Some Theorems on the Dynamic Response of Nonlinear Transistor Networks

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Relative to the huge body of theory of linear time-invariant systems, very little of a general and precise nature is known about the network-theoretic properties of transistor circuits operating under large-signal conditions. One basic property P which a transitor network might have is that if the input approaches a constant, then the output approaches a constant which is independent of the initial conditions. In this paper we prove a stability theorem concerning a nonlinear differential equation that governs the behavior of a large class of networks. A corollary of this theorem asserts that if a certain condition is satisfied, then property P holds.

We consider also the problem of estimating the rate of decay of transients in transistor networks and we prove theorems which allow us to make some often quite conservative, but definite, statements concerning limitations on switching speeds. A practical example considered shows that in some cases the bounds, which are frequently very easy to evaluate, can be quite useful.

The proofs depend in an interesting way on the relationship between the static diode characteristic and the nonlinear capacitance associated with a semiconductor junction.

I. INTRODUCTION AND DERIVATION OF THE DIFFERENTIAL EQUATION

We initially consider the network of Fig. 1, which contains transistors, linear resistors, voltage sources, and current sources. Each transistor is represented by a model of the type shown in Fig. 2 (see Gummel[1] and Koehler[2]) which takes into account nonlinear dc properties as well as the presence of nonlinear junction capacitances. Associated with this model are six parameters: α_f, α_r, τ_e, τ_c, c_e, and c_c (all positive constants; $\alpha_f < 1$, $\alpha_r < 1$) and two nonlinear functions $f_e(\cdot)$ and $f_c(\cdot)$.

Concerning $f_e(\cdot)$ and $f_c(\cdot)$, for our purposes it is necessary to assume only that

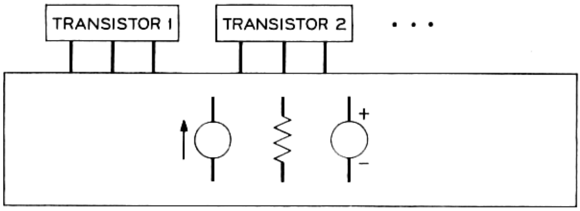

Fig. 1 — General network containing transistors, sources, and resistors.

Assumption 1: For each transistor: $f_e(\cdot)$ and $f_c(\cdot)$ are strictly-monotone increasing mappings of the real interval $(-\infty, \infty)$ into itself; $f_e(0) = f_c(0) = 0$, and $f_e(\cdot)$ and $f_c(\cdot)$ are continuously differentiable on $(-\infty, \infty)$.

The functions $f_e(\cdot)$ and $f_c(\cdot)$ of Gummel's model[1] are of simple exponential type and satisfy Assumption 1.

From Fig. 2:

$$i_e = \frac{d}{dt}\,[c_e v_e + \tau_e f_e(v_e)] + f_e(v_e) - \alpha_r f_c(v_c),$$

$$i_c = \frac{d}{dt}\,[c_c v_c + \tau_c f_c(v_c)] - \alpha_f f_e(v_e) + f_c(v_c).$$

Suppose that the network of Fig. 1 contains p transistors; for $k = 1, 2, \cdots, p$, let v_{2k-1} and v_{2k}, respectively, denote the emitter to base voltage and the collector to base voltage of the kth transistor. Simi-

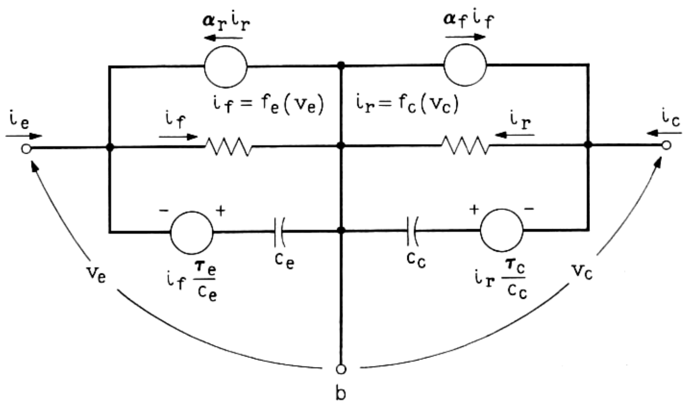

Fig. 2 — Transistor model.

larly, for $k = 1, 2, \cdots, p$, let i_{2k-1} and i_{2k}, respectively, denote the emitter current and the collector current of the kth transistor (with reference polarities as indicated in Fig. 2). Then, with $v = (v_1, v_2, \cdots, v_{2p})^{tr}$, $i = (i_1, i_2, \cdots, i_{2p})^{tr}$, $f_{2k-1}(\cdot)$ and c_{2k-1} the $f_e(\cdot)$ and c_e of the kth transistor, and $f_{2k}(\cdot)$ and c_{2k} the $f_c(\cdot)$ and c_c of the kth transistor,

$$i = \frac{d}{dt}[C(v)] + TF(v) \tag{1}$$

where, for $j = 1, 2, \ldots, 2p$,

$$[C(v)]_j = c_j v_j + \tau_j f_j(v_j) \tag{2}$$

$$[F(v)]_j = f_j(v_j), \tag{3}$$

and $T = T_1 \oplus T_2 \oplus \cdots \oplus T_p$, the direct sum of p 2×2 matrices T_k in which

$$T_k = \begin{bmatrix} 1 & -\alpha_r^{(k)} \\ -\alpha_f^{(k)} & 1 \end{bmatrix}$$

for $k = 1, 2, \cdots, p$.

We assume that the linear resistive portion of the structure of Fig. 1 introduces the constraint

$$i = -Gv + B \tag{4}$$

in which G is a conductance matrix and B is an element of the set $\mathcal{B}$ of all real bounded continuous $2p$-vector-valued functions of t on $[0, \infty)$.

