# Data Transmission Error Probabilities in the Presence of Low-Frequency Removal and Noise

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*Upper bounds on error probability are derived for data transmission systems which are subjected to gaussian noise and to the removal of the low-frequency components of the signal. This error probability can be quite low for random data, even though the eye pattern is closed. Both standard format and partial response signaling are considered, as are binary and multilevel alphabets. Numerical results are given for a high-pass filter containing a single pole and for a cascade of several such identical filters.*

## I. INTRODUCTION

It is frequently desirable, or unavoidable, that the low-frequency components of a data signal be eliminated. This may occur through the use of capacitor or transformer coupling in the terminal equipment or in the baseband transmission facilities. Another instance results from the necessity of removing low-frequency baseband components before modulation in order to provide a spectral guard band in the vicinity of the carrier frequency.

Since dc is usually completely attenuated, no linear operation can correct for low-frequency removal. One commonly used approach uses nonlinear feedback to restore the low-frequency components.[1] Another solution to this problem involves dc-free signal formats.[2, 3]

We evaluate the penalty resulting from the removal of low-frequency components from a standard format data signal (Nyquist I shaping) and a partial response signaling format (multilevel extension of duobinary with precoding.)[4] Clearly, in both of these cases, the degradation is most severe when the transmitted data sequence contains long strings of identical digits. In fact, when the system bandwidth is less than the signaling rate, which is usual in data communication systems, the received signal will be zero. This follows from the fact that for a periodic input impulse train the lowest frequency components are at dc and the signaling frequency, both of which are filtered out. However, the degradation of a random signal can be quite small when the cutoff frequency of the offending high pass filter is far below the signaling rate.

We consider binary and multilevel data-transmission systems with signaling formats as above, degraded by a single-pole high-pass filter or a cascade of such filters. The systems are evaluated for error probability in the presence of additive gaussian noise. A previously derived error probability bound[5] is used, which takes the form of a gaussian distribution of the signal to noise ratio, in which the larger intersymbol interference components subtract from the signal amplitude and the smaller ones add to the noise power.[5] In general, the optimum splitting of intersymbol interference terms between signal amplitude and noise power cannot be determined analytically. We show that for intersymbol interference components, related by a single exponential damping factor, an optimum subdivision can be explicitly specified. Where the eye is open, the error probability bound is given directly in terms of the eye opening to rms noise ratio.

We also discuss the refinements of the generalized bound in the case of interysmbol interference from a single exponential signal tail, and then apply the results to Nyquist I shaped and partial response signaling formats respectively. Single poles and a cascade of identical poles are considered, and numerical results are given for practical data system parameters.

## II. DERIVATION OF A SIMPLIFIED ERROR PROBABILITY BOUND FOR SINGLE EXPONENTIAL INTERSYMBOL INTERFERENCE

Reference 5 gives an upper bound for the probability of error in the reception of a random digital message perturbed by gaussian noise and intersymbol interference. This gives

$$P_e \leqq A \exp\left\{-\frac{[f_0 - (N-1)\sum_{k\epsilon K} f_k]}{\left[2\sigma_n^2 + \frac{N^2-1}{3}\sum_{k\not\epsilon K} f_k^2\right]}\right\} \tag{1}$$

which is subject to

$$\sum_{k\epsilon K} f_k < \frac{f_0}{N-1}$$

where

$N$ is the number of levels of the input random message.
$\sigma_n^2$ is the variance of the additive noise.
$f(t)$ is the signaling waveform.
$\frac{1}{T}$ is the signaling rate.

$$f_k = \begin{cases} |\, f(kT) \,| & \text{for standard format signaling} \\ |\, f[(k - \tfrac{1}{2})T] \,| & \text{for } N \text{ level partial response signaling with precoding*} \end{cases}$$

and

$$A = \begin{cases} \dfrac{2(N-1)}{N} & \text{for standard format signaling} \\ \dfrac{2(N^2-1)}{N^2} & \text{for } N \text{ level partial response signaling with precoding} \end{cases}$$

We notice that the applicability of the error probability bound to partial response signaling formats was not discussed in the original paper but is presented here as a further extension of the result.[5]

The sets $k \varepsilon K$ and $k \notin K$ include all members except $k = 0$. It is also shown in Ref. 5 that

$$\underset{\ell \varepsilon K}{f_\ell} > \underset{m \notin K}{f_m} \tag{2}$$

Thus, if the signal sample set $\{f_k\}$ excluding $k = 0$ is rearranged in order of decreasing magnitude to form a set $\{g_k\}$, then the sums in equation (1) may be replaced by

$$\begin{aligned} \sum_{k \varepsilon K} f_k &= \sum_{k=1}^{M} g_k \\ \sum_{k \notin K} f_k^2 &= \sum_{k=M+1}^{\infty} g_k^2 \,. \end{aligned} \tag{3}$$

For an arbitrary signaling waveform, $f(t)$, the optimum $M$ [in the sense of minimizing the right side of equation (1)] must be determined by a trial comparison method as decribed in Ref. 5.

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* In the partial response case, $f_1$ must be replaced by $f_1$—$f_0$ in both numerator and denominator summations of equation (1) since only the unintentional intersymbol interference should be included there.

