\gamma(t)$ and $\Phi(t)$ are real; and it follows that $a(u)$, $c(u, v, \tau)$, $\hat{c}(u, v, \tau)$ are also real. Furthermore, $H_u(-f) = H_u^*(f)$ and $\hat{c}(u, v, \tau) = -c(u, v, \tau)$.

VI. THE POWER SPECTRUM OF $K(t)$

The dc portion of the complex random process $K(t)$ defined by the integral in equation (11) is the complex constant

$$\langle K(t)\rangle_{av} = \langle K_1\rangle_{av} = \int_{-\infty}^{\infty} du\, \gamma(u)\langle J(u, o) - 1\rangle_{av}$$
$$= \int_{-\infty}^{\infty} du\, \gamma(u)\{\exp [a(u)] - 1\}. \tag{68}$$

This follows from equation (55) for $\langle K_1\rangle_{av}$ and the expression for $\langle J(u, o)\rangle_{av}$ given in Table I.

The power spectrum of $K(t)$ is the Fourier transform of $\langle K_1^* K_2\rangle_{av}$. The integral for $\langle K_1^* K_2\rangle_{av}$ given by equation (55) and the ensemble averages of $J^*(u, o)$, $J(v, \tau)$, and $J^*(u, o)J(v, \tau)$ given in Table I lead to

$$\langle K_1^* K_2\rangle_{av} = \langle K_1\rangle\langle K_1^*\rangle_{av} + \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma^*(u)\gamma(v) \exp [a^*(u) + a(v)]$$
$$\cdot \{\exp [2\hat{c}(u, v, \tau)] - 1\}. \tag{69}$$

Integrals of the type appearing in equations (68) and (69) may be expressed as infinite series involving the S functions [which depend only on $\Gamma(f)$] described in Appendix B and the more complicated functions S_n described in Appendix C. Only the first few terms need be considered when most of the distortion arises from second and third order modulation.

The definition [equation (167)] of the complex constant S_0 and its series expansion [equation (171)] give

$$\langle K(t)\rangle_{av} = S_0 - 1$$
$$= -\frac{1}{2} \int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho) S(\rho, -\rho)$$
$$+ \frac{1}{8} \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma) S(\rho, \sigma, -\rho, -\sigma) + O(\varphi^6). \tag{70}$$

Expanding $\exp [2\hat{c}(u, v, \tau)]$ in equation (69) in powers of $2\hat{c}(u, v, \tau)$, replacing each $\hat{c}(u, v, \tau)$ by its defining integral [equation (66)] with ρ in place of f, and integrating with respect to u and v with the help of

$$S_n(\rho_1, \cdots, \rho_n) = \int_{-\infty}^{\infty} dv\, \gamma(v) \exp [a(v)] \prod_{k=1}^{n} H_v(\rho_k) \tag{71}$$

leads to

$$\langle K_1^* K_2\rangle_{av} = |\langle K_1\rangle_{av}|^2 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n \left[\prod_{k=1}^{n} W_\varphi(\rho_k)\right]$$
$$\cdot \exp [i2\pi\tau(\rho_1 + \cdots + \rho_n)] S_n^*(\rho_1, \cdots, \rho_n) S_n(\rho_1, \cdots, \rho_n). \tag{72}$$

When this expression for $\langle K_1^* K_2 \rangle_{av}$ is put in the integral

$$W_K(f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau) \langle K_1^* K_2 \rangle_{av} \, d\tau, \qquad \omega = 2\pi f \tag{73}$$

for the power spectrum, $W_K(f)$, of $K(t)$ and the Fourier transform of unity denoted by $\delta(f)$, the result is

$$\begin{aligned} W_K(f) &= | \langle K(t) \rangle_{av} |^2 \, \delta(f) + \int_{-\infty}^{\infty} d\tau \exp(-i\omega\tau) \\ &\quad \cdot \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \, \gamma^*(u)\gamma(v) \exp[a^*(u)+a(v)] \{ \exp[2\hat{c}(u, v, \tau)] - 1 \} \\ &= | S_0 - 1 |^2 \, \delta(f) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n \\ &\quad \cdot \delta(f - \rho_1 - \cdots - \rho_n) \left[\prod_{k=1}^{n} W_\varphi(\rho_k) \right] | S_n(\rho_1, \cdots, \rho_n) |^2 \\ &= | S_0 - 1 |^2 \, \delta(f) + W_\varphi(f) \, | S_1(f) |^2 \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} d\rho \, W_\varphi(\rho) W_\varphi(f - \rho) \, | S_2(\rho, f - \rho) |^2 \\ &\quad + \frac{1}{6} \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma \, W_\varphi(\rho) W_\varphi(\sigma) W_\varphi(f - \rho - \sigma) \\ &\quad \cdot | S_3(\rho, \sigma, f - \rho - \sigma) |^2 + \cdots . \end{aligned} \tag{74}$$

The leading terms in the series for S_0, S_1, S_2, S_3 in terms of unsubscripted S's are given by equations at the end of Appendix C. The inequality for S_n given in Appendix C may be used to show that the last series in equation (74) converges when $W_\varphi(f)$ remains finite for all values of f and $\langle [\varphi(t)]^2 \rangle_{av}$ is finite. The convergence of similar series which will be encountered later will be tacitly assumed.

VII. "FIRST ORDER" APPROXIMATION FOR POWER SPECTRUM OF $\theta(t)$

Before taking up the problem of computing $W_\theta(f)$ from $\theta(t) = \text{Re}\,\Theta(t)$ and

$$\Theta(t) = \Phi(t) - iK(t) + \frac{i}{2} K^2(t) + O(\varphi^6),$$

which is the same as equation (13), we shall go through a similar, but simpler, calculation using the "first order" approximation

$$\Theta(t) = \Phi(t) - iK(t) + O(\varphi^4). \tag{75}$$

Neglecting $O(K^2)$ terms in equation (14) for θ_{dc}, and in the covariance expressions given in Section IV, shows that

$$\begin{aligned} \theta_{dc} &= \text{Im}\,\langle K(t)\rangle_{av} + O(\varphi^4) \\ W_\theta(f) &= 2^{-1}\,\text{Re}\,[P(f) + Q(f) + P^*(-f) + Q^*(-f)] + O(\varphi^4 W_\varphi) \end{aligned} \tag{76}$$

where $P(f)$, $Q(f)$ are the respective Fourier transforms of

$$\begin{aligned} A(\tau) &= \langle \Phi_1(2^{-1}\Phi_2 - iK_2) - 2^{-1}K_1K_2\rangle_{av} \\ B(\tau) &= \langle \Phi_1^*(2^{-1}\Phi_2 - iK_2) - 2^{-1}K_1^*K_2\rangle_{av} \end{aligned} \tag{77}$$

Expressions for $\langle K_1^*K_2\rangle_{av}$ have been obtained in the preceding section. Repeating the work with K_1 in place of K_1^* brings in $\gamma(u)$, $a(u)$, $c(u, v, \tau)$, $-H_u(-\rho)$, and $(-)^n S_n(-\rho_1, \cdots, -\rho_n)$ in place of $\gamma^*(u)$, $a^*(u)$, $\hat{c}(u, v, \tau)$, and $H_u^*(\rho)$, $S^*(\rho_1, \cdots, \rho_n)$. The result is

$$\begin{aligned} \langle K_1K_2\rangle_{av} &= [\langle K_1\rangle_{av}]^2 + \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma(u)\gamma(v) \exp[a(u) + a(v)] \\ &\quad \cdot \{\exp[2c(u, v, \tau)] - 1\} \\ &= [\langle K_1\rangle_{av}]^2 + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n \left[\prod_{k=1}^{n} W_\varphi(\rho_k)\right] \\ &\quad \cdot \exp[i2\pi\tau(\rho_1 + \cdots + \rho_n)] \\ &\quad \cdot S_n(-\rho_1, \cdots, -\rho_n) S_n(\rho_1, \cdots, \rho_n). \end{aligned} \tag{78}$$

The remaining portion of $A(\tau)$ in equation (77) is

$$\begin{aligned} &\langle \Phi_1(2^{-1}\Phi_2 - iK_2)\rangle_{av} \\ &= \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma(u)\gamma(v) \langle 2^{-1}\varphi(t-u)\varphi(t+\tau-v) - i\varphi(t-u)J(v, \tau)\rangle_{av} \\ &= \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma(u)\gamma(v) \\ &\quad \cdot \int_{-\infty}^{\infty} df\, W_\varphi(f) \exp[i\omega(\tau + u)]\{2^{-1}\exp[-i\omega v] + H_v(f)\exp[a(v)]\} \end{aligned} \tag{79}$$

where $\omega = 2\pi f$ and Table II has been used in going from the first equation to the second. Integration with respect to u brings in $\Gamma(-f)$, and integration with respect to v brings in both $\Gamma(f)$ and the function $S_1(f)$ of Appendix C.

$$\langle \Phi_1(2^{-1}\Phi_2 - iK_2)\rangle_{av} = \int_{-\infty}^{\infty} df\, W_\varphi(f) \exp(i\omega\tau)\Gamma(-f)[2^{-1}\Gamma(f) + S_1(f)]. \tag{80}$$

Replacing Φ_1, $\gamma(u)$ by Φ_1^*, $\gamma^*(u)$ causes $\Gamma^*(f)$ to appear in place of

$\Gamma(-f)$ and shows that the remaining portion of $B(\tau)$ in equation (77) is

$$\langle \Phi_1^*(2^{-1}\Phi_2 - iK_2)\rangle_{\text{av}} = \int_{-\infty}^{\infty} df\, W_\varphi(f) \exp(i\omega\tau)\Gamma^*(f)[2^{-1}\Gamma(f) + S_1(f)]. \tag{81}$$

The function $A(\tau)$ is the sum of $-2^{-1}\langle K_1K_2\rangle$, obtained from equation (78), and (80). Its Fourier transform is

$$P(f) = W_\varphi(f)\Gamma(-f)[2^{-1}\Gamma(f) + S_1(f)] - 2^{-1}[\langle K_1\rangle_{\text{av}}]^2\, \delta(f)$$
$$- \frac{1}{2}\int_{-\infty}^{\infty} d\tau \exp(-i2\pi f\tau)\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma(u)\gamma(v)$$
$$\cdot \exp[a(u) + a(v)]\{\exp[2c(u, v, \tau)] - 1\} \tag{82}$$

where the leading term follows immediately from equation (80) and the Fourier integral theorem.

The function $B(\tau)$ is the sum of $2^{-1}\langle K_1^*K_2\rangle_{\text{av}}$, obtained from equation (69), and (81). Its Fourier transform is

$$Q(f) = W_\varphi(f)\Gamma^*(f)[2^{-1}\Gamma(f) + S_1(f)] + 2^{-1}\,|\,\langle K_1\rangle_{\text{av}}\,|^2\, \delta(f)$$
$$+ \frac{1}{2}\int_{-\infty}^{\infty} d\tau \exp(-i2\pi f\tau)\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma^*(u)\gamma(v)$$
$$\cdot \exp[a^*(u) + a(v)]\{\exp[2\hat{c}(u, v, \tau)] - 1\}. \tag{83}$$

A first order approximation for $W_\theta(f)$ may be obtained by combining equations (76), (82), and (83). Deleting the terms multiplied by $\delta(f)$ and $W_\varphi(f)$ gives the first order approximation to the power spectrum of the nonlinear distortion $\theta_{nl}(t)$. This approximation is stated by equations (38), (39), and (40) in the section describing the results. We now proceed to express the first order approximation for $W_\theta(f)$ as the series given by equation (90).