From equations (1) and (4)

$$\frac{d}{dt}[C(v)] + TF(v) + Gv = B. \tag{5}$$

Let $u = C(v)$. Since all of the c_j and τ_j are positive, and each of the $f_j(\cdot)$ is continuous and monotone increasing, there exists a $C^{-1}(\cdot)$ such that $v = C^{-1}(u)$. Thus,

$$\frac{du}{dt} + TF[C^{-1}(u)] + GC^{-1}(u) = B. \tag{6}$$

The Jacobian matrix J_u of $TF[C^{-1}(u)] + GC^{-1}(u)$ is

$$T \operatorname{diag}\left\{\frac{f_j'[g_j(u_j)]}{c_j + \tau_j f_j'[g_j(u_j)]}\right\} + G \operatorname{diag}\left\{\frac{1}{c_j + \tau_j f_j'[g_j(u_j)]}\right\}$$

in which for all $j = 1, 2, \cdots, 2p$

$$g_j(u_j) = [C^{-1}(u)]_j$$

with each of the $g_j(\cdot)$ continuously differentiable.

Since J_u is continuously dependent on u, and $\| J_u \|$ ($\| \cdot \|$ any norm) is bounded from above uniformly in u, it follows that there exists a constant L such that

$$\| TF[C^{-1}(u_a)] + GC^{-1}(u_a) - TF[C^{-1}(u_b)] - GC^{-1}(u_b) \| \leqq L \| u_a - u_b \| \tag{7}$$

for all u_a and u_b belonging to real Euclidean $2p$-space E^{2p}. In particular, we have

$$\| TF[C^{-1}(u)] + GC^{-1}(u) - B \| \leqq L \| u \| + \|B \| \tag{8}$$

for all $t \geqq 0$ and all $u \in E^{2p}$. Therefore (see, for example, Nemytskii and Stepanov[3]), for any initial condition $u_0 \in E^{2p}$, there exists a unique continuous $2p$-vector-valued function $u(\cdot)$ such that $u(0) = u_0$ and equation (6) is satisfied for all $t > 0$. In other words, under the assumptions we have introduced, it makes sense to study the properties of the solution of the equation

$$\frac{du}{dt} + TF[C^{-1}(u)] + G[C^{-1}(u)] = B, \qquad t \geqq 0 \quad [u(0) = u_0] \tag{9}$$

II. STATEMENT OF RESULTS, AND EXAMPLES

We need the following definitions.

Definition 1: A real matrix M of arbitrary order n is *strongly column-sum dominant* if and only if for all $j = 1, 2, \ldots, n$

$$m_{jj} - \sum_{i \neq j} | m_{ij} | > 0.$$

An important property of T is that it is strongly column-sum dominant.

Definition 2: We shall say that a real matrix M of order $2p$ is an element of $\mathfrak{D}$ if and only if there exists a diagonal matrix diag $(d_1, d_2, \cdots, d_{2p})$ with each $d_j > 0$ such that

$$\alpha_f^{(k)} < \frac{d_{2k-1}}{d_{2k}} < \frac{1}{\alpha_r^{(k)}}$$

for $k = 1, 2, \cdots, p$, and diag $(d_1, d_2, \cdots, d_{2p})$ M is strongly column-sum dominant.

Our main result* concerning equation (9) is:

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* Proofs of all results in this section are given in Section III.

Theorem 1: If $G \in \mathfrak{D}$, and $u_a(\cdot)$ and $u_b(\cdot)$ satisfy

$$\frac{du_a}{dt} + TF[C^{-1}(u_a)] + G[C^{-1}(u_a)] = B_a(t), \qquad t \geqq 0 \tag{10}$$

$$\frac{du_b}{dt} + TF[C^{-1}(u_b)] + G[C^{-1}(u_b)] = B_b(t), \qquad t \geqq 0 \tag{11}$$

with $B_a \in \mathcal{B}$ and $B_b \in \mathcal{B}$, and if $[B_a(t) - B_b(t)] \to \theta$ (the zero vector of E^{2p}) as $t \to \infty$, then $[u_a(t) - u_b(t)] \to \theta$ as $t \to \infty$.

An interesting corollary of Theorem 1 is

Corollary 1: Referring to equation (9), if $G \in \mathfrak{D}$, and if there exists a constant vector B_∞ such that $[B(t) - B_\infty] \to \theta$ as $t \to \infty$, then there exists a constant vector u_∞ such that $[u(t) - u_\infty] \to \theta$ as $t \to \infty$, and u_∞ is independent of the initial condition u_0. In particular, if $B_\infty = \theta$, then $u_\infty = \theta$.

It is interesting to observe that $G \in \mathfrak{D}$ whenever the base leads of all transistors are connected together and there is a resistor between the emitter and base, and between the collector and base, of every transistor, for then G is strongly column-sum dominant. Also it is easy to give examples of conductance matrices which are not strongly column-sum dominant, and which belong to $\mathfrak{D}$. For instance, for the network of Fig. 3.

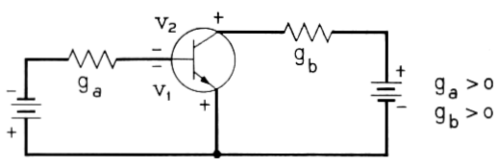

Fig. 3 — Single-transistor network.

$$G = \begin{pmatrix} g_a + g_b & -g_b \\ -g_b & g_b \end{pmatrix}$$

and diag $(d_1, d_2)G$ is strongly column-sum dominant for $d_2 = 1$ and some d_1 such that

$$\alpha_f < d_1 < \frac{1}{\alpha_r}.^*$$

* More generally, G of order $2p$ with positive diagonal elements belongs to $\mathfrak{D}$ whenever it is possible to obtain a strongly column-sum dominant matrix from G by adding an arbitrarily small positive quantity to a single diagonal element.

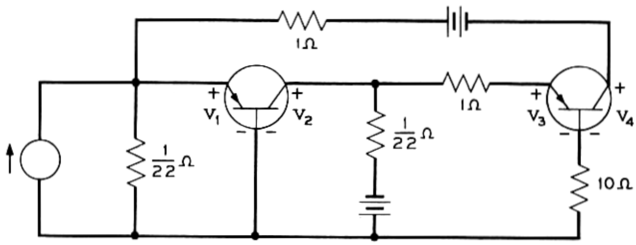

Fig. 4 — A two-transistor circuit.

As another example, consider the circuit of Fig. 4, for which

$$G = \frac{1}{21}\begin{bmatrix} 473 & -10 & 10 & -11 \\ -10 & 473 & -11 & 10 \\ 10 & -11 & 11 & -10 \\ -11 & 10 & -10 & 11 \end{bmatrix}.$$

Since diag (1, 1, 22, 22)G is strongly column-sum dominant, $G \in \mathfrak{D}$.