For an exponential signal tail,

$$f_k = rf_{k-1}, \qquad 0 < r < 1; \quad k = 2, 3, \cdots \tag{4}$$

Thus, since $f(t)$ is already monotonically decreasing for all $t \geqq T$, the ordered sets $\{f_k\}$ and $\{g_k\}$ are identical in this case.

$$\begin{aligned} \sum_{k=1}^{M} f_k &= \frac{f_1[1 - r^M]}{1 - r} \\ \sum_{k=M+1}^{\infty} f_k^2 &= \frac{f_1^2 r^{2M}}{1 - r^2}. \end{aligned} \tag{5}$$

To minimize the right side of equation (1), it is sufficient to maximize

$$Q = \frac{[f_0 - (N-1)\sum_{k\epsilon K} f_k]^2}{2\left[\sigma_n^2 + \frac{N^2-1}{3}\sum_{k\notin K} f_k^2\right]} = \frac{\left[f_0 - (N-1)\frac{f_1(1-r^M)}{1-r}\right]^2}{2\left[\sigma_n^2 + \frac{N^2-1}{3} f_1^2 \frac{r^{2M}}{1-r^2}\right]}. \tag{6}$$

Differentiating $Q$ with respect to $M$ gives

$$\begin{aligned} \frac{dQ}{dM} = {} & x \ln r\left[f_0 - \frac{(N-1)}{1-r} f_1(1-x)\right]\left[f_0 - \frac{(N-1)}{1-r} f_1\right]\left[\frac{f_1^2}{1-r^2}\right] \\ & \cdot \left[\frac{\sigma_n^2(N-1)}{\left(\frac{N^2-1}{3}\right)\left(\frac{f_1}{1+r}\right)\left(f_0 - \frac{(N-1)f_1}{1-r}\right)} - x\right]\left[\frac{N^2-1}{3}\right] \\ & \Big/ \left[\sigma_n^2 + \frac{N^2-1}{3}\frac{f_1^2}{1-r^2}x^2\right]^2 \end{aligned} \tag{7}$$

where

$$x = r^M \qquad (0 \leqq x \leqq 1)$$

and

$$\frac{(N-1)}{1-r} f_1(1-x) < f_0 . \tag{8}$$

Three separate cases must now be examined.

(*i*) If $f_0 - (N-1)f_1/(1-r) < 0$, then the eye is closed. From equation (7) it follows that $dQ/dM < 0$ for $0 < x \leqq 1$. Therefore the positive maximum of $Q$ occurs at the boundary $x = 1$, so the optimum value of $M$ is $M_{\text{opt}} = 0$.

(*ii*) If

$$\frac{\sigma_n^2(N-1)}{\left(\frac{N^2-1}{3}\right)\left(\frac{f_1}{1+r}\right)\left[f_0-\frac{(N-1)f_1}{1-r}\right]} > 1,$$

it is implicit that $f_0 - (N-1)f_1/(1-r) > 0$, and the eye is open. In this case it is again true that $dQ/dM < 0$ for $0 < x \leqq 1$, and $M_{\text{opt}} = 0$.

(*iii*) If

$$0 < \frac{\sigma_n^2(N-1)}{\left(\frac{N^2-1}{3}\right)\left(\frac{f_1}{1+r}\right)\left[f_0-\frac{(N-1)f_1}{(1-r)}\right]} < 1,$$

it is again implicit that $f_0 - (N-1)f_1/(1-r) > 0$, and the eye is open. In this case a positive maximum for $Q$ occurs in the interval $0 < x < 1$. Solving for the point where $dQ/dM = 0$, we obtain

$$r^{M_{\text{opt}}} = \frac{\sigma_n^2(N-1)}{\left(\frac{N^2-1}{3}\right)\left(f_0-\frac{(N-1)f_1}{1-r}\right)\left(\frac{f_1}{1+r}\right)}. \tag{9}$$

Notice that condition (8) is automatically satisfied.

Since the solution for $M_{\text{opt}}$ as given by equation (9) is not necessarily integer, the error probability bound as given by equation (1) must be modified in terms of the actual choice of an integer $M$. We will arbitrarily use the next higher integer. Letting $[M_{\text{opt}}]$ denote the next higher integer to $M_{\text{opt}}$ and

$$z = 3\left(\frac{N-1}{N+1}\right)\left(\frac{1+r}{1-r}\right),$$

equation (6) may be expressed as:

$$Q = \frac{[S_p - I_{\max}(1 - r^{[M_{\text{opt}}]})]^2}{2[\sigma_n^2 + I_{\max}^2 r^{2[M_{\text{opt}}]}/z]} \tag{10}$$

where

$$\begin{aligned} r^{[M_{\text{opt}}]} &= b\,\frac{\sigma_n^2 z}{I_{\max}(S_p - I_{\max})}; \\ b &= r^{[M_{\text{opt}}]-M_{\text{opt}}}, \qquad r < b < 1. \end{aligned} \tag{11}$$

$I_{\max} = (N-1)f_1/(1-r)$ denotes the maximum intersymbol interference and $S_p = f_0$ denotes the signal amplitude. Combining equations (10) and (11),

$$2Q = \frac{\left\{S_p - \left[\dfrac{I_{\max}(S_p - I_{\max}) - b\sigma_n^2 z}{S_p - I_{\max}}\right]\right\}^2}{\sigma_n^2 + \dfrac{b^2\sigma_n^4 z}{(S_p - I_{\max})^2}} = \frac{[(S_p - I_{\max})^2 + b\sigma_n^2 z]^2}{\sigma_n^2[(S_p - I_{\max})^2 + b^2\sigma_n^2 z]}.$$