When equation (82) for $P(f)$ is added to equation (83) for $Q(f)$ and the triple integrals replaced by their series, namely, the Fourier transforms of the series appearing in equations (78) and (72), the result is

$$P(f) + Q(f) = W_\varphi(f)[\Gamma(-f) + \Gamma^*(f)]$$
$$\cdot[2^{-1}\Gamma(f) + S_1(f)] + 2^{-1}(|\,\langle K_1\rangle_{\text{av}}\,|^2 - [\langle K_1\rangle_{\text{av}}]^2)\, \delta(f)$$
$$+ \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n\, \delta(f - \rho_1 - \cdots - \rho_n)\left[\prod_{k=1}^{n} W_\varphi(\rho_k)\right]$$
$$\cdot S_n(\rho_1, \cdots, \rho_n)[-(-)^n S_n(-\rho_1, \cdots, -\rho_n) + S_n^*(\rho_1, \cdots, \rho_n)]. \tag{84}$$

Changing the signs of f and the variables of integration $\rho_1, \cdots, \rho_n$, and then taking the conjugate complex shows that the series in the expression for $P^*(-f) + Q^*(-f)$ differs from equation (84) only in that the S_n factors are replaced by

$$S_n^*(-\rho_1, \cdots, -\rho_n)[-(-)^n S_n^*(\rho_1, \cdots, \rho_n) + S_n(-\rho_1, \cdots, -\rho_n)]. \tag{85}$$

Taking $(-)^{n-1}$ out of the square brackets in equation (85) and then adding the series term in $P(f) + Q(f)$ to the series term in $P^*(-f) + Q^*(-f)$ gives

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n\, \delta(f - \rho_1 - \cdots - \rho_n)\left[\prod_{k=1}^{n} W_\varphi(\rho_k)\right] \cdot | S_n(\rho_1, \cdots, \rho_n) - (-)^n S_n^*(-\rho_1, \cdots, -\rho_n) |^2 \tag{86}$$

for the series term in $P(f) + Q(f) + P^*(-f) + Q^*(-f)$.

The term for $n = 1$ in the series of equation (86) is

$$\tfrac{1}{2} W_\varphi(f) \mid S_1(f) + S_1^*(-f) \mid^2. \tag{87}$$

From the first line in equation (84), the sum of the other terms in $P(f) + Q(f) + P^*(-f) + Q^*(-f)$ containing the factor $W_\varphi(f)$ is

$$W_\varphi(f)[\Gamma(-f) + \Gamma^*(f)] \cdot [2^{-1}\Gamma(f) + 2^{-1}\Gamma^*(-f) + S_1(f) + S_1^*(-f)]. \tag{88}$$

The real part of the sum of equations (87) and (88) may be written as

$$\tfrac{1}{2} W_\varphi(f) \mid \Gamma(f) + \Gamma^*(-f) + S_1(f) + S_1^*(-f)|^2 \tag{89}$$

These results and equation (76) for $W_\theta(f)$ lead to

$$W_\theta(f) = \theta_{dc}^2\, \delta(f) + 4^{-1} W_\varphi(f) \mid \Gamma(f) + S_1(f) + \Gamma^*(-f) + S_1^*(-f) \mid^2 + 4^{-1}\sum_{n=2}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n\, \delta(f - \rho_1 - \cdots - \rho_n)\left[\prod_{k=1}^{n} W_\varphi(\rho_k)\right] \cdot | S_n(\rho_1, \cdots, \rho_n) - (-)^n S_n^*(-\rho_1, \cdots, -\rho_n) |^2 + O(\varphi^4 W_\varphi). \tag{90}$$

The remainder in equation (90) for $W_\theta(f)$ is $O(\varphi^4 W_\varphi)$ while the one in the main result, that is, equation (15), is $O(\varphi^6 W_\varphi)$. The result of neglecting all $O(\varphi^4 W_\varphi)$ terms in equation (90) agrees with the result obtained by neglecting the $O(\varphi^4 W_\varphi)$ terms in the main result. This may be verified with the help of

$$
\begin{aligned}
S_1(f) &= -\frac{1}{2}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho) S(\rho, -\rho, f) + O(\varphi^4) \\
S_2(\rho, \nu) &= S(\rho, \nu) + O(\varphi^2)
\end{aligned}
\tag{91}
$$

which follow from the equations at the end of Appendix C.

VIII. "SECOND ORDER" APPROXIMATION FOR THE POWER SPECTRUM OF $\theta(t)$

In Section VII the "first order" approximation to $W_\theta(f)$ is computed using the approximation

$$\Theta(t) = \Phi(t) - iK(t) + O(\varphi^4) \tag{92}$$

for the complex phase angle $\Theta(t)$. In this section the "second order" approximation to $W_\theta(f)$ will be computed using the approximation given by equation (13),

$$\Theta(t) = \Phi(t) - iK(t) + \frac{i}{2} K^2(t) + O(\varphi^6). \tag{93}$$

The equations needed are given in Section IV. Portions of the ensemble averages $A(\tau)$, $B(\tau)$ defined by equation (49) have already been obtained in Sections VI and VII. The remaining portions needed are

$$\langle i2^{-1}\Phi_1 K_2^2\rangle_{\text{av}}, \qquad \langle 2^{-1} K_1 K_2^2\rangle_{\text{av}}, \tag{94}$$

for $A(\tau)$ and

$$\langle i2^{-1}\Phi_1^* K_2^2\rangle_{\text{av}}, \qquad \langle -2^{-1} K_1^* K_2^2\rangle_{\text{av}} \tag{95}$$

for $B(\tau)$.

From Table II,

$$
\begin{aligned}
\langle i2^{-1}\Phi_1 K_2^2\rangle_{\text{av}} &= i2^{-1}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, \gamma(u)\gamma(v)\gamma(w) \\
&\quad \cdot \langle \varphi(t-u)[J(v,\tau)J(w,\tau) - J(v,\tau) - J(w,\tau) + 1]\rangle_{\text{av}} \\
&= -2^{-1}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, \gamma(u)\gamma(v)\gamma(w) \\
&\quad \cdot \int_{-\infty}^{\infty} df\, W_\varphi(f) \exp\,[i\omega(\tau+u)][2H_v(f)][VWz - V], \\
&\qquad\qquad \omega = 2\pi f.
\end{aligned}
\tag{96}
$$

In going from the first to the second equation, symmetry in v and w has been used to replace $H_v(f) + H_w(f)$ by $2H_v(f)$ and we have introduced part of the notation

$$
\begin{aligned}
U &= \exp [a(u)], & V &= \exp [a(v)], & W &= \exp [a(w)] \\
x &= \exp [2c(u, v, \tau)], & y &= \exp [2c(u, w, \tau)], & z &= \exp [2c(w, v, o)] \\
\hat{x} &= \exp [2\hat{c}(u, v, \tau)], & \hat{y} &= \exp [2\hat{c}(u, w, \tau)], & U^* &= \exp [a^*(u)].
\end{aligned} \tag{97}
$$

As in equation (80), integration with respect to u brings in $\Gamma(-f)$, and integration with respect to v and w brings in the functions $S_{10}(f;)$, $S_1(f)$ of Appendix C:

$$\langle i2^{-1}\Phi_1 K_2^2\rangle_{\text{av}} = -\int_{-\infty}^{\infty} df \exp (i\omega\tau) W_\varphi(f)\Gamma(-f)[S_{10}(f;) - S_1(f)]. \tag{98}$$

The corresponding portion of $B(\tau)$, $\langle i2^{-1}\Phi_1^* K_2^2\rangle_{\text{av}}$, is equal to the expressions obtained when $\gamma(u)$ and $\Gamma(-f)$ are replaced by $\gamma^*(u)$ and $\Gamma^*(f)$ in the right sides of equations (96) and (98).

The last portion of $A(\tau)$ is

$$
\begin{aligned}
\langle 2^{-1}K_1K_2^2\rangle_{\text{av}} &= 2^{-1} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, \gamma(u)\gamma(v)\gamma(w) \\
&\quad \cdot\langle [J(u, o) - 1][J(v, \tau) - 1][J(w, \tau) - 1]\rangle_{\text{av}} \\
&= C_1 + D_1(\tau)
\end{aligned} \tag{99}
$$

where C_1 is independent of τ and represents the value of equation (99) at $\tau = \infty$. With the help of Table I and the notation defined in equation (97), the ensemble average in the integrand may be written as

$$UVWxyz - UVx - UWy - VWz + U + V + W - 1. \tag{100}$$

The only variables in this expression which contain τ are x and y. When $\tau \to \infty$, $c(u, v, \tau)$ tends to 0 and x and y tend to 1. Therefore the portion of equation (100) which contributes to C_1 is

$$UVWz - UV - UW - VWz + U + V + W - 1$$

and the portion contributing to $D_1(\tau)$ is the remainder

$$UVWz(xy - 1) - UV(x - 1) - UW(y - 1). \tag{101}$$

The portion contributing to C_1 will be ignored since the Fourier transform of C_1, namely $C_1\, \delta(f)$, is part of $\theta_{dc}^2\, \delta(f)$, and θ_{dc} will be treated by itself.

When $xy - 1$ is written as $(x - 1)(y - 1) + (x - 1) + (y - 1)$

equation (101) becomes

$$UVWz(x-1)(y-1)+UV(Wz-1)(x-1)+UW(Vz-1)(y-1).$$

The symmetry in v and w allows the last summand to be replaced by the second and hence

$$D_1(\tau)=\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dv\int_{-\infty}^{\infty}dw\,\gamma(u)\gamma(v)\gamma(w)$$
$$\cdot[2^{-1}UVWz(x-1)(y-1)+UV(Wz-1)(x-1)]. \quad (102)$$

Expanding $(x-1)$, $(y-1)$ in powers of $c(u, v, \tau)$, $c(u, w, \tau)$, respectively, and integrating termwise in much the same way as in the passage from equation (69) to equation (72) leads to

$$D_1(\tau)=\sum_{n=1}^{\infty}\frac{(-)^n}{n!}\int_{-\infty}^{\infty}d\rho_1\cdots\int_{-\infty}^{\infty}d\rho_n\left[\prod_{k=1}^{n}W_\varphi(\rho_k)\right]$$
$$\cdot\exp\,[i2\pi\tau(\rho_1+\cdots+\rho_n)]$$
$$\cdot S_n(-\rho_1,\cdots,-\rho_n)[S_{n0}(\rho_1,\cdots,\rho_n;)-S_n(\rho_1,\cdots,\rho_n)]$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-)^{n+m}}{n!\,m!}\int_{-\infty}^{\infty}d\rho_1\cdots\int_{-\infty}^{\infty}d\rho_n$$
$$\cdot\int_{-\infty}^{\infty}d\sigma_1\cdots\int_{-\infty}^{\infty}d\sigma_m\left[\prod_{k=1}^{n}W_\varphi(\rho_k)\right]\left[\prod_{\ell=1}^{m}W_\varphi(\sigma_k)\right]$$
$$\cdot\exp\,[i2\pi\tau(\rho_1+\cdots+\rho_n+\sigma_1+\cdots+\sigma_m)]$$
$$\cdot S_{n+m}(-\rho_1,\cdots,-\rho_n,-\sigma_1,\cdots,-\sigma_m)$$
$$\cdot S_{nm}(\rho_1,\cdots,\rho_n;\sigma_1,\cdots,\sigma_m). \quad (103)$$

When the last portion of $B(\tau)$ is written as

$$\langle -2^{-1}K_1^*K_2^2\rangle_{\text{av}}=C_2+D_2(\tau) \quad (104)$$

the work goes through much as for $C_1+D_1(\tau)$. The functions $\gamma^*(u)$, $J^*(u, o)$, U^*, $\hat{x}$, and $\hat{y}$ replace $\gamma(u)$, $J(u, o)$, U, x, and y, respectively. The functions $H_u(-\rho_k)$, and $H_u(-\sigma_\ell)$ in $c(u, v, \tau)$, $c(u, w, \tau)$ are replaced by $-H_u^*(\rho_k)$, and $-H_u^*(\sigma_\ell)$. This carries x, y into $\hat{x}$, $\hat{y}$ and causes $S_n(\rho_1, \cdots, -\rho_n)$ to be replaced by $(-)^nS_n^*(\rho_1, \cdots, \rho_n)$. A similar replacement holds for S_{n+m}.

The resulting expression for $D_2(\tau)$ is obtained by changing the sign (because $-K_1^*$ replaces K_1) of the expression (103) for $D_1(\tau)$, and then replacing $S_n(-\rho_1, \cdots -\rho_n)$ and $S_{n+m}(-\rho_1, \cdots, -\sigma_m)$ by $(-)^nS_n^*(\rho_1, \cdots, \rho_n)$ and $(-)^{n+m}S_{n+m}^*(\rho_1, \cdots, \sigma_m)$, respectively.

Now that expressions for the portions (94) and (95) of $A(\tau)$ and $B(\tau)$ have been obtained (in effect), there remain two problems:

(*i*) taking their Fourier transforms to get their contributions to $P(f)$ and $Q(f)$, and

(*ii*) adding these to the first order approximation for $W_\theta(f)$ given by equation (90).