Finally, for the network shown in Fig. 5,

$$G = \frac{1}{21}\begin{bmatrix} 11 & -10 & 10 & -11 \\ -10 & 11 & -11 & 10 \\ 10 & -11 & 11 & -10 \\ -11 & 10 & -10 & 11 \end{bmatrix}.$$

In this case, G is obviously singular and hence does not belong to $\mathfrak{D}$. Suppose that the source current of Fig. 5 $i_0(t)$ is a constant and that the transistor functions $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$, and $f_4(\cdot)$ are all bounded from below by the constant b (this is certainly an assumption consistent with our earlier assumptions and with the character of transistor models

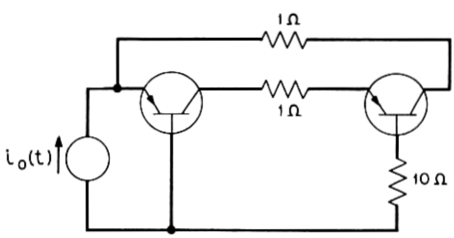

Fig. 5 — Transistor circuit for which the dc equations may have no solution.

ordinarily used.) We wish to show that here for sufficiently small i_0, there does not exist a constant vector u_∞ such that $[u(t) - u_\infty] \to \theta$ as $t \to \infty$.

Suppose that $u(t) \to u_\infty$, a constant vector, as $t \to \infty$. Then there would exist a $2p$-vector v_∞ such that $u_\infty = C(v_\infty)$ and

$$TF(v_\infty) + Gv_\infty = B$$

with $B = (i_0, 0, 0, \cdots, 0)^{tr}$. Let η denote the $2p$-row-vector $(1, 1, 1, \cdots, 1)$. Then

$$\eta TF(v_\infty) + \eta Gv_\infty = \eta B.$$

But $\eta Gv_\infty = 0$, and hence

$$i_0 = \sum_{k=1}^{p} [1 - \alpha_f^{(k)}] f_{2k-1}(v_{\infty 2k-1}) + \sum_{k=1}^{p} [1 - \alpha_r^{(k)}] f_{2k}(v_{\infty 2k})$$

which does not possess a solution v_∞ if

$$i_0 < b \sum_{k=1}^{p} [1 - \alpha_f^{(k)}] + [1 - \alpha_r^{(k)}].$$

2.1 *Estimation of the Rate of Decay of Transients*

Theorem 2: If the hypotheses of Corollary 1 are satisfied with $B(t) = B_\infty$ *for* $t \geqq 0$, *then*

$$\sum_{j=1}^{2p} d_j \mid u_j(t) - u_{\infty j} \mid \leqq \exp(-\underline{K}t) \sum_{j=1}^{2p} d_j \mid u_j(0) - u_{\infty j} \mid, \qquad t \geqq 0$$

for every set of positive constants $d_1, d_2, \cdots, d_{2p}$ *such that*

$$0 < \underline{K} \triangleq \min_j \min \left\{ \frac{1}{\tau_j} (1 - \tilde{d}_j d_j^{-1} \alpha_j), \frac{1}{c_j} (g_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid g_{ij} \mid) \right\}$$

in which $-\alpha_j$ *is the nonzero off-diagonal term in the jth column of* T, *and* $\tilde{d}_j = d_{j+1}$ *for j odd and* $\tilde{d}_j = d_{j-1}$ *for j even.*

It is easy to show that $G \in \mathfrak{D}$ implies that there are positive constants d_j, $j = 1, 2, \cdots, 2p$, such that $\underline{K} > 0$.

As an example of the application of Theorem 2, consider the problem of estimating the switching time of the single-transistor inverter circuit of Fig. 6 in which $\alpha_f = 0.968$, $c_e = 2 \times 10^{-12}$ fd, $\tau_e = 1.7 \times 10^{-10}$ second, $\alpha_r = 0.583$, $c_c = 1.7 \times 10^{-12}$ fd, and $\tau_c = 2.62 \times 10^{-8}$ second. Here (in mhos)

$$G = \begin{bmatrix} 1.1886 \times 10^{-3} & -1.01215 \times 10^{-3} \\ -1.01215 \times 10^{-3} & 1.01215 \times 10^{-3} \end{bmatrix}$$

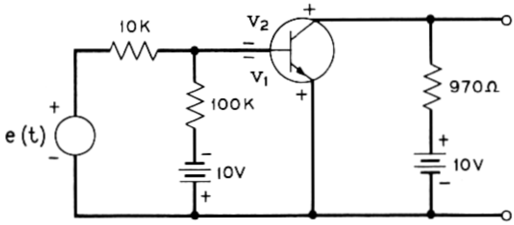

Fig. 6 — Practical logical-inverter circuit.

which takes into account a bulk base resistance of 280 ohms and a bulk collector resistance of 18 ohms. The circuit is initially at steady state with $e(t) = 0.3$ volt for $t < 0$. For $t \geqq 0$, $e(t) = 10$ volts, and as $t \to \infty$, $u(t) \to u_\infty$, some constant vector. With $d_2 = 1$, the number $\underline{K}$ is the smallest of the four quantities: $0.58(1 - 0.968d_1^{-1}) \times 10^{10}$, $0.5(1.1886 - 1.01215d_1^{-1}) \times 10^9$, $0.3815(1 - 0.583\text{d}_1) \times 10^8$, and $0.58(1.01215)(1 - d_1) \times 10^9$.

It is clear that d_1 must satisfy $0.968 < d_1 < 1$ in order that $\underline{K} > 0$. Then optimal choice of d_1 (that is, the choice that yields the largest value of $\underline{K}$) is approximately 0.9709. For $d_1 = 0.9709$, $\underline{K} = 1.66 \times 10^7$. Let the "charge switching time" t_s denote the smallest value of t such that $\sum_{i=1}^{2} | u_i(t) - u_{\infty i} |$ is less than or equal to two percent of $\sum_{i=1}^{2} | u_i(0) - u_{\infty i} |$ for all $t \geqq t_s$. Then our upper bound on t_s is approximately $4 \times (1.66)^{-1} \times 10^{-7} \approx 241$ nanoseconds. The actual value of t_s, as determined by numerically integrating the system of two nonlinear differential equations is approximately 57 nanoseconds. Thus, for this circuit, Theorem 2 provides a very easily evaluated and useful upper bound on t_s.