Since $r < b < 1$,

$$2Q > \frac{[(S_p - I_{\max}) + b\sigma_n^2 z]^2}{\sigma_n^2[(S_p - I_{\max})^2 + b\sigma_n^2 z]} = \left(\frac{S_p - I_{\max}}{\sigma_n}\right)^2 + bz > \left(\frac{S_p - I_{\max}}{\sigma_n}\right)^2 + rz.$$

In terms of the error probability,

$$P_e < A \exp[-Q] \tag{12}$$

$$< A \exp\left\{-\frac{\left(\dfrac{S_p - I_{\max}}{\sigma_n}\right)^2 + 3r\left(\dfrac{N-1}{N+1}\right)\left(\dfrac{1+r}{1-r}\right)}{2}\right\}.$$

For the situations where $M_{\text{opt}} = 0$ (that is, cases *i* and *ii*) equation (1) becomes

$$P_e < A \exp\left\{-\frac{S_p^2}{2\left[\sigma_n^2 + \dfrac{1}{3}\left(\dfrac{N+1}{N-1}\right)\left(\dfrac{1-r}{1+r}\right)I_{\max}^2\right]}\right\}. \tag{13}$$

## III. ERROR PROBABILITY PERFORMANCE WITH A STANDARD FORMAT INPUT DATA SIGNAL

Figure 1 is a block diagram of the system considered. Although a baseband system is shown, a system using linear modulation and demodulation can readily be fit to this model. $P(\omega)$ is the basic shaping filter and it is assumed that the receiver is matched to this shaping filter. For simplicity, $P(\omega)$ is chosen to be real. The added noise is white gaussian. $H(\omega)$ is the narrow high-pass causal filter whose effects are considered. Since $H(\omega)$ is narrow, it makes little difference whether the noise is added ahead of, behind, or somewhere in the middle of this filter.

The source generates symbols randomly from an $N$-ary alphabet at a rate of $1/T$ symbols per second. The transmitted signal may be represented by

$$s(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT)$$

where the $a_k$'s are independent, zero-mean random variables which take one of $N$ equally spaced values with equal probability, and $p(t)$ is the impulse response of $P(\omega)$.

It is assumed that there is no distortion other than $H(\omega)$ and that $Q(\omega) = P^2(\omega)$ is a Nyquist shaped filter of bandwidth less than $1/T$, so that

$$q(kT) = 0, \quad \text{all} \quad k \neq 0. \tag{14}$$

If we let $P(0) = 1$, then

$$q(0) = \frac{1}{2\pi} \int Q(\omega)\, d\omega = T/P. \tag{15}$$

The power of the transmitted signal is

$$S = \frac{\langle a_k^2 \rangle_{\text{av}}}{2\pi T} \int P^2(\omega)\, d\omega = \frac{\sigma_a^2}{T^2} \tag{16}$$

where $\sigma_a^2$ is the variance of $a_k$.

The signal presented to the sampler may be written in the form

$$r(t) = \sum_{k=-\infty}^{\infty} a_k[q(t - kT) + e(t - kT)] + n(t)$$

where $e(t)$ is the error signal caused by the low frequency removal, $H(\omega)$. From equations (14) and (15),

$$r(mT) = a_m\left[\frac{1}{T} + e(0)\right] + \sum_{k \neq m} a_k e[(m - k)T] + n(mT). \tag{17}$$

The effect of the low frequency removal is both the reduction of the signal amplitude [since $e(0)$ is negative] and, more important, the introduction of intersymbol interference.

The Fourier transform of the error signal is

$$E(\omega) = Q(\omega)[H(\omega) - 1] \tag{18}$$

so that

$$e(t) = \int_{-\infty}^{\infty} q(t - x)h_{-1}(x)\, dx$$

where $h_{-1}(t)$ is the inverse Fourier transform of $[H(\omega) - 1]$.

In all cases of interest, $H(\omega) - 1$ is much narrower than $Q(\omega)$. The time function $h_{-1}(t)$ therefore is virtually constant over a time interval equal to the effective duration of $q(t)$. We may therefore approximate $q(t)$ by a delta function, whose area is unity since $Q(0) = 1$.

$$e(t) = \int_{-\infty}^{\infty} \delta(t - x)h_{-1}(x)\, dx.$$

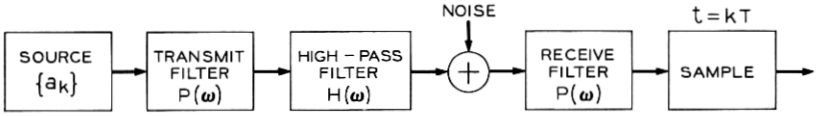


Fig. 1 — System block diagram.