The Fourier transform of $\langle i2^{-1}\Phi_1 K_2^2\rangle_{\text{av}}$ follows from equation (98) and the Fourier integral theorem. The Fourier transform of $\langle i2^{-1}\Phi_1^* K_2^2\rangle_{\text{av}}$ may be obtained in much the same way and consequently the contribution of these terms to $P(f) + Q(f)$ is

$$\int_{-\infty}^{\infty} d\tau \exp(-i\omega\tau)[\langle i2^{-1}\Phi_1 K_2^2\rangle_{\text{av}} + \langle i2^{-1}\Phi_1^* K_2^2\rangle_{\text{av}}]$$
$$= -W_\varphi(f)[\Gamma(-f) + \Gamma^*(f)][S_{10}(f;) - S_1(f)]. \quad (105)$$

Consequently their contribution to the right side of

$$W_\theta(f) = 2^{-1}\,\text{Re}\,[P(f) + Q(f) + P^*(-f) + Q^*(-f)] + O(\varphi^6 W_\varphi) \quad (106)$$

[from equation (50)] is

$$2^{-1}W_\varphi(f)\,\text{Re}\,[\Gamma(-f) + \Gamma^*(f)][S_1(f) - S_{10}(f;) + S_1^*(-f) - S_{10}^*(-f;)]. \quad (107)$$

The Fourier transform of the portion $D_1(\tau)$ of $\langle 2^{-1}K_1K_2^2\rangle_{\text{av}}$ is obtained by replacing $\exp[i2\pi\tau(\rho_1 + \cdots + \rho_n)]$ and $\exp[i2\pi\tau(\rho_1 + \cdots + \sigma_m)]$ in equation (103) for $D_1(\tau)$ by $\delta(f - \rho_1 - \cdots - \rho_n)$ and $\delta(f - \rho_1 - \cdots - \sigma_m)$, respectively. The Fourier transform of $D_2(\tau)$, from $\langle -2^{-1}K_1^*K_2^2\rangle_{\text{av}}$, can be obtained similarly. The sum of these two Fourier transforms gives the contribution of $D_1(\tau) + D_2(\tau)$ to $P(f) + Q(f)$. Changing the signs of f and the variables of integration $\rho_1, \cdots, \sigma_m$, and then taking the conjugate complex, gives the contribution of $D_1(\tau) + D_2(\tau)$ to $P^*(-f) + Q^*(-f)$. When the two contributions are added, it is found that the contribution of $D_1(\tau) + D_2(\tau)$ to $P(f) + Q(f) + P^*(-f) + Q^*(-f)$ is

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n\left[\prod_{k=1}^{n} W_\varphi(\rho_k)\right]$$
$$\cdot\,\delta(f - \rho_1 - \cdots - \rho_n)[S_n^*(\rho_1, \cdots) - (-)^n S_n(-\rho_1, \cdots)]$$
$$\cdot\,[S_n(\rho_1, \cdots) - (-)^n S_n^*(-\rho_1, \cdots)$$
$$- S_{n0}(\rho_1, \cdots) + (-)^n S_{n0}^*(-\rho_1, \cdots)]$$

$$
\begin{aligned}
&- \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n!\, m!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\sigma_m \\
&\cdot \left[\prod_{k=1}^{n} W_\varphi(\rho_k) \right] \left[\prod_{l=1}^{m} W_\varphi(\sigma_l) \right] \delta(f - \rho_1 - \cdots - \sigma_m) \\
&\cdot [S^*_{n+m}(\rho_1 , \cdots) - (-)^{n+m} S_{n+m}(-\rho_1 , \cdots)] \\
&\cdot [S_{nm}(\rho_1 , \cdots) - (-)^{n+m} S^*_{nm}(-\rho_1 , \cdots)] \qquad (108)
\end{aligned}
$$

where the complete arguments of the S functions are shown in equation (103).

The desired expression corresponding to equation (106) for $W_\theta(f)$ is obtained by adding the significant terms in the first order approximation of equation (90) for $W_\theta(f)$ to the second order terms given by equations (107) and (108). The remainder term, $O(\varphi^4 W_\varphi)$, in equation (90) can be ignored because the significant terms are obtained from $\Phi(t) - iK(t)$ without approximation [compare equations (92) and (93) for $\Theta(t)$]. The result is

$$
\begin{aligned}
W_\theta(f) &= \theta_{dc}^2\, \delta(f) + 4^{-1} W_\varphi(f) \mid \Gamma(f) + S_1(f) + \Gamma^*(-f) + S_1^*(-f) \mid^2 \\
&+ 4^{-1} \sum_{n=2}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n\, \delta(f - \rho_1 - \cdots - \rho_n) \left[\prod_{k=1}^{n} W_\varphi(\rho_k) \right] \\
&\cdot \mid S_n(\rho_1 , \cdots , \rho_n) - (-)^n S_n^*(-\rho_1 \cdots -\rho_n) \mid^2 \\
&+ \text{expression (107)} + 2^{-1} \operatorname{Re}[\text{expression (108)}] + O(\varphi^6 W_\varphi). \qquad (109)
\end{aligned}
$$

The next section is concerned with the elimination of all $O(\varphi^6 W_\varphi)$ terms from the significant portion of equation (109). When these terms are eliminated, the result is the "main result" stated in equation (15).

IX. ELIMINATION OF HIGHER ORDER MODULATION TERMS FROM $W_\theta(f)$

In this section all terms of $O(\varphi^6 W_\varphi)$ in equation (109) for $W_\theta(f)$ will be discarded, that is modulation terms of order higher than three will be discarded. Since the integral of $W_\varphi(f)$ is $O(\varphi^2)$, all terms in equation (109) containing the product of four or more W_φ's may be dropped immediately.

First consider the terms which explicitly contain the product of three $W\varphi$'s. This corresponds to $n = 3$ in the single series in expressions (108) and (109), and to the pairs of values $n = 1$, $m = 2$; $n = 2$, $n = 1$ in the double series. The contribution of the double series can be discarded because it is $O(\int \int W_\varphi^3 \varphi^2) = O(\varphi^6 W_\varphi)$, the functions S_{21} and S_{12} being $O(\varphi^2)$ [from $S(f) = 0$, $S_1(f) = O(\varphi^2)$ and Appendix C]. The

$n = 3$ term in expression (108) can also be discarded because, from Appendix C, $S_3 - S_{30}$ is $O(\varphi^2)$. Using $S_3 - S = O(\varphi^2)$ in the $n = 3$ term in equation (109) shows that the contribution to $W_\theta(f)$ of the terms which explicitly contain the product of three W_φ's is

$$\frac{1}{24}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho)W_\varphi(\sigma)W_\varphi(\nu) \cdot | S(\rho, \sigma, \nu) + S^*(-\rho, -\sigma, -\nu) |^2 + O(\varphi^6 W_\varphi) \tag{110}$$

where $\nu = f - \rho - \sigma$.

Next consider terms which explicitly contain the product of two W_φ's, namely the terms $n = 2$ in the single series and $n = m = 1$ in the double series. When we put $\rho_1 = \rho$, $\rho_2 = f - \rho = \nu$ in the single series terms, and $\rho_1 = \rho$, $\sigma_1 = f - \rho = \nu$ in the double series term, all of the integrands contain the factor

$$\beta^* = [S_2^*(\rho, \nu) - S_2(-\rho, -\nu)]$$

and the contribution of their sum to $W_\theta(f)$ can be written as

$$\frac{1}{8}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho)W_\varphi(\nu) \operatorname{Re}[\beta^*(\beta + 2\gamma)]$$

where β is $O(1)$, and γ is not the earlier $\gamma(u)$. Here

$$\gamma = S_2(\rho, \nu) - S_{20}(\rho, \nu;) - S_{11}(\rho; \nu) - S_2^*(-\rho, -\nu) + S_{20}^*(-\rho, -\nu;) + S_{11}^*(-\rho; -\nu)$$

is $O(\varphi^2)$ since both $S_2 - S_{20}$ and S_{11} are $O(\varphi^2)$. Furthermore,

$$\begin{aligned} \operatorname{Re}[\beta^*(\beta + 2\gamma)] &= | \beta + \gamma |^2 - \gamma\gamma^* = | \beta + \gamma |^2 + O(\varphi^4), \\ \beta + \gamma &= \hat{T}(\rho, \nu) - \hat{T}^*(-\rho, -\nu), \\ \hat{T}(\rho, \nu) &= 2S_2(\rho, \nu) - S_{20}(\rho, \nu;) - S_{11}(\rho, \nu). \end{aligned} \tag{111}$$

The equations at the end of Appendix C may be used to show that $\hat{T}(\rho, \nu)$ is equal to $T(\rho, \nu) + O(\varphi^4)$ where

$$T(\rho, \nu) = S(\rho, \nu) + \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\sigma)[-\tfrac{1}{2}S(\sigma, -\sigma, \rho, \nu) + \tfrac{1}{2}S(\sigma, -\sigma)S(\rho, \nu) + S(\sigma, \rho)S(-\sigma, \nu)] \tag{112}$$

and consequently the contribution to $W_\theta(f)$ of the terms which explicitly contain the product of two W_φ's is

$$\frac{1}{8}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho)W_\varphi(\nu)\mid T(\rho,\nu) - T^*(-\rho,-\nu)\mid^2 + O(\varphi^6 W_\varphi) \tag{113}$$

where $\nu = f - \rho$.

Now consider the terms in $W_\theta(f)$ which are multiplied by $W_\varphi(f)$. From equations (109), (107), and the term for $n = 1$ in the single series in equation (108), the sum of these terms is $W_\varphi(f)$ multiplied by

$$\begin{aligned} & 4^{-1}\mid \Gamma(f) + S_1(f) + \Gamma^*(-f) + S_1^*(-f)\mid^2 \\ &\quad + 2^{-1}\,\mathrm{Re}\,[\Gamma(-f) + \Gamma^*(f)][S_1(f) - S_{10}(f;) + S_1^*(-f) - S_{10}^*(-f;)] \\ &\quad + 2^{-1}\,\mathrm{Re}\,[S_1^*(f) + S_1(-f)][S_1(f) - S_{10}(f;) + S_1^*(-f) - S_{10}^*(-f;)] \\ &= 4^{-1}\mid \alpha + \beta\mid^2 + 2^{-1}\,\mathrm{Re}\,(\alpha^* + \beta^*)(\gamma) \\ &= 4^{-1}[(\alpha+\beta)(\alpha^*+\beta^*) + (\alpha^*+\beta^*)\gamma + (\alpha+\beta)\gamma^* + \gamma\gamma^* - \gamma\gamma^*] \\ &= 4^{-1}\mid \alpha+\beta+\gamma\mid^2 - 4^{-1}\gamma\gamma^* \\ &= 4^{-1}\mid \hat{U}(f) + \hat{U}^*(-f)\mid^2 - 4^{-1}\gamma\gamma^*. \end{aligned} \tag{114}$$

Here, with β and γ different from those in equation (111),

$$\begin{aligned} \alpha &= \Gamma(f) + \Gamma^*(-f) = O(1) \\ \beta &= S_1(f) + S_1^*(-f) = O(\varphi^2) \\ \gamma &= S_1(f) - S_{10}(f;) + S_1^*(-f) - S_{10}^*(-f;) = O(\varphi^4) \\ \hat{U}(f) &= \Gamma(f) + 2S_1(f) - S_{10}(f;). \end{aligned} \tag{115}$$

The equations at the end of Appendix C may be used to show that $\hat{U}(f)$ is equal to $U(f) + O(\varphi^6)$ where

$$\begin{aligned} U(f) = \Gamma(f) &- \frac{1}{2}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho)S(\rho,-\rho,f) + \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho)W_\varphi(\sigma) \\ &\cdot[\tfrac{1}{8}S(\rho,\sigma,-\rho,-\sigma,f) - \tfrac{1}{4}S(\rho,-\rho,f)S(\sigma,-\sigma) \\ &\quad - \tfrac{1}{2}S(\rho,f)S(\sigma,-\sigma,-\rho) - \tfrac{1}{2}S(\rho,\sigma,f)S(-\rho,-\sigma)]. \end{aligned} \tag{116}$$

Since $\gamma\gamma^*$ is $O(\varphi^8)$, the terms in $W_\theta(f)$ which are multiplied by $W_\varphi(f)$ can be written as

$$W_\varphi(f)\,|U(f) + U^*(-f)|^2 + O(\varphi^6 W_\varphi). \tag{117}$$

Finally consider the dc spike, $\theta_{dc}^2\,\delta(f)$, in $W_\theta(f)$. From equation (14), the dc component of $\theta(t)$ is

$$\theta_{dc} = \mathrm{Im}\,\langle K(t) - 2^{-1}K^2(t)\rangle_{av} + O(\varphi^6). \tag{118}$$

The value of $\langle K(t)\rangle_{av}$ is given by equation (70) and from equation (78) with $\tau = 0$,

$$\langle K^2(t)\rangle_{av} = \langle K(t)\rangle_{av}^2 - \int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho) S_1(-\rho) S_1(\rho)$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma) S_2(-\rho, -\sigma) S_2(\rho, \sigma) + O(\varphi^6).$$

Since $S_1(\rho)$ is $O(\varphi^2)$, the single integral is $O(\varphi^6)$ and may be included in the remainder term. Squaring the leading term in equation (70) to get $\langle K(t)\rangle^2$, combining terms, and using $S_2(\rho, \sigma) = S(\rho, \sigma) + O(\varphi^2)$ leads to

$$\langle K(t) - 2^{-1}K^2(t)\rangle_{av} = D_c$$

$$D = -\frac{1}{2}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho) S(\rho, -\rho) + \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma)$$

$$\cdot [\tfrac{1}{8}S(\rho, \sigma, -\rho, -\sigma) - \tfrac{1}{8}S(\rho, -\rho)S(\sigma, -\sigma) - \tfrac{1}{4}S(\rho, \sigma)S(-\rho, -\sigma)]$$

$$+ O(\varphi^6). \tag{119}$$

The imaginary part of D_c gives θ_{dc}.