Finally, we state a result which provides an often rather conservative but easily evaluated *lower bound* on the rate of decay of transients.

Theorem 3: With B a constant real 2p-vector, let

$$\frac{du}{dt} + TF[C^{-1}(u)] + GC^{-1}(u) = B, \qquad t \geqq 0.$$

If there exists a constant 2p-vector u_∞ such that $[u(t) - u_\infty] \to \theta$ as $t \to \infty$, then for any choice of positive constants d_j, $j = 1, 2, \cdots, 2p$:

$$\sum_{j=1}^{2p} d_j \mid u_j(t) - u_{\infty j} \mid \geqq \exp(-\bar{K}t) \sum_{j=1}^{2p} d_j \mid u_j(0) - u_{\infty j} \mid, \qquad t \geqq 0$$

in which

$$\bar{K} = \max_j \max \left\{ \frac{1}{\tau_j} (1 + \tilde{d}_j d_j^{-1} \alpha_j), \frac{1}{c_j} \sum_{i=1}^{2p} d_i d_j^{-1} \mid g_{ij} \mid \right\}$$

where $-\alpha_j$ *is the nonzero off-diagonal element in the jth column of* T, *and* $\tilde{d}_j = d_{j+1}$ *for* j *odd, and* $\tilde{d}_j = d_{j-1}$ *for* j *even.*

The arguments used to prove the results stated in this section can be modified in a straightforward manner to prove far more general results concerning networks that contain diodes, capacitors, and inductors, in addition to the elements of the structure of Fig. 1. Some of these more general results are described in Section IV.

III. PROOFS

3.1 *Proof of Theorem 1*

We first show that

$$F[C^{-1}(u_a)] - F[C^{-1}(u_b)] = D_1(u_a - u_b), \; t \geqq 0 \tag{12}$$

and

$$C^{-1}(u_a) - C^{-1}(u_b) = D_2(u_a - u_b), \; t \geqq 0 \tag{13}$$

with D_1 and D_2 diagonal matrices dependent on t and possessing some special properties.

For $j = 1, 2, \cdots, 2p$, let $g_j(u_{aj}) = [C^{-1}(u_a)]_j$ and $g_j(u_{bj}) = [C^{-1}(u_b)]_j$. Then, using equation (2),

$$u_{aj} - u_{bj} = c_j[g_j(u_{aj}) - g_j(u_{bj})] + \tau_j\{f_j[g_j(u_{aj})] - f_j[g_j(u_{bj})]\}.$$

Thus if $u_{aj} \neq u_{bj}$,

$$\frac{f_j[g_j(u_{aj})] - f_j[g_j(u_{bj})]}{u_{aj} - u_{bj}} = \frac{r_j(u_{aj}, u_{bj})}{c_j + \tau_j r_j(u_{aj}, u_{bj})},$$

in which (for $u_{aj} \neq u_{bj}$)

$$r_j(u_{aj}, u_{bj}) = \frac{f_j[g_j(u_{aj})] - f_j[g_j(u_{bj})]}{g_j(u_{aj}) - g_j(u_{bj})}.$$

In a similar manner we find that for all $u_{aj} \neq u_{bj}$:

$$\frac{g_j(u_{aj}) - g_j(u_{bj})}{u_{aj} - u_{bj}} = \frac{1}{c_j + \tau_j r(u_{aj}, u_{bj})}.$$

Now, let us define for $j = 1, 2, \cdots, 2p$

$$r_j(u_{aj}, u_{bj}) = f_j'[g_j(u_{aj})]$$

when $u_{aj} = u_{bj}$. Then since u_{aj} and u_{bj} are continuous on $[0, \infty)$, it follows (see Appendix A) that $r_j(u_{aj}, u_{bj})$ is continuous on $[0, \infty)$. Since $r_j(u_{aj}, u_{bj})$ is nonnegative, it is clear that both

$$\frac{r_j(u_{aj}, u_{bj})}{c_j + \tau_j r_j(u_{aj}, u_{bj})}$$

and

$$\frac{1}{c_j + \tau_j r_j(u_{aj}, u_{bj})}$$

are continuous on $[0, \infty)$. Moreover equations (12) and (13) are satisfied with

$$D_1 = \operatorname{diag}\left\{\frac{r_j(u_{aj}, u_{bj})}{c_j + \tau_j r_j(u_{aj}, u_{bj})}\right\} \tag{14}$$

$$D_2 = \operatorname{diag}\left\{\frac{1}{c_j + \tau_j r_j(u_{aj}, u_{bj})}\right\}. \tag{15}$$

At this point we have

$$\frac{d}{dt}(u_a - u_b) + (TD_1 + GD_2)(u_a - u_b) = B_a - B_b, \qquad t \geqq 0 \tag{16}$$

with $TD_1 + GD_2$ continuous on $[0, \infty)$.

We need the following lemma.

Lemma 1: Let $M(\cdot)$ be a continuous real $n \times n$ matrix-valued function of t defined on $[0, \infty)$ such that there exist positive constants ϵ and $c_1, c_2, \cdots, c_n$, with the property that for $j = 1, 2, \cdots, n$ and all $t \geqq 0$*

$$m_{jj} - \sum_{i \neq j} c_i c_j^{-1} \mid m_{ij} \mid \geqq \epsilon.$$

Let x be a differentiable real n-vector-valued function on $[0, \infty)$ such that

$$\frac{dx}{dt} + Mx = 0, \qquad t \geqq 0.$$

Then there exists a constant k such that for $i = 1, 2, \cdots, n$, and all $t \geqq 0$

$$\mid x_i(t) \mid \leqq k \exp(-\epsilon t).$$

Moreover, k depends only on the c_i and the initial values $x_i(0)$.

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* In Ref. 4, Rosenbrock states a similar result, but does not give a rigorous proof. He considers the case in which $c_j = 1$ for $j = 1, 2, \cdots, n$.