If $H(\omega)$ is causal (which is the case we are interested in), then

$$e(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}h_{-1}(0^+) & t = 0 \\ h_{-1}(t) & t > 0 \end{cases} \tag{19}$$

where $e(t)$ is the negative of the impulse response of a narrow causal low-pass filter. The generalized bound given in equation (1) can be applied to this case as:

$$P_e < \frac{2(N-1)}{N} \exp\left\{-\frac{\left[\frac{1}{T} + e(0) - (N-1)\sum_{k \epsilon K} |e_k|\right]^2}{2\left[\sigma_n^2 + \frac{N^2-1}{3}\sum_{k \not\epsilon K} e_k^2\right]}\right\}. \tag{20}$$

The quantity $\sigma_n^2$ is the noise power at the sampler input and is also equal to the noise power at the receiver input, measured in a bandwidth equal to half the signaling rate. For the $N$-level system,

$$\sigma_a^2 = \frac{N^2-1}{3}.$$

In terms of the signal power, equation (16), equation (20) may be rewritten as

$$P_e < \frac{2(N-1)}{N} \exp\left\{-\frac{[1 + g(0) - (N-1)\sum_{k \epsilon K} |g_k|]^2}{2\sigma_a^2\left[\frac{\sigma_n^2}{S} + \sum_{k \not\epsilon K} g_k^2\right]}\right\} \tag{21}$$

where $g(t)$ is the normalized error signal

$$\begin{aligned} g(t) &= Te(t) \\ G(\omega) &= T[H(\omega) - 1]. \end{aligned} \tag{22}$$

To apply the simplified bounds derived in equations (12) and (13), we must first specify the high pass filter, $H(\omega)$.

3.1 *Single Pole Filter*

A very common type of low frequency removal results from the use of a single capacitor or transformer. The transfer function is

$$H(s) = \frac{s\tau}{s\tau + 1}$$

where $\tau$ is the time constant of the low frequency removal circuit. Its corner frequency is then $1/(2\pi\tau)$. From equation (22), the normalized error signal is

$$G(s) = -\frac{T}{s\tau + 1} \tag{23}$$

and

$$g(t) = -\frac{T}{\tau} \exp\left(-\frac{t}{\tau}\right) u(t)$$

where $u(t)$ is the unit step function. Introducing the normalized quantity

$$a = \frac{T}{\tau}, \tag{24}$$

then

$$g(kT) = \begin{cases} 0 & k < 0 \\ -\dfrac{a}{2}, & k = 0. \\ -a \exp(-ka), & k > 0 \end{cases} \tag{25}$$

Letting

$$Tf_0 = 1 + g(0)$$
$$Tf_k = | g_k |, \qquad k = 1, 2, \cdots \tag{26}$$

and

$$r = e^{-a},$$

the normalized eye opening becomes

$$Tf_0 - \frac{(N-1)Tf_1}{1-r} = 1 - \frac{a}{2} - \frac{(N-1)ae^{-a}}{1-e^{-a}} < 0. \tag{27}$$

Thus, $M_{\text{opt}} = 0$, and equation (13) becomes

$$P_e < \frac{2(N-1)}{N} \exp\left\{ - \frac{\left(1 - \frac{a}{2}\right)^2}{2\sigma_a^2\left[\frac{\sigma_n^2}{S} + \frac{a^2}{\exp(2a) - 1}\right]} \right\} \tag{28}$$

When $a \ll 1$, we may approximate equation (28) by

$$P_e < \frac{2(N-1)}{N} \exp\left[ - \frac{1}{2\sigma_a^2\left(\frac{\sigma_n^2}{S} + \frac{a}{2}\right)} \right]. \tag{29}$$

The error bounds for binary, 4-level and 8-level systems are plotted in Figs. 2, 3, and 4, respectively, as a function of the signal to noise ratio, $S/\sigma_n^2$, and the normalized reciprocal time constant, $a$. The dashed curves are the exact values for no low-frequency removal.

$$\left(\text{that is,}\quad P_e = \frac{2(N-1)}{N} \operatorname{erfc}\left\{\frac{1}{T\sigma_n}\right\} = \frac{2(N-1)}{N} \operatorname{erfc} \frac{(S)^{\frac{1}{2}}}{\sigma_a \sigma_n}, \quad \text{where} \quad \operatorname{erfc}(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_x^\infty e^{-t^2/2}\, dt \right).$$

It is seen that, in the region of $10^{-5}$ error probability, these exact

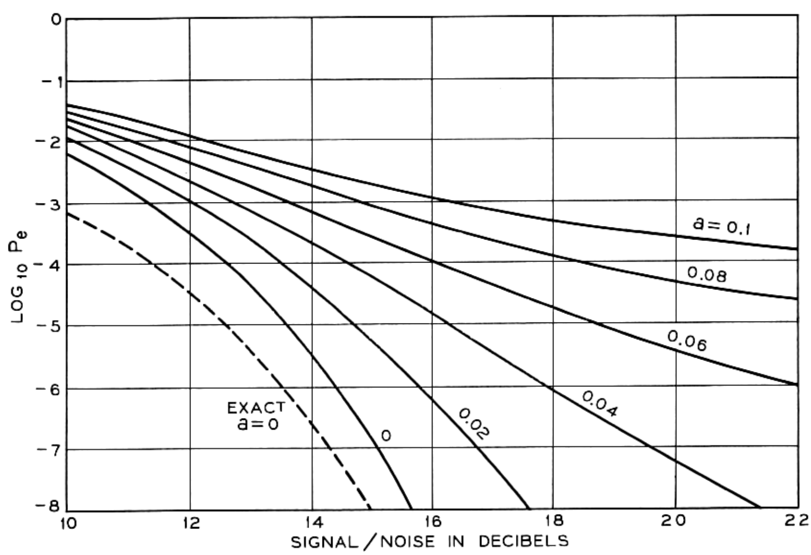


Fig. 2 — Upper bound of the error probability of a binary standard format system with a single-pole high-pass filter.