Addition of equations (110), (113), and (117) shows that

$$W_\theta(f) = \theta_{dc}^2\, \delta(f) + W_\varphi(f) \mid U(f) + U^*(-f) \mid^2$$

$$+ \frac{1}{8}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho) W_\varphi(f - \rho) \mid T(\rho, f - \rho) - T^*(-\rho, -f + \rho)\mid^2$$

$$+ \frac{1}{24}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma) W_\varphi(\nu)$$

$$\cdot \mid S(\rho, \sigma, \nu) + S^*(-\rho, -\sigma, -\nu) \mid^2 + O(\varphi^6 W_\varphi) \tag{120}$$

where $\nu = f - \rho - \sigma$. This is the same as equation (15) in the statement of results. However, the expressions for D_c, $T(\rho, f - \rho)$, and $U(f)$ given in Section III are simpler than the ones given in this section. The method of obtaining the simpler expressions will be outlined in Section X.

X. SIMPLIFIED EXPRESSIONS FOR θ_{dc}, $U(f)$, AND $T(\rho, \nu)$

The expressions obtained for θ_{dc}, $U(f)$, and $T(\rho, \nu)$ in Section IX may be put in forms better suited to calculation by writing the higher order S functions in terms of S functions of two arguments,

$$S(\rho, \sigma) = \Gamma(\rho + \sigma) - \Gamma(\rho)\Gamma(\sigma). \tag{121}$$

These simplified forms are the ones stated in equations (16), (17), and (18).

Since no really satisfactory procedure of reduction was found, the expressions given here may not be the simplest. The procedure is illustrated for the double integral

$$I = \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma \, W_\varphi(\rho) W_\varphi(\sigma)$$
$$\cdot [S(\rho, -\rho, \sigma, -\sigma) - S(\rho, -\rho) S(\sigma, -\sigma) - 2S(\rho, \sigma) S(-\rho, -\sigma)]$$

which appears in equation (119) for $\theta_{dc} = \text{Im}\, D_c$.

After some cancellation, the general equation (159) for $S(\rho, \sigma, \nu, \mu)$ shown in Appendix B gives

$$\begin{aligned} S(\rho, -\rho, \sigma, -\sigma) = 1 &- (\rho)(-\rho) - (\sigma)(-\sigma) + (\rho + \sigma)(-\rho)(-\sigma) \\ &+ (\rho - \sigma)(-\rho)(\sigma) + (-\rho + \sigma)(\rho)(-\sigma) \\ &+ (-\rho - \sigma)(\rho)(\sigma) - 3(\rho)(-\rho)(\sigma)(-\sigma). \end{aligned}$$

Here $\Gamma(x)$ has been written as (x) and $(0) = \Gamma(0) = 1$ has been used. When this expression is multiplied by $W_\varphi(\rho) W_\varphi(\sigma)$ and integrated with respect to ρ and σ, changes in the variables of integration show that the value of I is unchanged by the substitution

$$\begin{aligned} S(\rho, -\rho, \sigma, -\sigma) \to 1 - 2(\rho)(-\rho) &+ 4(\rho + \sigma)(-\rho)(-\sigma) \\ &- 3(\rho)(-\rho)(\sigma)(-\sigma). \end{aligned}$$

Here the arrow means "may be replaced in the double integral by".

Similarly,

$$\begin{aligned} -S(\rho, -\rho) S(\sigma, -\sigma) &= -[1 - (\rho)(-\rho)][1 - (\sigma)(-\sigma)] \\ &\to -1 + 2(\rho)(-\rho) - (\rho)(-\rho)(\sigma)(-\sigma) \\ -2S(\rho, \sigma) S(-\rho, -\sigma) &= -2[(\rho + \sigma) - (\rho)(\sigma)][(-\rho - \sigma) - (-\rho)(-\sigma)] \\ &\to -2(\rho + \sigma)(-\rho - \sigma) + 4(\rho + \sigma)(-\rho)(-\sigma) - 2(\rho)(-\rho)(\sigma)(-\sigma). \end{aligned}$$

Addition shows that the quantity within the square brackets in the integrand of I may be replaced by

$$\begin{aligned} &8(\rho + \sigma)(-\rho)(-\sigma) - 2(\rho + \sigma)(-\rho - \sigma) - 6(\rho)(-\rho)(\sigma)(-\sigma) \\ &= 6(-\rho)(-\sigma)[(\rho + \sigma) - (\rho)(\sigma)] - 2(\rho + \sigma)[(-\rho - \sigma) - (-\rho)(-\sigma)] \\ &= 6(-\rho)(-\sigma) S(\rho, \sigma) - 2(\rho + \sigma) S(-\rho, -\sigma) \end{aligned}$$

$$= 6[(-\rho - \sigma) - (-\rho - \sigma) + (-\rho)(-\sigma)]S(\rho, \sigma) - 2(\rho + \sigma)S(-\rho, -\sigma)$$

$$= 6(-\rho - \sigma)S(\rho, \sigma) - 6S(-\rho, -\sigma)S(\rho, \sigma) - 2(\rho + \sigma)S(-\rho, -\sigma)$$

$$\rightarrow 4(\rho + \sigma)S(-\rho, -\sigma) - 6S(-\rho, -\sigma)S(\rho, \sigma).$$

Hence

$$I = \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma) S(-\rho, -\sigma)[4\Gamma(\rho + \sigma) - 6S(\rho, \sigma)]$$

which is the form of I used in equation (16) for D (except that ρ and σ are interchanged).

The simplification of equations (112) and (116) for $T(\rho, \nu)$ and $U(f)$, respectively, proceeds along the same lines. In dealing with $U(f)$, the symbolic substitution

$$S(\rho, \sigma, -\rho, -\sigma, f) \rightarrow [1 + (\rho)(-\rho) - 2y^\rho(-\rho)] \cdot [1 + (\sigma)(-\sigma) - 2y^\sigma(-\sigma)][y^f - (f)]$$

was found helpful.

In addition to the simplified forms for $T(\rho, \nu)$ and $U(f)$ given in equations (17) and (18), we also have

$$S(\rho, \sigma, \nu) = S(\rho + \sigma, \nu) - \Gamma(\rho)S(\sigma, \nu) - \Gamma(\sigma)S(\rho, \nu). \qquad (122)$$

XI. THE "SMALL AND SLOW" DEVIATION APPROXIMATION TO $W_\theta(f)$

This section and the following one are concerned with approximations to $W_\theta(f)$ which are obtained by replacing the Γ's used in equations (15) to (20) by the first few terms in their power series expansions. These expansions are assumed to exist and to converge rapidly over the range of frequencies for which $W_\varphi(f)$ is effectively different from zero. Roughly speaking, the top baseband frequency is assumed to be small compared with the filter bandwidth.

When the top baseband frequency is small, the modulating frequency, $\varphi'(t)$, changes slowly and we have the quasistatic case. The name "small and slow deviation approximation" is used because (15) holds only for "small" rms frequency deviations (D small), and here the requirement of "slowness" is added.

Two series which play important roles are

$$\Gamma(f) = \sum_{n=0}^{\infty} \alpha_n f^n/n!, \qquad \alpha_0 = 1 \qquad (123)$$

$$\ln \Gamma(f) = \sum_{n=0}^{\infty} \lambda_n f^n/n!, \qquad \lambda_0 = 0 \qquad (124)$$

The first one is the series assumed for $\Gamma(f)$. Substituting (123) in (124), expanding the logarithm, and equating coefficients of powers of f leads to expressions for the λ's in terms of the α's:

$$\begin{aligned}
\lambda_1 &= \alpha_1 , \\
\lambda_2 &= \alpha_2 - \alpha_1^2 , \\
\lambda_3 &= \alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1^3 , \\
\lambda_4 &= \alpha_4 - 4\alpha_3\alpha_1 - 3\alpha_2^2 + 12\alpha_2\alpha_1^2 - 6\alpha_1^4 , \\
\lambda_5 &= \alpha_5 - 5\alpha_4\alpha_1 - 10\alpha_3\alpha_2 + 20\alpha_3\alpha_1^2 + 30\alpha_2^2\alpha_1 - 60\alpha_2\alpha_1^3 + 24\alpha_1^5 .
\end{aligned} \tag{125}$$

and so on. When the α_n's are the moments of a probability distribution, the λ_n's are the associated "cumulants" or semi-invariants. In our problem the α_n's are proportional to the moments of the normalized response $\gamma(t)$, a relation which follows when the series (123) for $\Gamma(f)$ is compared with the one obtained by expanding $\exp(-i2\pi ft)$ in the Fourier integral (4) for $\Gamma(f)$.

The small and slow deviation approximation obtained from (15) and the first few terms of (123) is

$$\begin{aligned}
W_\theta(f) &\to \theta_{dc}^2\ \delta(f) \\
&+ W_\varphi(f)[1 + f^2\{(\lambda_{1i} + 2^{-1}D^2\lambda_{3i} + 8^{-1}D^4\lambda_{5i})^2 + (\alpha_{2r} + 2^{-1}D^2A_r)\}] \\
&+ 2^{-1}(\lambda_{2i} + 2^{-1}D^2\lambda_{4i})^2 \int_{-\infty}^{\infty} d\rho\ W_\varphi(\rho)W_\varphi(f - \rho)\rho^2(f - \rho)^2 \\
&+ 6^{-1}(\lambda_{3i})^2 \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\ W_\varphi(\rho)W_\varphi(\sigma)W_\varphi(f - \rho - \sigma)\rho^2\sigma^2(f - \rho - \sigma)^2
\end{aligned} \tag{126}$$

Here the α_{ni}'s, λ_{ni}'s are the imaginary parts of the coefficients in the series (123), (124), $D^2 = \langle[\varphi'(t)/(2\pi)]^2\rangle$, D is the rms frequency deviation in Hz, and α_{2r}, A_r denote the real parts of α_2, A where

$$A = \alpha_4 - 2\alpha_3\alpha_1 - \alpha_2^2 + 2\alpha_2\alpha_1^2 .$$

The detailed derivation of equation (126) from equation (15) for $W_\theta(f)$ makes use of equation (162) which gives $S(x_1, x_2, \cdots x_n)$ for small values of the x's. The leading term in equation (162) gives

$$\begin{gathered}
S(\rho, \sigma) \to \rho\sigma\lambda_2 \\
S(\rho, \sigma, \nu) \to \rho\sigma\nu\lambda_3 \\
-S(\sigma, -\sigma, \rho, \nu) + S(\sigma, -\sigma)S(\rho, \nu) + 2S(\sigma, \rho)S(-\sigma, \nu) \to \sigma^2\rho\nu\lambda_4
\end{gathered} \tag{127}$$

where the left side of the last equation is proportional to the integrand in equation (112) for $T(\rho, \nu)$. There is a similar equation which shows that the integrand in the double integral in equation (116) for $U(f)$ tends to a quantity proportional to $\rho^2\sigma^2 f\lambda_5$. To deal with the single integral in $U(f)$ we use both terms in equation (162) to obtain

$$S(\rho, -\rho, f) \to -\rho^2 f\lambda_3 - 2^{-1}\rho^2 f^2 A. \tag{128}$$

Combining equations (127), (128), and the first three terms in the series for $\Gamma(f)$ gives

$$\begin{aligned} &S(\rho, \sigma, \nu) \to \rho\sigma\nu\lambda_3 \\ &\quad T(\rho, \nu) \to \rho\nu(\lambda_2 + 2^{-1}D^2\lambda_4) \\ &\qquad U(f) \to 1 + (\alpha_1 + 2^{-1}D^2\lambda_3 + 8^{-1}D^4\lambda_5)f + (2^{-1}\alpha_2 + 4^{-1}D^2 A)f^2. \end{aligned}$$

Substitution in the small deviation approximation (15) for $W_\theta(f)$ then gives the small and slow deviation approximation shown in equation (126).