Proof of Lemma 1: Let the functional s be defined in terms of an arbitrary continuously differentiable scalar function $\varphi(\cdot)$ by

$$\begin{aligned} s(\varphi)(t) &= 1 \quad \text{if} \quad \varphi(t) > 0 \quad \text{or} \quad \text{if} \quad \varphi(t) = 0 \quad \text{and} \quad \varphi'(t) > 0 \\ &= -1 \quad \text{if} \quad \varphi(t) < 0 \quad \text{or} \quad \text{if} \quad \varphi(t) = 0 \quad \text{and} \quad \varphi'(t) < 0 \\ &= 0 \quad \text{if} \quad \varphi(t) = 0 \quad \text{and} \quad \varphi'(t) = 0. \end{aligned}$$

Then for $t \geqq 0$,

$$\begin{aligned} \sum_i c_i s(x_i)(t) x_i'(t) &= -\sum_i c_i s(x_i)(t) \sum_j m_{ij} x_j \\ &= -\sum_j x_j \sum_i c_i s(x_i)(t) m_{ij} \\ &= -\sum_j x_j c_j s(x_j)(t) m_{jj} - \sum_j x_j \sum_{i \neq j} c_i s(x_i)(t) m_{ij} \\ &\leqq -\sum_j c_j m_{jj} \mid x_j \mid + \sum_j \mid x_j \mid \sum_{i \neq j} c_i \mid m_{ij} \mid \\ &\leqq -\epsilon \sum_j \mid c_j x_j \mid. \end{aligned}$$

But $\sum_i c_i s(x_i)(t) x_i'$ is equal to $\frac{d^+}{dt} \sum_i \mid c_i x_i \mid$, the right-hand derivative of $\sum_i \mid c_i x_i \mid$ [see Appendix B; the derivative of $\mid x_i \mid$ need not exist at points t at which $x_i(t) = 0$]. Therefore

$$\frac{d^+}{dt} \sum_j \mid c_j x_j \mid \leqq -\epsilon \sum_j \mid c_j x_j \mid, \qquad t \geqq 0$$

from which it follows that

$$\sum_j \mid c_j x_j(t) \mid \leqq \exp(-\epsilon t) \sum_j \mid c_j x_j(0) \mid, \qquad t \geqq 0. \ \square$$

If $M(\cdot)$ satisfies the conditions of Lemma 1, then it is easy to show that the unique continuously differentiable $n \times n$ matrix-valued function X defined on $[0, \infty)$ which satisfies

$$\frac{dX}{dt} + MX = 0, \qquad t \geqq 0 \quad [X(0) = I]$$

possesses the property that (for any norm $\| \cdot \|$ on E^n) there exists a constant K_1 such that

$$\| X(t) X(\tau)^{-1} \| \leqq K_1 \exp[-\epsilon(t - \tau)]$$

for all $t \geqq \tau$.

Returning now to equation (16), assume that $[TD_1 + GD_2]$ satisfies the conditions on $M(\cdot)$ of Lemma 1. Then with Y the solution of

$$\frac{dY}{dt} + [TD_1 + GD_2]Y = 0, \qquad t \geqq 0 \quad [Y(0) = I]$$

we have

$$u_a(t) - u_b(t) = Y(t)[u_a(0) - u_b(0)] + \int_0^t Y(t)Y(\tau)^{-1}[B_a(\tau) - B_b(\tau)]\,d\tau, \qquad t \geqq 0.$$

Therefore, for $t \geqq 0$

$$\begin{aligned} \| u_a(t) - u_b(t) \| &\leqq \| Y(t)[u_a(0) - u_b(0)] \| \\ &\quad + \int_0^t \| Y(t)Y(\tau)^{-1} \| \cdot \| B_a(\tau) - B_b(\tau) \|\,d\tau \\ &\leqq \| Y(t)[u_a(0) - u_b(0)] \| \\ &\quad + K_2 \int_0^t \exp[-\epsilon(t - \tau)]\, \| B_a(\tau) - B_b(\tau) \|\,d\tau \end{aligned}$$

for some positive constant K_2. Since $\| B_a(\tau) - B_b(\tau) \| \to 0$ as $\tau \to \infty$, it follows that $\| u_a(t) - u_b(t) \| \to 0$ as $t \to \infty$.

It remains only to prove that $[TD_1 + GD_2]$ meets the conditions imposed on $M(\cdot)$ of Lemma 1. Since $G \in \mathfrak{D}$, there exists a diagonal matrix diag $(d_1, d_2, \cdots, d_{2p})$ with $d_j > 0$ for $j = 1, 2, \cdots, 2p$ and

$$\alpha_f^{(k)} < \frac{d_{2k-1}}{d_{2k}} < \frac{1}{\alpha_r^{(k)}}$$

for $k = 1, 2, \cdots, p$ such that both

$$\text{diag}\,(d_1, d_2, \cdots, d_{2p})G$$

and

$$\text{diag}\,(d_1, d_2, \cdots, d_{2p})T$$

are strongly column-sum dominant. Thus for $j = 1, 2, \cdots, 2p$

$$t_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid t_{ij} \mid > 0$$

$$g_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid g_{ij} \mid > 0.$$

Let $W = TD_1 + GD_2$. Then, for $j = 1, 2, \cdots, 2p$,

$$w_{jj} = t_{jj} \frac{r_j}{c_j + \tau_j r_j} + g_{jj} \frac{1}{c_j + \tau_j r_j}$$

and

$$\sum_{i \neq j} d_i d_j^{-1} \mid w_{ij} \mid = \sum_{i \neq j} d_i d_j^{-1} \left| t_{ij} \frac{r_j}{c_j + \tau_j r_j} + g_{ij} \frac{1}{c_j + \tau_j r_j} \right|.$$

Therefore

$$w_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid w_{ij} \mid \geqq \frac{r_j}{c_j + \tau_j r_j} (t_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid t_{ij} \mid) + \frac{1}{c_j + \tau_j r_j} (g_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid g_{ij} \mid). \tag{17}$$

Since $r_j \geqq 0$, the right side of equation (17) is bounded from below by some positive constant ϵ uniformly in t and j. □

3.2 *Proof of Corollary 1*

By Corollary 3 of Ref. 5 there exists a unique $v \varepsilon E^{2p}$ such that

$$TF(v) + Gv = B_\infty \tag{18}$$

whenever G is such that all principal minors of $T^{-1}G$ are positive. In Reference 5 it is proved that $T^{-1}G$ will have this property if $T^{-1}G$ can be written as $A^{-1}B$ with both A and B stongly column-sum dominant.