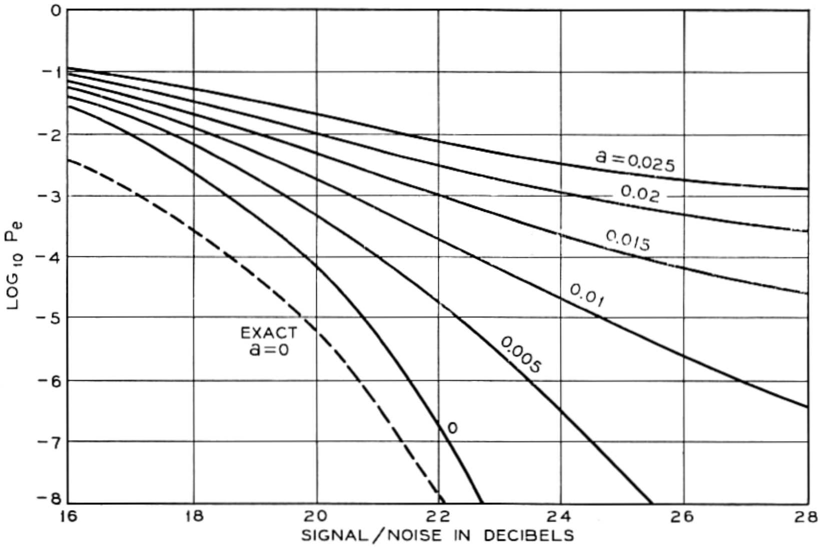


Fig. 3 — Upper bound of the error probability of a 4-level standard format system with a single-pole high-pass filter.

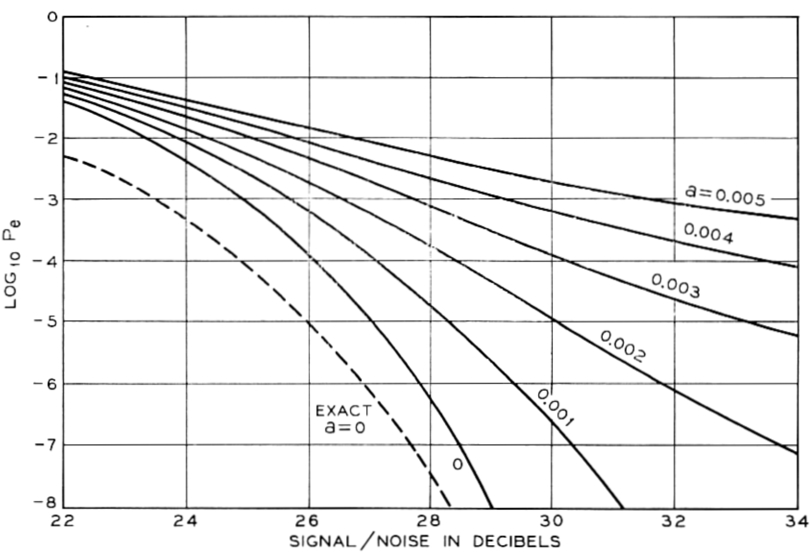


Fig. 4 — Upper bound of the error probability of an 8-level standard format system with a single-pole high-pass filter.

curves differ from the corresponding bounds by approximately a factor of 10 in error probability, or 1 decibel in signal to noise ratio.

In the binary case, it is seen that a simple high-pass filter with a time constant of 50 bit intervals introduces a degradation of only about 1 decibel in the region of $10^{-5}$ error probability. On the other hand, a time constant of 10 bit intervals leads to totally unacceptable performance. For the same amount of degradation and the same symbol rate, the 4- and 8-level systems must have high-pass time constants respectively 5 and 21 times that of the binary system.

### 3.2 *Cascaded Single Pole Filters*

In many cases, several single-pole high-pass filters are contained in the transmission path of the system. If $n$ identical networks are used, then the overall high-pass transfer function is

$$H_n(s) = \left(\frac{s\tau}{s\tau + 1}\right)^n. \tag{30}$$

In many cases, a transfer function containing a large number of real poles of different values can be approximated by a transfer function of the form of equation (30).[6]

The Laplace transform of the error signal is

$$G_n(s) = T\left[\left(\frac{s\tau}{s\tau + 1}\right)^n - 1\right].$$

To find $g_n(t)$, we first evaluate

$$\mathcal{L}\left[\frac{1}{T}\exp\left(\frac{t}{\tau}\right)g_n(t)\right] = \frac{(s - 1/\tau)^n}{s^n} - 1 = \frac{1}{s^n}\sum_{k=1}^{n}\binom{n}{k}\left(-\frac{1}{\tau}\right)^k s^{n-k}$$

$$\frac{1}{T}\exp\left(\frac{t}{\tau}\right)g_n(t) = \sum_{k=1}^{n}\binom{n}{k}\left(-\frac{1}{\tau}\right)^k \frac{t^{k-1}}{(k-1)!}, \qquad t > 0$$

$$g_n(t) = -\frac{T}{\tau}\exp\left(-\frac{t}{\tau}\right)\sum_{k=0}^{n-1}\frac{1}{k!}\binom{n}{k+1}\left(-\frac{t}{\tau}\right)^k, \qquad t > 0.$$

At the sampling times,

$$g_n(mT) = \begin{cases} 0, & m < 0 \\ -\dfrac{na}{2}, & m = 0 \\ -a\exp(-ma)\displaystyle\sum_{k=0}^{n-1}\binom{n}{k+1}\frac{(-ma)^k}{k!}, & m > 0 \end{cases} \tag{31}$$

where again $a = T/\tau$.