A form of the small and slow approximation which is more complete than (126) may be obtained by starting with the quasistatic form of equation (7) for $\Theta(t)$ instead of from the small deviation approximation (15) for $W_\theta(t)$. In the quasistatic case the instantaneous frequency $\Omega = \omega_0 + \varphi'(t)$ changes slowly and hence rms $\varphi''(t)$ is small. This leads us to replace $\varphi(t - u)$ in (7) by the equivalent expression $\varphi(t) - u\varphi'(t) + 2^{-1}u^2\varphi''(\xi)$ where ξ lies between $t - u$ and t. Let F denote the filter bandwidth and suppose that the impulse response $\gamma(u)$ is effectively 0 outside an interval of length $1/F$. Then, heuristically, the integral in (7) is given by

$$\begin{aligned} \int_{-\infty}^{\infty} \gamma(u) \exp\,[i\varphi(t - u)]\, du &= [1 + O(2^{-1}F^{-2} \text{ rms } \varphi'')] \\ &\quad \cdot \int_{-\infty}^{\infty} \gamma(u) \exp\,[i\varphi(t) - iu\varphi'(t)]\, du. \end{aligned}$$

The integral on the right is the desired quasistatic approximation. It is almost equal to the integral on the left when $2^{-1}F^{-2}$ rms $\varphi'' \ll 1$. However, for small rms frequency deviations, the contribution of $u\varphi'(t)$ may be less than the term $2^{-1}F^{-2}$ rms φ'' even though

(*i*) the latter may be $\ll 1$, and

(*ii*) despite the fact that when $\varphi(t)$ is band-limited with top frequency B we always have rms $\varphi'' \leqq (2\pi B)$ rms φ'. Therefore, in order to make the quasistatic approximation meaningful for small (as well as large) deviations, we impose the additional restriction $2^{-1}F^{-2}$ rms $\varphi''/(F^{-1}$ rms $\varphi') \ll 1$. Then, if

$$\text{rms } \varphi''/(2F^2) \ll 1, \qquad \text{rms } \varphi''/(2F \text{ rms } \varphi') \ll 1$$

the Fourier transform (4) gives the quasistatic approximations

$$\begin{aligned} \int_{-\infty}^{\infty} \gamma(u) \exp [i\varphi(t-u)]\, du &\approx \Gamma[\varphi'(t)/(2\pi)] \exp [i\varphi(t)] \\ \Theta(t) &\approx \varphi(t) - i \ln \Gamma[\varphi'(t)/(2\pi)] \end{aligned} \tag{129}$$

which are equivalent to the usual quasistatic approximation for the filter output, namely

$$s_0(t) \approx G(\Omega) \exp [i\omega_0 t + i\varphi(t)]. \tag{130}$$

D. T. Hess[13] has given a rigorous bound, roughly equivalent to rms $\varphi''/(2F^2) \ll 1$, for the error in (130).

For the flat FM baseband case discussed in Section 3.3 the above restrictions go into

$$10\, DB/F^2 \ll 1, \qquad 2B/F \ll 1$$

where D is the rms frequency deviation in Hz and B is the top baseband frequency in Hz. Notice that although the term "quasistatic" implies that rms φ'' tends to 0 in some sense or other, the requirements that the quasistatic approximations (129) and (130) hold differ from the requirement that the deviation ratio be large, a condition used in calculating the quasistatic approximation to the power spectrum of cos $[\omega_0 t + \varphi(t)]$[14] Thus, for the flat baseband case, the deviation ratio can be taken to be D/B, and this does not have to be large for (129) and (130) to hold.

To continue with the derivation of the more complete form of (126), we substitute the series (124) for ln $\Gamma(f)$ in (129) and take the real part. This gives

$$\begin{aligned} \theta(t) &\approx \varphi(t) + B(t) \\ B(t) &= \sum_{n=1}^{\infty} \lambda_{ni} [\varphi'(t)/(2\pi)]^n / n!. \end{aligned}$$

Since $B(t)$ depends only on $\varphi'(t)$, the power spectrum of $\varphi(t) + B(t)$ is $W_\varphi(f) + W_B(f)$ (Ref. 15). The covariance of $B(t)$ is

$$\langle B_1 B_2 \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \lambda_{ni} \lambda_{mi} \langle \varphi_1'^n \varphi_2'^m \rangle (2\pi)^{-n-m} / (n!\, m!) \tag{131}$$

where subscripts 1 and 2 denote arguments t and $t + \tau$, respectively.

From the characteristic function of the joint gaussian distribution function of φ_1', φ_2' we have

$$\frac{\langle \varphi_1'^n \varphi_2'^m \rangle}{n!\,m!} = (2\pi i)^{-2} \int^{(0+)} \frac{du}{u} \int^{(0+)} \frac{dv}{v} \frac{\exp[-\psi_0(u^2+v^2)2^{-1} - \psi_\tau uv]}{i^{n+m}u^n v^m}$$

$$= \sum_{k=0}^{\infty} \frac{[1+(-)^{n-k}][1+(-)^{m-k}]}{k!\,4\Gamma[(n-k)2^{-1}+1]\Gamma[(m-k)2^{-1}+1]} \left(\frac{2\psi_\tau}{\psi_0}\right)^k \left(\frac{\psi_0}{2}\right)^{(n+m)/2} \tag{132}$$

where $\psi_\tau = \langle \varphi_1' \varphi_2' \rangle$, $\psi_0 = (2\pi D)^2$, and we have used, for integer l,

$$(2\pi i)^{-1} \int^{(0+)} u^{-\ell-1} \exp[-\psi_0 u^2/2]\, du = \frac{(i^\ell + i^{-\ell})(\psi_0/2)^{\ell/2}}{2\Gamma(2^{-1}\ell+1)}.$$

Actually, instead of ∞ the upper limit of summation for k in (132) is the smaller of n, m. Also the sum is 0 unless n, m are both even or both odd. When n is even, k runs over even integers; and when n is odd, k runs over odd integers. When (132) is substituted in (131), the Fourier transform of the resulting series for $\langle B_1B_2 \rangle$ gives a series for $W_B(f)$ which leads to the more complete form of (126) we have been seeking, namely

$$W_\theta(f) \approx W_\varphi(f) + \sum_{k=0}^{\infty} \frac{1}{k!} \left[\sum_{n=0}^{\infty} \frac{\lambda_{2n+k,i} D^{2n}}{n!\,2^n} \right]^2 \cdot (2\pi)^{-2k} \int_{-\infty}^{\infty} \psi_\tau^k \exp(-i2\pi f\tau)\, d\tau \tag{133}$$

The integral in (133) can be expressed as a $(k-1)$-fold convolution of the power spectrum $W_{\varphi'}(f) = (2\pi f)^2 W_\varphi(f)$ of $\varphi'(t)$. This gives the first few terms of (126), except for the term $(\alpha_{2r} + 2^{-1}D^2A_r)\, W_\varphi(f)$ which arises from terms neglected by (133).

Equation (133) is useful only when D is small because the summation with respect to n usually diverges. To illustrate this, consider the single pole filter for which $\Gamma(f)$ is $(1 + iff_c^{-1})^{-1}$ and λ_n is $(n-1)!(-if_c^{-1})^n$. Equation (133), with k replaced by $2k+1$, gives

$$W_\theta(f) \approx W_\varphi(f) + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left[\sum_{n=0}^{\infty} \frac{(-)^n(2n+2k)!\,D^{2n}}{n!\,2^n f_c^{2n+2k+1}} \right]^2 \cdot (2\pi)^{-4k-2} \int_{-\infty}^{\infty} \psi_\tau^{2k+1} \exp(-i2\pi f\tau)\, d\tau. \tag{134}$$

The quasistatic approximation for $W_\theta(f)$ obtained by starting with (130) is (see Ref. 13)

$$W_\theta(f) \approx W_\varphi(f) + \sum_{k=0}^{\infty} \frac{I_{2k}^2(2\pi D)^{-4k-2}}{(2k+1)!} \int_{-\infty}^{\infty} \psi_\tau^{2k+1} \exp(-i2\pi f\tau)\, d\tau,$$

$$I_{2k} = \int_0^\infty x^{2k} \exp\left[-\frac{x^2}{2} - \frac{x}{D} f_c\right] dx.$$

This series, which converges for all D, is of the same form as (134) in that it has I_{2k}/D^{2k+1} in place of the divergent sum with respect to n. When D becomes small, the two expressions for $W_\theta(f)$ approach equality in the sense that the sum with respect to n is the asymptotic expansion of I_{2k}/D^{2k+1}.

XII. LIOU'S APPROXIMATION FOR SECOND AND THIRD ORDER INTERCHANNEL MODULATION

It is instructive to relate our main result, equation (15) for $W_\theta(f)$, to an approximation for the interchannel modulation given by Liou.[5] Liou's approximation is equivalent to taking additional terms in the small frequency deviation approximation given in Section XI.

The interchannel modulation is represented by the portions of $W_\theta(f)$ in equation (15) which contains $T(\rho, f - \rho)$ and $S(\rho, \sigma, f - \rho - \sigma)$. Liou's approximation may be obtained by (i) approximating $T(\rho, f - \rho)$ by the leading term, namely $S(\rho, f - \rho)$, in equation (17) and (ii) expanding $S(\rho, f - \rho)$, $S(\rho, \sigma, f - \rho - \sigma)$ in powers of ρ, σ, and f out to and including degree 4. This leads to

$$\begin{aligned} T(\rho, f - \rho) \approx S(\rho, f - \rho) &= \Gamma(f) - \Gamma(\rho)\Gamma(f - \rho) \\ &= \rho(f - \rho)[\lambda_2 + f\ell_2 + f^2\ell_3 + \rho(f - \rho)\ell_4] + O(f^5) \\ S(\rho, \sigma, f - \rho - \sigma) &= \rho\sigma(f - \rho - \sigma)[\lambda_3 + f\ell_1] + O(f^5) \end{aligned} \tag{135}$$

where

$$\begin{aligned} \Gamma(f) &= \sum_{n=0}^{\infty} \alpha_n f^n/n!, \qquad \alpha_0 = 1 \\ \lambda_2 &= \alpha_2 - \alpha_1^2, \qquad \lambda_3 = \alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1^3 \\ \ell_1 &= (\alpha_4 - 2\alpha_3\alpha_1 - \alpha_2^2 + 2\alpha_2\alpha_1^2)/2 \\ \ell_2 &= (\alpha_3 - \alpha_2\alpha_1)/2 \\ \ell_3 &= (\alpha_4 - \alpha_3\alpha_1)/6 \\ \ell_4 &= 4^{-1}(\alpha_4 - \alpha_2^2) - 3^{-1}(\alpha_4 - \alpha_3\alpha_1). \end{aligned} \tag{136}$$

Equation (162) of Appendix B gives the approximation for $S(\rho, \sigma,$

$f - \rho - \sigma$) shown in equation (135). It does not give the higher order terms in $S(\rho, f - \rho)$ shown in equation (135). These must be calculated from the series for $\Gamma(f)$. Although the ℓ's and λ's used here are not precisely the same as those used by Liou, they are of the same general character.

When the expressions [equation (135)] for $T(\rho, f - \rho)$ and $S(\rho, \sigma, f - \rho - \sigma)$ are used in equation (15) for $W_\theta(f)$, the second and third order interchannel modulation terms are found to be

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho) W_\varphi(f-\rho)\rho^2(f-\rho)^2 \\ \cdot\{[\lambda_{2i} + f^2\ell_{3i} + \rho(f-\rho)\ell_{4i}]^2 + f^2\ell_{2r}\} \\ + \frac{1}{6}[\lambda_{3i}^2 + f^2\ell_{1r}^2]\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma \\ \cdot W_\varphi(\rho)W_\varphi(\sigma)W_\varphi(f-\rho-\sigma)\rho^2\sigma^2(f-\rho-\sigma)^2 \end{aligned} \tag{137}$$

where the second subscripts r and i denote "real part" and "imaginary part." The basic approximation used by Liou [his Eqs. (29) and (30)] may be put in this form by expressing his Fourier transforms as convolution integrals and combining terms.