Let $H = \text{diag}\,(d_1, d_2, \cdots, d_{2p})G$ be strongly column-sum dominant with all $d_i > 0$ and

$$\alpha_f^{(k)} < \frac{d_{2k-1}}{d_{2k}} < \frac{1}{\alpha_r^{(k)}}$$

for $k = 1, 2, \cdots, p$. Then $U \triangleq \text{diag}\,(d_1, d_2, \cdots, d_{2p})T$ is strongly column-sum dominant, and $T^{-1}G = U^{-1}H$, which proves that equation (18) possesses a unique solution v.

With v the solution of equation (18), let $u_\infty = C(v)$. Clearly if $B_\infty = \theta$, then $u_\infty = \theta$. Let u_b satisfy

$$\frac{du_b}{dt} + TF[C^{-1}(u_b)] + G[C^{-1}(u_b)] = B_\infty, \qquad t \geqq 0$$

with $u_b(0) = u_\infty$. Of course, $u_b(t) = u_\infty$ for all $t \geqq 0$. By Theorem 1, $[u(t) - u_\infty] \rightarrow \theta$ as $t \rightarrow \infty$, independent of u_0.

3.3 *Proof of Theorem 2*

Following the proofs of Theorem 1 and Corollary 1,

$$\frac{d}{dt}(u - u_\infty) + (TD_1 + GD_2)(u - u_\infty) = 0, \qquad t \geqq 0$$

in which

$$D_1 = \operatorname{diag}\left\{\frac{r_j(u_j , u_{\infty j})}{c_j + \tau_j r_j(u_j , u_{\infty j})}\right\}$$

and

$$D_2 = \operatorname{diag}\left\{\frac{1}{c_j + \tau_j r_j(u_j , u_{\infty j})}\right\}.$$

Therefore

$$\frac{d^+}{dt} \sum_j d_j \mid u_j(t) - u_{\infty j} \mid \leqq -\underline{K} \sum_j d_j \mid u_j(0) - u_{\infty j} \mid , \qquad t \geqq 0$$

in which

$$\underline{K} = \min_j \min \left\{\frac{1}{\tau_j}\left(t_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid t_{ij} \mid\right), \frac{1}{c_j}\left(g_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid g_{ij} \mid\right)\right\}.$$

But for $j = 1, 2, \cdots , 2p$

$$t_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid t_{ij} \mid = 1 - \tilde{d}_j d_j^{-1} \alpha_j . \square$$

3.4 *Proof of Theorem 3*

Since $TF[C^{-1}(z)] + GC^{-1}(z)$ depends continuously on $z \in E^{2p}$, u_∞ satisfies (see Ref. 6)

$$TF[C^{-1}(u_\infty)] + GC^{-1}(u_\infty) = B.$$

Therefore, following the proofs of Theorem 1 and Corollary 1,

$$\frac{d}{dt}(u - u_\infty) + (TD_1 + GD_2)(u - u_\infty) = 0, \qquad t \geqq 0$$

in which

$$D_1 = \operatorname{diag}\left\{\frac{r_j(u_j , u_{\infty j})}{c_j + \tau_j r_j(u_j , u_{\infty j})}\right\}$$

and

$$D_2 = \operatorname{diag}\left\{\frac{1}{c_j + \tau_j r_j(u_j , u_{\infty j})}\right\}.$$

For any $z \varepsilon E^{2p}$, let $\| z \|$ denote $\sum_i d_i |z_i|$. Then, for $t \geqq 0$

$$\left\| \frac{d}{dt}(u - u_\infty) \right\| = \| (TD_1 + GD_2)(u - u_\infty) \|$$

$$\leqq \max_j \left\{ (1 + \tilde{d}_j d_j^{-1} \alpha_j) \frac{r_j(u_j, u_{\infty j})}{c_j + \tau_j r_j(u_j, u_{\infty j})} + \sum_{i=1}^{2p} d_i d_j^{-1} | g_{ij} | \frac{1}{c_j + \tau_j r_j(u_j, u_{\infty j})} \right\} \| u(t) - u_\infty \|.$$

But, since $r_j(u_j, u_{\infty j}) \geqq 0$,

$$\bar{K} \geqq \max_j \left\{ (1 + \tilde{d}_j d_j^{-1} \alpha_j) \frac{r_j(u_j, u_{\infty j})}{c_j + \tau_j r_j(u_j, u_{\infty j})} + \sum_{i=1}^{2p} d_i d_j^{-1} | g_{ij} | \frac{1}{c_j + \tau_j r_j(u_j, u_{\infty j})} \right\}.$$

Thus

$$\left\| \frac{d}{dt}(u - u_\infty) \right\| \leqq \bar{K} \| u - u_\infty \|, \qquad t \geqq 0. \tag{19}$$

Clearly,

$$\left\| \frac{d}{dt}(u - u_\infty) \right\| = \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \| u(t + \epsilon) - u_\infty - u(t) + u_\infty \|, \qquad t \geqq 0$$

Also, for $t \geqq 0$, the limit

$$\lim_{\epsilon \to 0+} \frac{1}{\epsilon} [\| u(t) - u_\infty \| - \| u(t + \epsilon) - u_\infty \|]$$

exists and is equal to $-\frac{d^+}{dt} \| u - u_\infty \|$ in which as before $\frac{d^+}{dt}$ denotes the right-hand derivative (see Appendix B). But, since for any $\epsilon > 0$ and $t \geqq 0$,

$$\| u(t) - u_\infty \| - \| u(t + \epsilon) - u_\infty \| \leqq \| u(t + \epsilon) - u_\infty - u(t) + u_\infty \|,$$

we have

$$-\frac{d^+}{dt} \| u - u_\infty \| \leqq \left\| \frac{d}{dt}(u - u_\infty) \right\|, \qquad t \geqq 0. \tag{20}$$

Therefore, using equations (19) and (20),

$$\frac{d^+}{dt} \| u - u_\infty \| \geqq -\bar{K} \| u - u_\infty \|, \qquad t \geqq 0$$

and, for $t \geqq 0$,

$$\| u - u_\infty \| \geqq \exp(-\bar{K}t) \, \| u(0) - u_\infty \|. \ \square$$

IV. A SIGNIFICANT EXTENSION

We can easily extend our results to cover an interesting class of networks containing diodes, capacitors (not necessarily linear), and (not necessarily linear) inductors, in addition to the elements of the Fig. 1 network.