This function may also be expressed in terms of the generalized Laguerre polynomial,[7]

$$L_n^{(-1)}(x) = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} \frac{(-x)^k}{(k-1)!}$$

$$g_n(mT) = \frac{n}{m} \exp(-ma) L_n^{(-1)}(ma), \qquad m > 0. \tag{32}$$

It has been found empirically in several numerical examples that the best error probability bound was obtained when all intersymbol interference terms were added to the noise (that is, $M_{\text{opt}} = 0$). The resultant bound is therefore

$$P_e < \frac{2(N-1)}{N} \exp\left\{ - \frac{\left(1 - \frac{na}{2}\right)^2}{2\sigma_a^2\left[\frac{\sigma_n^2}{S} + \sum_{m=1}^{\infty}\left(\frac{n}{m}\exp(-ma)L_n^{(-1)}(ma)\right)^2\right]}\right\}. \tag{33}$$

An example of practical interest is the evaluation of the performance of a baseband binary 50,000 bits per second data set without dc restoration, operating over a transmission facility using transformer coupled repeaters. The transformers each have a corner frequency of 15 Hz, and therefore a time constant of

$$\tau = \frac{1}{2\pi \times 15} = 10.6 \text{ msec.}$$

so that

$$a = \frac{2 \times 10^{-5}}{0.0106} = 0.00188.$$

The results of Fig. 2 indicates a degradation of only about 0.1 decibel when a single transformer is introduced. However, several transformers are usually present in actual systems. The error signals, $g_n(t)$, and error probability bounds have been computed for both 14 and 28 transformers. The error signals for these two cases are shown in Fig. 5. Remember that one millisecond is equal to 50 bit intervals.

Figure 6 shows the error probability bounds for these situations; 28 transformers lead to unacceptable performance while 14 transformers introduce a degradation of 3 decibels at $10^{-5}$ error rate. It is significant that $n$ transformers produce more degradation than a single transformer with a corner frequency $n$ times greater. Also, under the assumptions of this paper, all of the above results apply independently of the roll-off characteristic of $Q(\omega)$, as long as it is a member of the Nyquist I class.

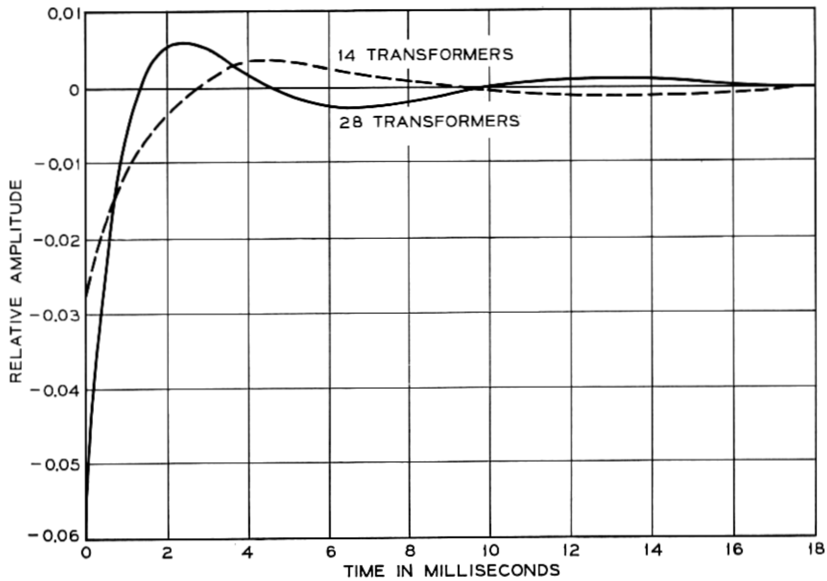


Fig. 5 — Errors signal for a cascade of transformers with 15 Hz corner frequencies.

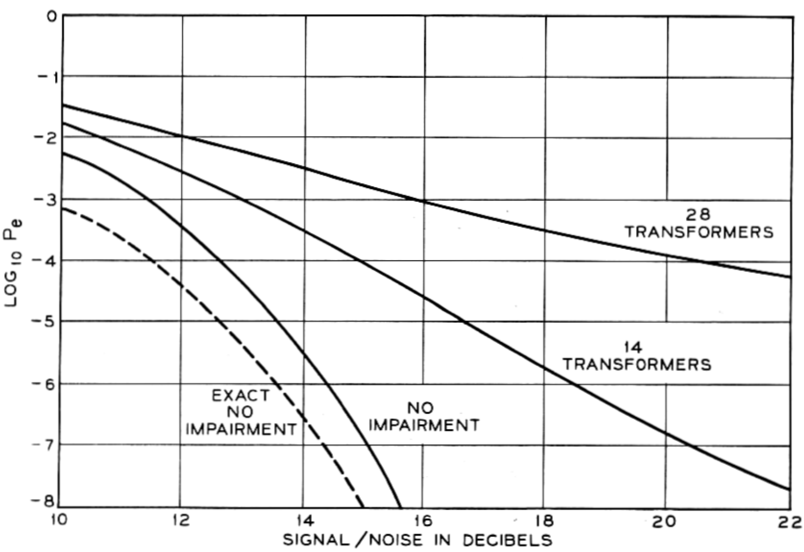


Fig. 6 — Upper bound of the error probability of a 50,000 bit/sec binary system with a cascade of transformers with 15 Hz corner frequencies.