APPENDIX A

Power Spectra of Real and Imaginary Parts of a Complex Random Process

Let $z(t)$ be a complex, stationary, ergodic, random process [for example the complex phase angle $\Theta(t)$] and let $x(t)$, $y(t)$ be its real and imaginary parts. We seek convenient expressions for the power spectra $W_x(t)$, $W_y(t)$ of $x(t)$ and $y(t)$ when $z(t)$ is the sum of several correlated complex random processes, say $a(t)$, $b(t)$, $c(t)$, $\cdots$. For illustration we take

$$z(t) = a(t) + b(t) + c(t) \tag{138}$$

which corresponds to equation (13) for $\Theta(t)$ with $a(t)$, $b(t)$, and $c(t)$ in place of $\Phi(t)$, $-iK(t)$, and $i2^{-1}K^2(t)$, respectively.

Denoting functions with arguments t, $t + \tau$ by subscripts 1, 2 and using relations of the type

$$x_1 = \text{Re } z_1 = (z_1 + z_1^*)/2 \tag{139}$$

leads to the following expression for the ensemble average $\langle x_1 x_2 \rangle_{\text{av}}$

$$\begin{aligned}\langle x_1x_2\rangle_{av} &= \langle (z_1 + z_1^*)(z_2 + z_2^*)\rangle_{av}/4 \\ &= \langle (z_1z_2 + z_1^*z_2^*) + (z_1^*z_2 + z_1z_2^*)\rangle_{av}/4 \\ &= 2^{-1}\,\mathrm{Re}\,(\langle z_1z_2\rangle_{av} + \langle z_1^*z_2\rangle_{av}).\end{aligned} \tag{140}$$

It is convenient to write $\langle z_1z_2\rangle_{av}$ as

$$\langle z_1z_2\rangle_{av} = A(\tau) + A(-\tau) \tag{141}$$

where

$$A(\tau) = \langle \tfrac{1}{2}(a_1a_2 + b_1b_2 + c_1c_2) + a_1b_2 + a_1c_2 + b_1c_2\rangle_{av}. \tag{142}$$

This is suggested when the product z_1z_2 is multiplied out and terms of the type $a_1b_2 + b_1a_2$ are considered. Thus, if

$$\langle a_1b_2\rangle_{av} = \langle a(t)b(t + \tau)\rangle_{av} = f(\tau), \tag{143}$$

setting $t = t' - \tau$ and making use of stationarity leads to

$$\langle b_1a_2\rangle_{av} = \langle a(t + \tau)b(t)\rangle_{av} = \langle a(t')b(t' - \tau)\rangle_{av} = f(-\tau) \tag{144}$$

and hence to equation (141).

Similarly, replacing a_1, b_1, c_1 by a_1^*, b_1^*, c_1^* leads to writing the second ensemble average in equation (140) for $\langle x_1x_2\rangle_{av}$ as

$$\langle z_1^*z_2\rangle_{av} = B(\tau) + B^*(-\tau) \tag{145}$$

where

$$B(\tau) = \langle \tfrac{1}{2}(a_1^*a_2 + b_1^*b_2 + c_1^*c_2) + a_1^*b_2 + a_1^*c_2 + b_1^*c_2\rangle_{av}. \tag{146}$$

For terms of the type $a_1^*b_2 + b_1^*a_2$, the analogues of equation (143) and (144) are

$$\begin{aligned}\langle a_1^*b_2\rangle_{av} &= \langle a^*(t)b(t + \tau)\rangle_{av} = f(\tau), \\ \langle b_1^*a_2\rangle_{av} &= \langle a(t + \tau)b^*(t)\rangle_{av} = \langle a(t')b^*(t' - \tau)\rangle_{av} \\ &= \langle a^*(t')b(t' - \tau)\rangle_{av}^* = f^*(-\tau).\end{aligned} \tag{147}$$

Comparing equation (13) for $\Theta(t)$ with equation (138) for $z(t)$ suggests setting $a(t) = \Phi(t)$, $b(t) = -iK(t)$, and $c(t) = i2^{-1}K^2(t)$; this leads to equation (49) for $A(\tau)$, $B(\tau)$ given in Section IV.

Equation (140) for the autocovariance $\langle x_1x_2\rangle_{av}$ of $x(t)$ now takes the form

$$\langle x_1x_2\rangle_{av} = 2^{-1}\,\mathrm{Re}\,[A(\tau) + A(-\tau) + B(\tau) + B^*(-\tau)], \tag{148}$$

and the power spectrum of $x(t)$ is

$$W_x(f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau)\langle x_1 x_2\rangle_{av}\, d\tau, \qquad \omega = 2\pi f.$$

This may be written as

$$W_x(f) = 2^{-1}\,\mathrm{Re}\,[P(f) + Q(f) + P^*(-f) + Q^*(-f)] \tag{149}$$

where

$$P(f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau)A(\tau)\, d\tau,$$
$$Q(f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau)B(\tau)\, d\tau. \tag{150}$$

Equation (149) for $W_x(f)$ may be derived from

$$2^{-1}\,\mathrm{Re}\,[A(\tau) + A(-\tau)] = 4^{-1}[A(\tau) + A^*(\tau) + A(-\tau) + A^*(-\tau)]$$
$$2^{-1}\,\mathrm{Re}\,[B(\tau) + B^*(-\tau)] = 4^{-1}[B(\tau) + B^*(\tau) + B(-\tau) + B^*(-\tau)] \tag{151}$$

and relations of the type

$$P^*(f) = \int_{-\infty}^{\infty} \exp(i\omega\tau)A^*(\tau)\, d\tau = \int_{-\infty}^{\infty} \exp(-i\omega\tau)A^*(-\tau)\, d\tau$$
$$P(-f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau)A(-\tau)\, d\tau, \tag{152}$$
$$P^*(-f) = \int_{-\infty}^{\infty} \exp(-i\omega\tau)A^*(\tau)\, d\tau.$$

The power spectrum $W_y(f)$ of the imaginary part $y(t)$ of $z(t)$ may be computed in much the same way, starting with

$$\langle y_1 y_2\rangle_{av} = \langle (z_1 - z_1^*)(z_2 - z_2^*)\rangle_{av}/(2i)^2$$
$$= 2^{-1}\,\mathrm{Re}\,[-\langle z_1 z_2\rangle_{av} + \langle z_1^* z_2\rangle_{av}].$$

This differs from equation (140) for $\langle x_1 x_2\rangle_{av}$ only in the sign of $\langle z_1 z_2\rangle_{av}$. Therefore only the signs of $A(\tau)$ and $P(f)$ need be changed in the earlier work, and we get

$$W_y(f) = 2^{-1}\,\mathrm{Re}\,[-P(f) + Q(f) - P^*(-f) + Q^*(-f)] \tag{153}$$

APPENDIX B

The Functions $S(\rho, \sigma)$, $S(\rho, \sigma, \nu)$, $\cdots$

The function

$$S(x_1, x_2, \cdots, x_n) = \int_{-\infty}^{\infty} du\, \gamma(u) H_u(x_1) H_u(x_2) \cdots H_u(x_n) \tag{154}$$

where

$$H_u(x) = \exp(-i2\pi xu) - \Gamma(x) \tag{155}$$

is a symmetrical function of the x's. It may be expressed as the sum of products of Γ's by replacing the H_u's by their definitions, multiplying out, and using the fact that $\Gamma(f)$ is the Fourier transform of $\gamma(t)$. This evaluation of the integral shows that $S(x_1, x_2, \cdots, x_n)$ is given symbolically by

$$S(x_1, x_2, \cdots, x_n) = \prod_{k=1}^{n} [y^{x_k} - \Gamma(x_k)] \tag{156}$$

where, after multiplying out, the various powers of y are replaced by Γ's according to the rule $y^z \to \Gamma(z)$.

For example,

$$\begin{aligned} S(\rho) &= 0, \\ S(\rho, \sigma) &= [y^\rho - \Gamma(\rho)][y^\sigma - \Gamma(\sigma)] \\ &= y^{\rho+\sigma} - y^\rho\Gamma(\sigma) - \Gamma(\rho)y^\sigma + \Gamma(\rho)\Gamma(\sigma) \\ &= \Gamma(\rho + \sigma) - \Gamma(\rho)\Gamma(\sigma) \end{aligned} \tag{157}$$

$$\begin{aligned} S(\rho, \sigma, \nu) = \Gamma(\rho + \sigma + \nu) &- \Gamma(\rho + \sigma)\Gamma(\nu) - \Gamma(\rho + \nu)\Gamma(\sigma) \\ &- \Gamma(\sigma + \nu)\Gamma(\rho) + 2\Gamma(\rho)\Gamma(\sigma)\Gamma(\nu). \end{aligned} \tag{158}$$

For four variables, writing (x) for $\Gamma(x)$,

$$\begin{aligned} S(\rho, \sigma, \nu, \mu) = (\rho + \sigma + \nu + \mu) &- (\rho + \sigma + \nu)(\mu) - (\rho + \sigma + \mu)(\nu) \\ &- (\rho + \nu + \mu)(\sigma) - (\sigma + \mu + \nu)(\rho) + (\rho + \sigma)(\nu)(\mu) \\ &+ (\rho + \nu)(\sigma)(\mu) + (\rho + \mu)(\sigma)(\nu) + (\sigma + \nu)(\rho)(\mu) \\ &+ (\sigma + \mu)(\rho)(\nu) + (\nu + \mu)(\rho)(\sigma) - 3(\rho)(\sigma)(\mu)(\nu). \end{aligned} \tag{159}$$

$S(x_1, x_2, \cdots, x_n)$ vanishes when one or more of its arguments are zero because $H_u(0)$ is zero. Of interest is the form taken by $S(x_1, \cdots, x_n)$ when the x's are small and $\Gamma(f)$ may be expanded as a power series in f. Let the power series be

$$\Gamma(f) = \sum_{n=0}^{\infty} \frac{f^n}{n!} \alpha_n , \qquad \alpha_0 = 1. \tag{160}$$

Since

$$\begin{aligned} \Gamma(f) &= \int_{-\infty}^{\infty} \gamma(u) \exp (\xi f)\, du, \qquad \xi = -i2\pi u \\ &= \sum_{n=0}^{\infty} \frac{f^n}{n!} \int_{-\infty}^{\infty} \gamma(u)\xi^n\, du \end{aligned}$$

it follows that

$$\alpha_n = \int_{-\infty}^{\infty} \gamma(u)\xi^n\, du. \tag{161}$$

When x is small, equation (155) for $H_u(x)$ gives

$$\begin{aligned} H_u(x) &= \exp (\xi x) - \Gamma(x) \\ &= x[(\xi - \alpha_1) + 2^{-1}x(\xi^2 - \alpha_2)] + O(x^3). \end{aligned}$$

Then

$$\prod_{k=1}^{n} H_u(x_n) = (x_1 x_2 \cdots x_n)\left[(\xi - \alpha_1)^n + 2^{-1}(\xi - \alpha_1)^{n-1}(\xi^2 - \alpha_2) \sum_{k=1}^{n} x_k \right] + O(x^{n+2})$$

and substitution in the integral [equation (154)] defining $S(x_1, \cdots, x_n)$ leads to

$$\begin{aligned} &S(x_1, \cdots, x_n) \\ &= (x_1 x_2 \cdots x_n) \int_{-\infty}^{\infty} du\, \gamma(u) \left[\sum_{\ell=0}^{n} \binom{n}{\ell} \xi^{\ell}(-\alpha_1)^{n-\ell} \right. \\ &\quad \left. + 2^{-1} \sum_{1}^{n} x_k \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\xi^{\ell+2} - \xi^{\ell}\alpha_2)(-\alpha_1)^{n-\ell-1} \right] + O(x^{n+2}) \\ &= (x_1 x_2 \cdots x_n) \left[\sum_{\ell=0}^{n} \binom{n}{\ell} \alpha_\ell (-\alpha_1)^{n-\ell} \right. \\ &\quad \left. + 2^{-1} \sum_{1}^{n} x_k \sum_{\ell=1}^{n-1} \binom{n-1}{\ell} (\alpha_{\ell+2} - \alpha_\ell \alpha_2)(-\alpha_1)^{n-\ell-1} \right] + O(x^{n+2}). \end{aligned} \tag{162}$$

This is the approximation needed to examine the form taken by $W_\theta(f)$

when the bandwidth of $\varphi(t)$ becomes small, as it does in Section XI.