Let each diode be represented by a model of the type shown in Fig. 7 in which

$$i_d = \frac{d}{dt}\,[c_d v_d + \tau_d f_d(v_d)] + f_d(v_d),$$

with c_d and τ_d positive constants. Assume that $f_d(\cdot)$ satisfies the conditions placed on $f_e(\cdot)$ and $f_c(\cdot)$ of the transistor model. Let there be q diodes and let v_{2p+k} and i_{2p+k} ($k = 1, 2, \cdots, q$) be the voltage and current associated with the kth diode.

Suppose that the kth capacitor (we assume that there are r capacitors) is governed by

$$\frac{d}{dt}\,[c_{2p+q+k}(v_{2p+q+k})] = i_{2p+q+k}$$

for $k = 1, 2, \cdots, r$, where $c_{2p+q+k}(\cdot)$ is a strictly-monotone-increasing continuously-differentiable mapping of E^1 *onto* itself such that $c_{2p+q+k}(0) = 0$ and the slope of $c_{2p+q+k}(\cdot)$ is uniformly bounded from above and from below by positive constants.

Finally, let there be s inductors which introduce constraints

$$\frac{d}{dt}\,[l_{2p+q+r+k}(i_{2p+q+r+k})] = v_{2p+q+r+k}$$

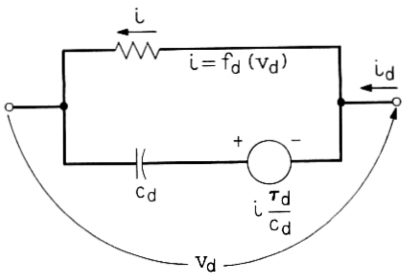

Fig. 7 — Diode model.

for $k = 1, 2, \cdots, s$, in which each $l_{2p+q+r+k}(\cdot)$ is a function of the same type as the $c_{2p+q+k}(\cdot)$.

Assume that the linear resistive portion of the network introduces the constraint

$$\tilde{\imath} = -H\tilde{v} + B, \qquad B \varepsilon \mathcal{B}$$

in which $\tilde{\imath} = (i_1, i_2, \cdots, i_{2+p+q+r}, v_{2p+q+r+1}, \cdots, v_{2p+q+r+s})^{tr}$, $\tilde{v} = (v_1, v_2, \cdots, v_{2p+q+r}, i_{2p+q+r+1}, \cdots, i_{2p+q+r+s})^{tr}$, and H is a constant hybrid-parameter matrix of order $(2p + q + r + s)$. Then

$$\frac{d}{dt}[\tilde{C}(\tilde{v})] + \tilde{T}\tilde{F}(\tilde{v}) + H\tilde{v} = B$$

where

$$\begin{aligned}
[\tilde{C}(\tilde{v})]_j &= [C(v)]_j, && j = 1, 2, \cdots, 2p \\
&= c_j v_j + \tau_j f_j(v_j), && j = 2p+1, 2p+2, \cdots, 2p+q \\
&= c_j(v_j), && j = 2p+q+1, \cdots, 2p+q+r \\
&= l_j(i_j), && j = 2p+q+r+1, \cdots, 2p+q+r+s;
\end{aligned}$$

$\tilde{T}$ is the direct sum of matrices $T \oplus I_q \oplus 0_{r+s}$, in which I_q is the identity matrix of order q and 0_{r+s} is the zero matrix of order $(r + s)$, and

$$\begin{aligned}
[\tilde{F}(\tilde{v})]_j &= [F(v)]_j, && j = 1, 2, \cdots, 2p \\
&= f_j(v_j), && j = 2p+1, \cdots, 2p+q.
\end{aligned}$$

Under our assumptions $\tilde{C}(\cdot)^{-1}$ exists and, with $\tilde{u} = \tilde{C}(\tilde{v})$,

$$\frac{d\tilde{u}}{dt} + \tilde{T}\tilde{F}[\tilde{C}^{-1}(\tilde{u})] + H\tilde{C}^{-1}(\tilde{u}) = B. \tag{21}$$

Let $\tilde{\mathfrak{D}}$ denote the set of all real matrices M of order $(2p + q + r + s)$ such that there exist positive constants $d_1, d_2, \cdots, d_{2p+q+r+s}$ with the property that

$$\alpha_f^{(k)} < \frac{d_{2k-1}}{d_{2k}} < \frac{1}{\alpha_r^{(k)}}$$

for $k = 1, 2, \cdots, p$ (when $p \neq 0$) and diag $(d_1, d_2, \cdots, d_{2p+q+r+s})M$ is strongly column-sum dominant.

With straightforward modifications of the arguments already presented, we can prove (*i*) that for each $\tilde{u}_0 \varepsilon E^{(2p+q+r+s)}$ equation (21) possesses a unique solution defined on $[0, \infty)$ such that $\tilde{u}(0) = \tilde{u}_0$, and (*ii*) the analogs of Theorems 1, 2, and 3 and Corollary 1. To be more specific, the analogs of Theorem 1, Corollary 1, and Theorem 2 are:

Theorem 1′: If $H \in \tilde{\mathfrak{D}}$, and $\tilde{u}_a$ and $\tilde{u}_b$ are solutions of equation (21) *with $B = B_a$ and $B = B_b$, respectively, for $t \geqq 0$, and if $[B_a(t) - B_b(t)] \to \theta$ [the zero vector of $E^{(2p+q+r+s)}$] as $t \to \infty$ with $B_a \in \mathcal{B}$ and $B_b \in \mathcal{B}$, then $[\tilde{u}_a(t) - \tilde{u}_b(t)] \to \theta$ as $t \to \infty$.*