## IV. ERROR PROBABILITY PERFORMANCE WITH AN $N$-LEVEL EXTENSION OF A DUOBINARY INPUT DATA SIGNAL

The system model considered here is identical to Fig. 1 except that (*i*) a precoder which converts the input $N$-level sequence $\{a_k\}$ to another $N$-level sequence $\{b_k\}$ according to the relation

$$b_n = [a_{n-1} - b_{n-1}](\text{mod } N) \tag{34}$$

is inserted between the source and the transmitting filter, $P(\omega)$, and (*ii*) a decoder follows the sampler which decodes the received levels modulo $N$ to recover the original symbols $a_n$. The important point for our application is that by including precoding at the transmitter, no knowledge of any symbol or sample other than the received sample, $r_k$, is involved in deciding $a_k$.

Instead of the Nyquist shaping characteristic, the cosine filter is used for the composite signal shaping characteristic, $Q(\omega) = P^2(\omega)$, that is,

$$Q(\omega) = \cos \frac{T}{2} \omega, \qquad |\omega| \leqq \frac{\pi}{T}. \tag{35}$$

The system impulse response is given by

$$q(t) = \frac{2}{\pi T} \left[ \frac{\cos \pi t/T}{1 - 4t^2/T^2} \right],$$

so its values at the sampling instant are

$$q[(k - \tfrac{1}{2})T] = \begin{cases} \dfrac{1}{2T} & k = 0, 1 \\ 0 & k \neq 0, 1. \end{cases} \tag{36}$$

The power of the transmitted signal is

$$S = \frac{\langle b_k^2 \rangle_{\text{av}}}{2\pi T} \int_{-\pi/T}^{\pi/T} Q(\omega)\, d\omega = \frac{2\sigma_b^2}{\pi T^2} \tag{37}$$

where $\sigma_b^2$ is the variance of $b_k$. If the input symbols $a_k$ are equally likely and independent, then so are the precoded symbols $b_k$. Thus, $\sigma_b^2 = \sigma_a^2$. The sampler input waveform, $r(t)$, may be expressed as

$$r(t) = \sum_{k=-\infty}^{\infty} b_k[q(t - kT) + e(t - kT)] + n(t) \tag{38}$$

where once again $e(t)$ is the degradation caused by the low frequency removal, $H(\omega)$. Substituting equation (36) in equation (38),

$$r[(m - \tfrac{1}{2})T] = r_m = (b_m + b_{m-1}) \frac{1}{2T} + b_m e\left(-\frac{T}{2}\right) + \sum_{k \neq m} b_k e[(m - k - \tfrac{1}{2})T] + n[(m - \tfrac{1}{2})T]. \quad (39)$$

If $H(\omega)$ is causal as before, then $e(-T/2)$ will be zero. Making the same assumptions as in the standard signal format case, we arrive at an expression for error probability analogous to equation (21)

$$P_e < 2\left(\frac{N^2 - 1}{N^2}\right) \exp\left\{-\frac{[\frac{1}{2} - (N - 1) \sum_{k \varepsilon K} | g_k |]^2}{2\sigma_a^2\left[\frac{2\sigma_n^2}{\pi S} + \sum_{k \varepsilon K} g_k^2\right]}\right\}. \quad (40)$$

Here we consider only the single pole high-pass filter for $H(\omega)$. The result for a cascade of $n$ identical poles follows immediately.

### 4.1 *Single Pole Filter*

We start by examining the normalized eye. Letting

$$\begin{aligned} Tf_0 &= 1/2 \\ Tf_k &= | g_k |; \qquad k = 1, 2, \cdots \end{aligned} \quad (41)$$

and $r = e^{-a}$

$$Tf_0 - \frac{(N - 1)f_1}{1 - r} = \frac{1}{2} - \frac{(N - 1)ae^{-a}}{1 - e^{-a}} < 0. \quad (42)$$

Thus, $M_{\text{opt}} = 0$ and equation (13) becomes for $a \ll 1$

$$P_e < 2\left(\frac{N^2 - 1}{N^2}\right) \exp\left\{-\frac{1/4}{2\sigma_a^2\left(\frac{2\sigma_n^2}{\pi S} + \frac{a}{2}\right)}\right\}. \quad (43)$$

Figures 7, 8, and 9 illustrate the behavior of the error probability bounds versus $S/\sigma_n^2$ for binary, 4-level and 8-level partial response signals with the normalized reciprocal time constant, $a$, as a parameter. The dotted curves give the exact values of $P_e$ for the case $a = 0$

$$\left[\text{that is, } P_e = 2\left(\frac{N^2 - 1}{N^2}\right) \text{erfc}\left(\frac{(S\pi/8)^{\frac{1}{2}}}{\sigma_a \sigma_n}\right)\right].$$

We once again observe that in the neighborhood of $10^{-5}$ error probability, the exact curves for $a = 0$ differ from the corresponding bounds by approximately a factor of 10 in error probability, or 1 decibel in signal to noise ratio.