When $\Gamma(f)$ is such that $| \Gamma(f) - 1 | < \epsilon \ll 1$ for all values of f, it may be shown that the symbolic form of equation (156) for $S(x_1, \cdots, x_n)$ becomes

$$S(x_1, \cdots, x_n) = \prod_{k=1}^{n} (y^{x_k} - 1) + O(\epsilon^2) \tag{163}$$

and the Γ's appear only linearly. Furthermore, $S(x_1, \cdots, x_n)$ is $O(\epsilon)$. For example

$$\begin{aligned} S(\rho, \sigma) &= y^{\rho+\sigma} - y^{\rho} - y^{\sigma} + 1 + O(\epsilon^2) \\ &= \Gamma(\rho + \sigma) - \Gamma(\rho) - \Gamma(\sigma) + 1 + O(\epsilon^2). \end{aligned}$$

This result is of interest in connection with the first order approximation discussed in Appendix D.

To establish equation (163) let

$$z_k = y^{x_k} - 1, \qquad \epsilon_k = 1 - \Gamma(x_k)$$

so that equation (156) for S becomes

$$\begin{aligned} S(x_1, \cdots, x_n) &= \prod_{k=1}^{n} (z_k + \epsilon_k) \\ &= \prod_{k=1}^{n} z_k + \prod_{k=1}^{n} \epsilon_k \prod_{\ell=1}^{n}{}' z_\ell + O(\epsilon^2). \end{aligned} \tag{164}$$

Here the factor z_k is omitted from $\prod'$. When the product

$$z_1 z_2 \cdots z_m = \prod_{k=1}^{m} (y^{x_k} - 1)$$

is multiplied out and the y^u's are replaced by $\Gamma(u)$'s, the result is the sum of $2^m \Gamma$'s $[(1 = \Gamma(0)]$. Half of the Γ's will have plus signs and the other half will have minus signs. Adding $+1$ for each $-\Gamma$ and -1 for each $+\Gamma$ shows that the entire sum is $O(\epsilon)$. Hence $z_1 z_2 \cdots z_m$ is $O(\epsilon)$, and when this is used to show that the $\prod'$ in equation (164) is $O(\epsilon)$, the result stated in equation (163) follows.

APPENDIX C

The Functions S_n and S_{nm}

The functions $S_n(x_1, x_2, \cdots, x_n)$ and $S_{nm}(x_1, \cdots, x_n; y_1, \cdots, y_m)$ are defined by the integrals, for $n \geqq 1$ and $m \geqq 1$,

$$S_n(x_1, \cdots, x_n) = \int_{-\infty}^{\infty} du\, \gamma(u) \exp [a(u)] \prod_{k=1}^{n} H_u(x_k), \tag{165}$$

$$\begin{aligned} &S_{nm}(x_1, \cdots, x_n; y_1, \cdots, y_m) \\ &\quad = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, \gamma(v)\gamma(w) \exp [a(v) + a(w) + 2c(w, v, o)] \\ &\qquad \cdot \left[\prod_{k=1}^{n} H_v(x_k)\right]\left[\prod_{l=1}^{m} H_w(y_l)\right] \end{aligned}$$

where $H_u(x)$, $a(u)$, $c(w, v, o)$ are given by

$$\begin{aligned} H_u(x) &= \exp(-i2\pi xu) - \Gamma(x) \\ a(u) &= -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f) H_u(-f) H_u(f) \\ c(w, v, o) &= -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f) H_w(-f) H_v(f) = c(v, w, o) \end{aligned} \tag{166}$$

[see equation (66)]. For $n = 0$, S_0 is defined as

$$S_0 = \int_{-\infty}^{\infty} du\, \gamma(u) \exp [a(u)] \tag{167}$$

and for the double subscripts,

$$\begin{aligned} &S_{n0}(x_1, \cdots, x_n;) \\ &\quad = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, \gamma(v)\gamma(w) \exp [a(v) + a(w) + 2c(w, v, o)] \prod_{k=1}^{n} H_v(x_k) \\ &S_{n0}(x_1, \cdots, x_n;) = S_{0n}(; x_1, \cdots, x_n) \\ &S_{00}(;) = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dw\, \gamma(v)\gamma(w) \exp [a(v) + a(w) + 2c(w, v, o)]. \end{aligned} \tag{168}$$

The functions S_n and S_{nm} depend upon both $W_\varphi(f)$ and $\Gamma(f)$ [through $H_u(f)$]. This is in contrast with the function $S(x_1, \cdots, x_n)$, defined in Appendix B, which depends only on $\Gamma(f)$ and is independent of $W_\varphi(f)$.

The function S_n may be expressed as the sum of multiple integrals involving the functions S. Expanding $\exp [a(u)]$ in powers of $a(u)$ and replacing each $a(u)$ by its integral [equation (166)] with ρ in place of the variable of integration f leads to

$$S_0 = 1 + \sum_{j=1}^{\infty} \frac{(-\frac{1}{2})^j}{j!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_j$$

$$\cdot\left[\prod_{k=1}^{j} W_{\varphi}(\rho_k)\right] S(\rho_1, \cdots, \rho_j, -\rho_1, \cdots, -\rho_j)$$

$$S_n(x_1, x_2, \cdots, x_n)$$
$$= S(x_1, x_2, \cdots, x_n) + \sum_{j=1}^{\infty} \frac{(-\frac{1}{2})^j}{j!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_j$$
$$\cdot\left[\prod_{k=1}^{j} W_{\varphi}(\rho_k)\right] S(\rho_1, \cdots, \rho_j, -\rho_1, \cdots, -\rho_j, x_1, \cdots, x_n) \tag{169}$$

where $n \geqq 1$.

Similarly, expanding $\exp[2c(w, v, o)]$ in the integrand of the integrals defining the S_{nm} functions leads to

$$S_{00}(;) = S_0^2 + \sum_{j=1}^{\infty} \frac{(-)^j}{j!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_j$$
$$\cdot\left[\prod_{k=1}^{j} W_{\varphi}(\rho_k)\right] S_j(\rho_1, \cdots, \rho_j) S_j(-\rho_1, \cdots, -\rho_j)$$

$$S_{n0}(x_1, \cdots, x_n;)$$
$$= S_0 S_n(x_1, \cdots, x_n) + \sum_{j=1}^{\infty} \frac{(-)^j}{j!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_j \left[\prod_{k=1}^{j} W_{\varphi}(\rho_k)\right]$$
$$\cdot S_{j+n}(\rho_1, \cdots, \rho_j, x_1, \cdots, x_n) S_j(-\rho_1, \cdots, -\rho_j)$$

$$S_{nm}(x_1, \cdots, x_n; y_1, \cdots, y_m) = S_n(x_1, \cdots, x_n) S_m(y_1, \cdots, y_m)$$
$$+ \sum_{j=1}^{\infty} \frac{(-)^j}{j!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_j \left[\prod_{k=1}^{j} W_{\varphi}(\rho_k)\right]$$
$$\cdot S_{j+n}(\rho_1, \cdots, \rho_j, x_1, \cdots, x_n) S_{j+m}(-\rho_1, \cdots, -\rho_j, y_1, \cdots, y_m) \tag{170}$$

when $n \geqq 1$ and $m \geqq 1$.

In order to obtain an inequality for $S_n(x_1, \cdots, x_n)$ assume that (*i*) the termwise integration in the derivation of the series in equation (169) is legitimate, and (*ii*) an $M > 1$ exists such that for all real values of f

$$M > |\Gamma(f)| = |G(f + f_0)/G(f_0)|.$$

Since $S(x_1, \cdots, x_n)$ may be expressed as the sum of 2^n terms, each of which is a product of not more than n of the Γ's,

$$|S(x_1, \cdots, x_n)| < 2^n M^n.$$

Furthermore, since the integral of $W_\varphi(f)$ from $-\infty$ to $+\infty$ is equal to $\langle\varphi^2(t)\rangle_{av}$, the terms of the series in equation (169) for $S_n(x_1, \cdots, x_n)$ are dominated by the terms in

$$\sum_{j=0}^{\infty} \frac{(\frac{1}{2})^j}{j!} \langle\varphi^2\rangle_{av}^j (2M)^{2j+n} = (2M)^n \exp [2M^2\langle\varphi^2\rangle_{av}].$$

Therefore the series in equation (169) converges and

$$| S_n(x_1, \cdots, x_n) | < (2M)^n \exp [2M^2\langle\varphi^2\rangle].$$

This inequality may be used to show that the series in equation (170) for S_{nm} converges and that

$$| S_{nm}(x_1, \cdots, x_n; y_1, \cdots, y_m) | < (2M)^{n+m} \exp [8M^2\langle\varphi^2\rangle_{av}].$$

The leading terms in the series required to handle the second and third order modulation are

$$S_0 = 1 - \frac{1}{2}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho) S(\rho, -\rho)$$
$$+ \frac{1}{8}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma) S(\rho, \sigma, -\rho, -\sigma) + O(\varphi^6), \quad (171)$$

$$S_1(f) = 0 - \frac{1}{2}\int_{-\infty}^{\infty} d\rho\, W_\varphi(\rho) S(\rho, -\rho, f)$$
$$+ \frac{1}{8}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma) S(\rho, \sigma, -\rho, -\sigma, f) + O(\varphi^6), \quad (172)$$

$$S_2(\rho, \nu) = S(\rho, \nu) - \frac{1}{2}\int_{-\infty}^{\infty} d\sigma\, W_\varphi(\sigma) S(\sigma, -\sigma, \rho, \nu) + O(\varphi^4), \quad (173)$$

$$S_3(\rho, \sigma, \nu) = S(\rho, \sigma, \nu) + O(\varphi^2), \quad (174)$$

$$S_{00}(;) = S_0^2 + \frac{1}{2}\int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma) S(\rho, \sigma) S(-\rho, -\sigma)$$
$$+ O(\varphi^6), \quad (175)$$

$$S_{10}(f;) = S_1(f) + \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\sigma\, W_\varphi(\rho) W_\varphi(\sigma)[4^{-1} S(\rho, -\rho, f) S(\sigma, -\sigma)$$
$$+ 2^{-1} S(\rho, f) S(\sigma, -\sigma, -\rho) + 2^{-1} S(\rho, \sigma, f) S(-\rho, -\sigma)] + O(\varphi^6),$$
$$(176)$$

$$S_{20}(\rho, \nu;) = S_2(\rho, \nu) - \frac{1}{2}\int_{-\infty}^{\infty} d\sigma\, W_\varphi(\sigma) S(\sigma, -\sigma) S(\rho, \nu) + O(\varphi^4).$$

$$S_{30}(\rho, \sigma, \nu;) = S_3(\rho, \sigma, \nu) + O(\varphi^2),$$

$$S_{11}(\rho; \nu) = -\int_{-\infty}^{\infty} d\sigma\, W_\varphi(\sigma) S(\sigma, \rho) S(-\sigma, \nu) + O(\varphi^4),$$

$$S_{21}(\rho, \sigma; \nu) = O(\varphi^2).$$

In obtaining the leading terms in $S_{00}(;)$ and $S_{10}(f;)$ the leading terms in S_0, $S_1(f)$ and $S_2(\rho, f)$ were used. $S_1(f)$ is $O(\varphi^2)$ in contrast with S_0, S_2, S_3, $\cdots$ which are $O(1)$.

APPENDIX D

Derivation of Earlier First Order Approximation by Present Procedure

The first order approximations which are given in Section VII are somewhat more complicated, as well as more accurate, than the ones which have appeared in the literature.[3, 4, 9–12] Here the relation between the earlier and present work will be brought out by applying the procedure of Section VII to obtain a first order approximation, which is the same as the given in Ref. 10 [the $\theta(t)$ of Ref. 10 is $\theta(t) - \varphi(t)$ in the present notation].

The derivation starts from the initial equations (5) and (7) for the filter output $s_0(t)$,

$$s_0(t) = \{\exp[-\alpha_0 - i\beta_0 + i\Theta(t)]\} \exp(i\omega_0 t)$$
$$\Theta(t) = -i \ln \left\{ \int_{-\infty}^{\infty} du\, \gamma(u) \exp[i\varphi(t-u)] \right\}. \tag{177}$$

The difference between the output phase angle $\theta(t) = \text{Re}\,\Theta(t)$ and the input phase angle $\varphi(t)$ is assumed to be small, and the filter delay is usually taken to be zero at the carrier frequency, that is, $\text{Im}\,[d\Gamma(f)/df]$ is zero at $f = 0$.

Adding and subtracting $i\varphi(t)$ [instead of the linear portion $\Phi(t)$ of the output] in the exponent appearing in equation (177) for $\Theta(t)$ gives

$$\Theta(t) = \varphi(t) - i \ln[1 + k(t)],$$
$$k(t) = \int_{-\infty}^{\infty} du\, \gamma(u) \{\exp[i\varphi(t-u) - i\varphi(t)] - 1\}. \tag{178}$$

The first order approximations to the complex phase angle $\Theta(t)$ and the output phase angle $\theta(t)$ are now

$$\Theta(t) \approx \varphi(t) - ik(t)$$

and

$$\theta(t) \approx \varphi(t) + \text{Re}\,[-ik(t)], \tag{179}$$

respectively.