Corollary 1′: Referring to equation (21), *if $H \in \tilde{\mathfrak{D}}$, and if there exists a constant vector B_∞ such that $[B(t) - B_\infty] \to \theta$ as $t \to \infty$, then there exists a constant vector $\tilde{u}_\infty$ such that $[\tilde{u}(t) - \tilde{u}_\infty] \to \theta$ as $t \to \infty$, and $\tilde{u}_\infty$ is independent of the initial condition $\tilde{u}_0$. In particular, if $B_\infty = \theta$, then $\tilde{u}_\infty = \theta$.*

Theorem 2′: If the hypotheses of Corollary 1′ are satisfied with $B(t) = B_\infty$ for $t \geqq 0$, then with $j_0 = (2p + q + r + s)$, we have

$$\sum_{j=1}^{j_0} d_j \mid \tilde{u}_j(t) - u_{\infty j} \mid \leqq \exp(-\tilde{\underline{K}}t) \sum_{j=1}^{j_0} d_j \mid \tilde{u}_j(0) - \tilde{u}_{\infty j} \mid, \qquad t \geqq 0$$

for every set of positive constants $d_1, d_2, \cdots, d_{2p+q+r+s}$ such that $0 < \tilde{\underline{K}} = \min\{\tilde{\underline{K}}_1, \tilde{\underline{K}}_2, \tilde{\underline{K}}_3\}$ where

$$\tilde{\underline{K}}_1 = \min_{1 \leq j \leq 2p} \min \left\{ \frac{1}{\tau_j} (1 - \tilde{d}_j d_j^{-1} \alpha_j), \frac{1}{c_j} (g_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid g_{ij} \mid) \right\}$$

$$\tilde{\underline{K}}_2 = \min_{2p+1 \leq j \leq 2p+q} \min \left\{ \frac{1}{\tau_j}, \frac{1}{c_j} (g_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid g_{ij} \mid) \right\}$$

$$\tilde{\underline{K}}_3 = \min_{2p+q+1 \leq j \leq 2p+q+r+s} \left\{ \frac{1}{s_j} (g_{jj} - \sum_{i \neq j} d_i d_j^{-1} \mid g_{ij} \mid) \right\}$$

in which $s_j = \sup c_j'(\cdot)$ for $j = 2p + q + 1, \cdots, 2p + q + r$; $s_j = \sup l_j'(\cdot)$ for $j = 2p + q + r + 1, \cdots, 2p + q + r + s$; $-\alpha_j$ is the nonzero off-diagonal term in the jth column of T; and $\tilde{d}_j = d_{j+1}$ for j odd and $\tilde{d}_j = d_{j-1}$ for j even. Moreover there exists one such set of constants $\{d_j\}$.

V. FINAL COMMENTS

The results presented here are quite encouraging in that they are concerned with the equations of reasonably realistic nonlinear network models, and provide some understanding of a precise nature in an area where there is a great need for many results of similar type.

VI. ACKNOWLEDGMENT

The writer is indebted to H. Shichman for supplying the data and computer results for the practical circuit of Fig. 6.

APPENDIX A

Proof that $r_j(u_{aj}, u_{bj})$ is continuous.

It is clear that $r_j(u_{aj}, u_{bj})$ is continuous at each point t such that $u_{aj}(t) \neq u_{bj}(t)$. Suppose now that t is such that $u_{aj}(t) = u_{bj}(t)$, and let $\epsilon > 0$ be given. Since u_{aj}, u_{bj}, g and f_j' are continuous, there exists $\delta_1 > 0$ such that

$$| f_j' \{g_j[u_{aj}(t + \eta)]\} - f_j' \{g_j[u_{aj}(t)]\} | \leqq \epsilon$$

for all $| \eta | \leqq \delta_1$. Then for $| \eta | \leqq \delta_1$ either $u_{aj}(t + \eta) = u_{bj}(t + \eta)$ in which case

$$| r_j[u_{aj}(t + \eta), u_{bj}(t + \eta)] - r_j[u_{aj}(t), u_{bj}(t)] | \leqq \epsilon$$

or $u_{bj}(t + \eta) \neq u_{bj}(t + \eta)$ and (using the mean-value theorem)

$$r_j[u_{aj}(t + \eta), u_{bj}(t + \eta)] = \frac{f_j\{g_j[u_{aj}(t + \eta)]\} - f_j\{g_j[u_{bj}(t + \eta)]\}}{g_j[u_{aj}(t + \eta)] - g_j[u_{bj}(t + \eta)]}$$

$$= f_j'(\xi)$$

in which

$$| \xi - g_j[u_{aj}(t)] | \leqq \max \{ | g_j[u_{aj}(t + \eta)] - g_j[u_{aj}(t)] |, | g_j[u_{bj}(t + \eta)] - g_j[u_{aj}(t)] | \}.$$

In the latter case, there exists $\delta_2 > 0$ such that $| f'(\xi) - f' \{g_j[u_{aj}(t)]\} | \leqq \epsilon$ for all $| \eta | \leqq \delta_2$. Thus for all $| \eta | \leqq \min \{\delta_1, \delta_2\}$, we have

$$| r_j[u_{aj}(t + \eta), u_{bj}(t + \eta)] - r[u_{aj}(t), u_{bj}(t)] | \leqq \epsilon.$$

APPENDIX B

Proof that the Right-Hand Derivative of $| x_j |$ exists and is equal to $s(x_j)(t)x_j'$.

If t is a point such that $x_j(t) \neq 0$, then it is clear that

$$\frac{d^+}{dt} | x_j | = s(x_j)(t)x_j'(t).$$

At t such that $x_j(t) = 0$ and $x_j'(t) \neq 0$,

$$s(x_j)(t)x_j' = \lim_{\epsilon \to 0} s(x_j)(t) \frac{x_j(t + \epsilon)}{\epsilon} = \frac{d^+}{dt} | x_j |.$$

Finally if $x_i(t) = 0$ and $x_i'(t) = 0$, then

$$0 = \lim_{\substack{\epsilon \to 0+ \\ t \leq \xi \leq t+\epsilon}} | x_i'(\xi) | = \lim_{\epsilon \to 0+} \frac{| x_i(t + \epsilon) |}{\epsilon} = \frac{d^+}{dt} | x_i |,$$

since x_i is continuously differentiable.

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