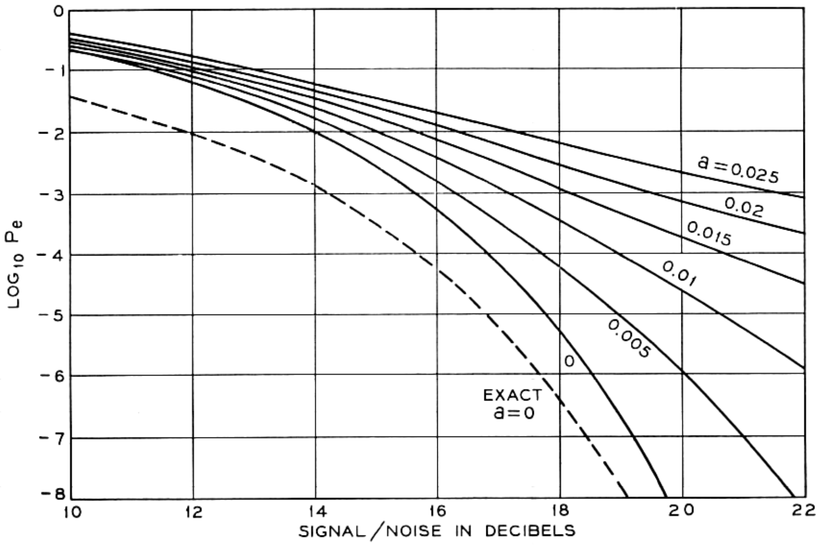


Fig. 7 — Upper bound of the error probability of a binary partial response system with a single-pole high-pass filter.

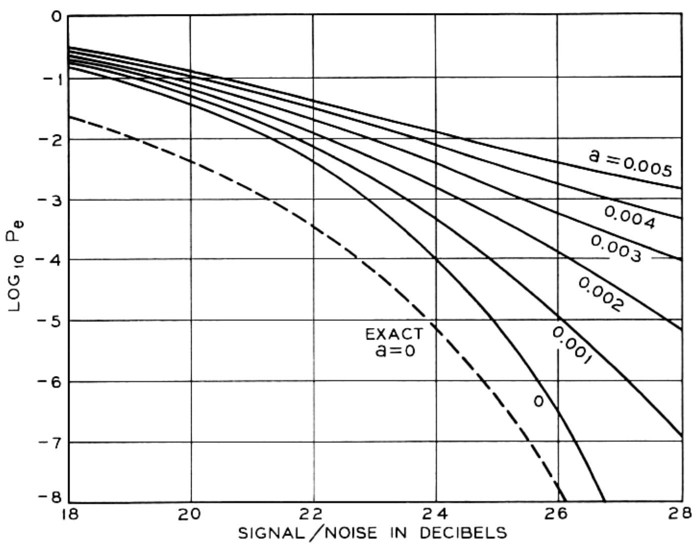


Fig. 8 — Upper bound of the error probability of a 4-level partial response system with a single-pole high-pass filter.

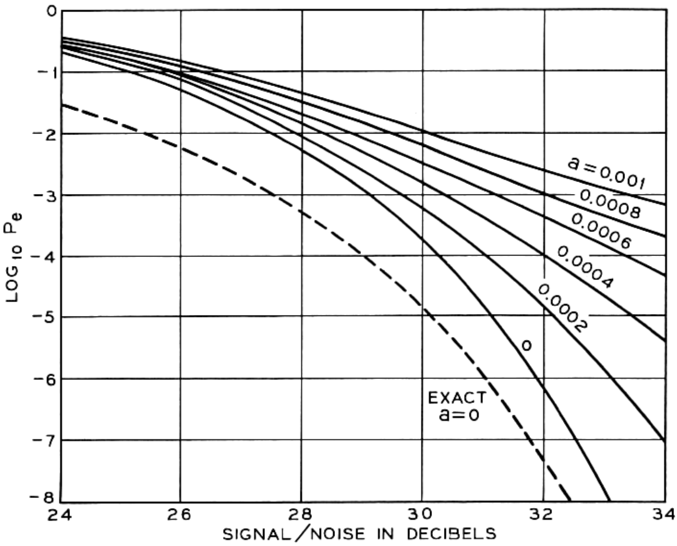


Fig. 9 — Upper bound of the error probability of an 8-level partial response system with a single-pole high-pass filter.

However, to achieve a S/N degradation of only 1 decibel in the region of $10^{-5}$ error probability with a simple high-pass filter, the time constant must be about four times that needed for the standard format signal. The above statement is true for the binary, 4-level, and 8-level cases. This more stringent requirement on the location of the low frequency cutoff may be viewed as a tradeoff for the saving in bandwidth associated with partial response signaling.

## V. CONCLUSIONS

Although a high-pass filter will always close the eye pattern of *i* a standard format data signal (Nyquist I shaping) or *iii* a multilevel partial response signal (duobinary format), the error probability may still be quite low for random data provided that the high-pass filter is sufficiently narrow. This effect permits the use of capacitor or transformer coupling in the data terminals or transmission facilities. Multilevel systems require a longer time constant for these networks than do binary systems for the same performance.

Upper bounds of error probability have been given for binary, 4-level, and 8-level systems with gaussian noise and a single-pole high-pass filter (exponential time response). A binary system with a standard format input signal is degraded by only about 1 decibel by a simple high-pass filter whose time constant is 50 bit intervals. Four-level and 8-level systems require time constants of 250 and 1000 baud intervals, respectively, for the same performance.

A data system whose input is a binary, 4-level, or 8-level partial response signal must have a low frequency cutoff which is two octaves lower in order to achieve the same performance as a standard format system.

The error signal for a multiple-order pole is an exponential multiplied by a generalized Laguerre polynomial. The performance of a system with an $n$th order pole high-pass filter is worse than one with a single pole $n$ times as large.

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