The analysis of the earlier sections goes through much as before with $k(t)$ in place of $K(t)$, and

$$h_u(f) = \exp(-i2\pi fu) - 1 \tag{180}$$

in place of

$$H_u(f) = \exp(-i2\pi fu) - \Gamma(f).$$

An illustration of how $h_u(f)$ enters the analysis is furnished by the computation of $\langle \exp[i\varphi(t-u) - i\varphi(t)]\rangle_{\text{av}}$. As suggested by equation (56) for $\varphi(t-u) - \Phi(t)$, let

$$\begin{gathered} L\varphi(t) - \varphi(t-u) - \varphi(t) \\ \ell(f) = \exp(-i2\pi fu) - 1 = h_u(f) \\ -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)\ell(f)\ell(-f) = -\frac{1}{2}\int_{-\infty}^{\infty} df\, W_\varphi(f)h_u(f)h_u(-f) = a'(u) \end{gathered} \tag{181}$$

$$\langle \exp[i\varphi(t-u) - i\varphi(t)]\rangle_{\text{av}} = \exp[a'(u)].$$

Of most interest in practice is the power spectrum $W_\xi(f)$ of $\xi(t)$,

$$\begin{gathered} \xi(t) = \text{Re}\,[-ik(t)] \\ \theta(t) \approx \varphi(t) + \xi(t) \end{gathered} \tag{182}$$

where $\xi(t)$ is an approximation to the distortion. The power spectrum $W_\xi(f)$ is the Fourier transform of the covariance $\langle \xi_1\xi_2\rangle$ where, as before, subscripts 1, 2 refer to times t, $t + \tau$, respectively. By putting $\xi(t)$, $-ik(t)$ for $\theta(t)$, $\Theta(t)$ in equation (47), or directly,

$$\langle \xi_1\xi_2\rangle_{\text{av}} = 2^{-1}\,\text{Re}\,[-\langle k_1k_2\rangle_{\text{av}} + \langle k_1^*k_2\rangle_{\text{av}}]. \tag{183}$$

It may be shown that

$$\langle k_1\rangle_{\text{av}} = \int_{-\infty}^{\infty} du\, \gamma(u) \exp[a'(u)]$$

$$\langle k_1k_2\rangle_{\text{av}} = \langle k_1\rangle_{\text{av}}^2 + \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma(u)\gamma(v) \exp[a'(u) + a'(v)] \cdot \{\exp[2c'(u, v, \tau)] - 1\}$$

$$\langle k_1^* k_2 \rangle_{av} = | \langle k_1 \rangle_{av} |^2 + \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, \gamma^*(u)\gamma(v) \exp [a'(u) + a'(v)] \cdot \{\exp [-2c'(u, v, \tau)] - 1\} \tag{184}$$

where

$$\begin{aligned} c'(u, v, \tau) &= -\frac{1}{2} \int_{-\infty}^{\infty} df\, W_\varphi(f) h_u(-f) h_v(f) \exp (i2\pi f\tau) \\ a'(u) &= c'(u, u, o) \\ c'(u, v, -\tau) &= c'(v, u, \tau). \end{aligned} \tag{185}$$

Since $h(-f)$ is equal to $h^*(f)$, $a'(u)$ and $c'(u, v, \tau)$ are real. Furthermore,

$$\begin{aligned} c'(u, v, \tau) &= -\tfrac{1}{2}[R_\varphi(\tau + u - v) - R_\varphi(\tau - v) - R_\varphi(\tau + u) + R_\varphi(\tau)] \\ a'(u) &= -R_\varphi(o) + R_\varphi(u) \end{aligned} \tag{186}$$

where $R_\varphi(\tau)$ is the covariance $\langle \varphi(t)\varphi(t + \tau)\rangle_{av}$ of $\varphi(t)$.

The expression for $\langle \xi_1 \xi_2 \rangle_{av}$ obtained by combining equations (183) and (184) is similar to equation (8) of Ref. 10.

The power spectrum of $\xi(t)$ may be written as

$$W_\xi(f) = \xi_{dc}^2\, \delta(f) + 2^{-1} \operatorname{Re} [P(f) + Q(f) + P^*(-f) + Q^*(-f)] \tag{187}$$

in which ξ_{dc} is equal to $\operatorname{Im} \langle k_1 \rangle_{av}$ and

$$\begin{aligned} P(f) &= \int_{-\infty}^{\infty} d\tau \exp [-i2\pi f\tau][-\tfrac{1}{2}(\langle k_1 k_2 \rangle_{av} - \langle k_1 \rangle_{av}^2)], \\ Q(f) &= \int_{-\infty}^{\infty} d\tau \exp (-i2\pi f\tau)[\tfrac{1}{2}(\langle k_1^* k_2 \rangle_{av} - | \langle k_1 \rangle_{av} |^2)]. \end{aligned} \tag{188}$$

Addition gives

$$\begin{aligned} P(f) + Q(f) = \int_{-\infty}^{\infty} d\tau \exp (-i2\pi f\tau) \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \exp [a'(u) + a'(v)] \\ \cdot (-\tfrac{1}{2}\gamma(u)\gamma(v)\{\exp [2c'(u, v, \tau)] - 1\} \\ + \tfrac{1}{2}\gamma^*(u)\gamma(v)\{\exp [-2c'(u, v, \tau)] - 1\}) \end{aligned} \tag{189}$$

which, when used in equation (187), leads to an expression for $W_\xi(f)$ which is similar to the main result given in Ref. 10 [equation (16) of Ref. 10].

The relation $c'(u, v, -\tau) = c'(v, u, \tau)$ may be used to show that $Q(f)$ is real and $P(f)$ is even. When $\gamma(u)$ is real, $\Gamma(-f) = \Gamma^*(f)$, and the expression for $W_\xi(f)$ may be simplified.

Expanding $\exp [\pm 2c'(u, v, \tau)]$ in powers of $c'(u, v, \tau)$ leads to

$$W_\xi(f) = \xi_{dc}^2 \, \delta(f) + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_n \, \delta(f - \rho_1 - \cdots - \rho_n)$$
$$\cdot \left[\prod_{k=1}^{n} W_\varphi(\rho_k) \right] \mid S_n'(\rho_1, \cdots, \rho_n) - (-)^n S_n'^*(-\rho_1, \cdots, -\rho_n) \mid^2 \tag{190}$$

where, as in Appendix C for the unprimed S's,

$$S_n'(x_1, \cdots, x_n)$$
$$= \int_{-\infty}^{\infty} du \, \gamma(u) \exp [a'(u)] \prod_{k=1}^{n} [\exp (-i2\pi u x_k) - 1]$$
$$= S'(x_1, \cdots, x_n) + \sum_{j=1}^{\infty} \frac{(-\frac{1}{2})^j}{j!} \int_{-\infty}^{\infty} d\rho_1 \cdots \int_{-\infty}^{\infty} d\rho_j \left[\prod_{k=1}^{j} W_\varphi(\rho_k) \right]$$
$$\cdot S'(\rho_1, \cdots, \rho_j, -\rho_1, \cdots, -\rho_j, x_1, \cdots, x_n).$$

The series in equation (190) is analogous to the series in equation (90) for the more accurate first order approximation based on $\varphi(t - u) - \Phi(t)$. The function $S'(x_1, \cdots, x_n)$ is the analogue of $S(x_1, \cdots, x_n)$ discussed in Appendix B and is defined by

$$S'(x_1, \cdots, x_n) = \int_{-\infty}^{\infty} du \, \gamma(u) \prod_{k=1}^{n} h_u(x_k)$$
$$= \prod_{k=1}^{n} (y^{x_k} - 1).$$

The second equation is symbolic in that y^z is to be replaced by $\Gamma(z)$ after expansion of the product. It is shown in Appendix B that when $\mid \Gamma(f) - 1 \mid < \epsilon \ll 1$ for all real values of f,

$$S(x_1, \cdots, x_n) = S'(x_1, \cdots, x_n) + O(\epsilon^2).$$

APPENDIX E

Power Spectrum of $a\theta(t) + b \ln R(t)$

Section III states that an expression for the power spectrum $W_x(f)$ of $x(t) = a\theta(t) + b \ln R(t)$ may be obtained from equation (15) for $W_\theta(f)$ by replacing $U(f)$, $T(\rho, f-\rho)$, and $S(\rho, \sigma, f-\rho-\sigma)$ by $(a+ib)U(f)$, $(a + ib)T(\rho, f - \rho)$, and $(a + ib)S(\rho, \sigma, f - \rho - \sigma)$, respectively. Here the steps leading to this result are outlined.

From equation (6) for the complex phase angle $\Theta(t)$ we have

$$i\Theta(t) = \ln R(t) + i\Theta(t),$$

and it follows that, for arbitrary real values of a and b,

$$a\theta + b \ln R(t) = \text{Re}\,[(a + ib)\Theta(t)].$$

Consequently $W_x(f)$ may be obtained by replacing $\Theta(t)$ by $(a + ib)\Theta(t)$ in the analysis which led to equation (15) for $W_\theta(f)$.

The functions $A(\tau)$ and $B(\tau)$ appearing in equation (48) are replaced by $(a + ib)^2 A(\tau)$ and $| a + ib |^2 B(\tau)$, and their respective Fourier transforms $P(f)$ and $Q(f)$ are replaced by $(a+ib)^2 P(f)$ and $| a+ib |^2 Q(f)$. Each factor in $(a + ib)^2 = (a + ib)(a + ib)$ can be associated with factors in $P(f)$, and each factor in $(a + ib)(a + ib)^*$ with factors in $Q(f)$, in such a way that $U(f)$ becomes multiplied by $(a + ib)$ and $U^*(-f)$ by $(a + ib)^*$, and so on. This may be verified by repeating the analysis of Sections VIII and IX with the modified expressions.

APPENDIX F

Results Obtained from the Series for ln[1 + K(t)] may be Asymptotic

For gaussian $\varphi(t)$ with average 0 and rms value σ, the following considerations suggest that results obtained from equations (10) and (12), namely

$$\begin{aligned} \Theta(t) &= \Phi(t) - i \ln [1 + K(t)] \\ &= \Phi(t) + i \sum_1^\infty n^{-1}[-K(t)]^n \end{aligned} \tag{191}$$

represent the first few terms of an asymptotic series when $\sigma \to 0$. Since $K(t)$ is difficult to handle, we replace it by $a[a^{i\varphi(t)} - 1]$ where a is somewhat like the integral of $\gamma(u)$ between $-\infty$ and ∞. The value of this integral is 1, and we regard a as being near 1. The series for $\ln [1 + K(t)]$ behaves somewhat like the series

$$\ln (1 + a\{\exp [i\varphi(t)] - 1\}) = -\sum_1^\infty \frac{(-a)^n}{n} \{\exp [i\varphi(t)] - 1\}^n \tag{192}$$

in which the mean square value of the modulus of the nth term is

$$\int_{-\infty}^{\infty} \frac{\exp [-\varphi^2/(2\sigma^2)]}{\sigma(2\pi)^{\frac{1}{2}}} n^{-2}[2a \sin \varphi/2]^{2n} \, d\varphi. \tag{193}$$

When $\sigma \ll 1$ and n is not too large, most of the contribution to the integral (193) arises from the region around $\varphi = 0$, and the integral is approximately

$$1 \cdot 3 \cdots (2n - 1)(a\sigma)^{2n}/n^2.$$

Consequently, the first few terms decrease rapidly when $\sigma \ll 1$. However, when n is very large most of the contribution arises from the regions around $\varphi = \pm\pi, \pm 3\pi, \cdots$, where $[\sin(\varphi/2)]^{2n}$ is a narrow pulse of height 1 and area $2(\pi/n)^{\frac{1}{2}}$. When $\sigma \ll 1$ only the regions around $\varphi = \pm\pi$ are important and the integral (193) is approximately

$$\sigma^{-1} 2n^{3/2-5/2}(2a)^{2n} \exp\left[-\pi^2/(2\sigma^2)\right]$$

which tends to ∞ when a is near 1 and $n \to \infty$.

The fact that the rms values of the terms of the series in equation (192) decrease rapidly at first and then increase without limit suggests that results attained from the somewhat similar series in equation (191) may be asymptotic in nature as $\sigma \to 0$.

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